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C. A CHARACTERISTIC PROPERTY OF CERTAIN POLYNOMIALS OF E. F. MOORE AND C. E. SHANNON

Let $\overline{L}_n$ be the set of all Boolian functions, and L_n the subset of all Boolian functions not involving the negation operation, in the n variates $\alpha(i)$ $(i=1,2,\ldots,n)$. For any $\mu \in \overline{L}_n$, if the $\alpha(i)$ are random independent variates with $\Pr(\alpha(i) = 1) = p$, then $\Pr(\mu=1)$ is a polynomial (1), $h(\mu)$ in p. We give an elementary necessary and sufficient condition for the existence of at least a $\lambda \in L_n$ for which $h(\mu) = h(\lambda)$. As is well known (2), there is a natural one-to-one correspondence between L_n and the set of all simplicial complexes with, at most, n vertices. Consequently, this condition is also a characterization of the sequences of integers $\{a_j\}$ that can be the number of j-simplexes contained in a complex and its boundary. Because of this interpretation, it is unlikely that the condition is new, but I have not been able to find any relevant reference to it.

With the help of this condition and of the corresponding extremal functions $\omega(g)$ and $\Sigma P_{n-j}^{a_j}$, defined below, more elementary proofs can be given for Yamamoto's inequality (2) on the number of prime implicants of $\lambda \in L_n$ and for the Moore-Shannon lower bound (1) on the value of the derivative of $h(\lambda)$ $(\lambda \in L_n)$.

1. Notations

i. Let P_m^x be the set of the x first products of m of the variates $\alpha(i)$ when these products are taken in lexicographic order with respect to the indices i. We write P_m, instead of P_m^x, when x has its maximal value $\begin{bmatrix} n \\ m \end{bmatrix}$ and $P = \bigcup_m P_m$. For any subset $P' \subset P$, $\Sigma P'$ denotes the Boolian function (belonging to L_n) which is the sum of all the products, β, satisfying $\beta \leq \beta'$ for some $\beta' \in P'$. Conversely, for any $\lambda \in L_n$, $P_m\lambda$ is defined as the set of all the $\beta \in P_m$ that are such that $\beta \leq \lambda$. Thus, $\lambda = \Sigma P\lambda$ for any $\lambda \in L_n$. The set of all products, $\beta \in P'$, of the form $\beta = \beta'\alpha(i)$, with $\beta' \in P'$ and with $\alpha(i)$ not a factor of β', is denoted by $\Delta P'$ [cf. Yamamoto (2)].

ii. To every pair of positive integers x and m there corresponds one and only one strictly decreasing sequence of $m' \leq m$ positive integers: $y_1, y_2, \ldots, y_{m'}$ with the property that

$$x = \begin{bmatrix} y_1 \\ m \end{bmatrix} + \begin{bmatrix} y_2 \\ m-1 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} y_{m'} \\ m - m' + 1 \end{bmatrix}$$

Consequently, the function

$$D_m(x) = \begin{bmatrix} y_1 \\ m-1 \end{bmatrix} + \begin{bmatrix} y_2 \\ m-2 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} y_{m'} \\ m-m' \end{bmatrix}$$

is well determined for all non-negative integers x and m if we define $D_m(0)$ and $D_0(x)$ as zero.

For any x and m, $x + D_m(x) \geq D_{m+1}(x)$ (with a strict inequality if and only if $x > m + 1$). For all x, $D_m(x) + D_{m-1}(x') \geq D_m(x+x')$ if and only if $x' \leq D_m(x)$.

iii. For any $\mu \in \overline{L}_n$, we define the polynomial $g(\mu)$ as the product by $(1+t)^n$ of the function obtained when $(1+t)^{-1}$ is substituted for p in $h(\mu)$. The coefficient a_j of t^j in $g(\mu)$ is the number of monomials with n - j asserted, and j negated, variates $\alpha(i)$ in the canonical expansion of μ; when $\mu \in L_n$, a_j is also the number of elements in $P_{n-j}\mu$.

2. Statement of the Condition

For any $\mu \in \overline{L}_n$, a necessary and sufficient condition that there exist a $\lambda \in L_n$ for which $g(\mu) = g(\lambda)\left(=a_o+a_1t+\ldots+a_mt^m\right)$ is that

$$\begin{bmatrix} n \\ j-1 \end{bmatrix} \geq a_{j-1} \geq D_j(a_j), \qquad \text{for all } j > 0$$

3. Verification

The condition is sufficient. since, for any polynomial $g(\mu)$ that fulfills it, we can define a function $\omega(g) \in L_n$ as

$$\omega(g) = \Sigma\left(\bigcup_j P_{n-j}^{a_j}\right)$$

and $\omega(g)$ satisfies $g(\omega(g)) = g$ because $\Delta P_m^x = P_{m+1}^{x'}$, when $x' = D_{n-m}(x)$.

It can be remarked that the functions $\Sigma P_{n-j}^{a_j}$ are the only functions in L_n for which $a_{j'-1} = D_{j'}(a_{j'})$ for all $j' \leq j$.

The condition is necessary. The first inequality is obvious. With respect to the proof of the second inequality it is enough to consider a truncated function $\lambda = \Sigma P_{n-j}\lambda$ with a_j prime implicants. Let α and α' be any two $\alpha(i)$'s. Then, λ can be written as $\alpha\alpha'A + \alpha(B+C) + \alpha'(B+C') + D$, where A, B, C, C', and D are sums of products not involving α and α', and where, furthermore, $P_{n-j+2}C$ and $P_{n-j+2}C'$ are disjoint sets. It is readily checked that the function $\lambda' = \alpha\alpha'A + \alpha(B+C+C') + \alpha'(B) + D$ is such that the set $P_{n-j}\lambda'$ has a_j elements and that the set $P_{n-j+1}\Delta\lambda'$ has, at most, as many elements as $P_{n-j+1}\Delta\lambda$. By taking successively $\alpha = \alpha(i)$ and $\alpha' = \alpha(i+1)$ for all i, we can reduce the function λ to a function $\Sigma P_j^{a_j}$ and the result is proved.

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References

1. E. F. Moore and C. E. Shannon, J. Franklin Inst. 262, 191-208; 281-297 (1956).
2. K. Yamamoto, J. Math. Soc. Japan 6, 343-353 (1954).