ON A FAMILY OF SUBMONOIDS

by

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§ 1. Introduction

As it is well known, only few of the properties of the subgroups of a group are still enjoyed by all the submonoids of a monoid [1] and in the applications it is sometimes useful to consider more restricted families of stable subsets (i. e. of subsets A which are such that $A^2 \subset A$).

In remote connection with a problem in communication theory (Cf. [12]) one encounters a family $\mathfrak{R}(F)$ of submonoids of a monoid F that is characterized by extremal properties and that, consequently, admits several slightly different definitions. When F is a group, $\mathfrak{R}(F)$ reduces to the lattice of the subgroups of F ; in the general case, it is not necessarily a lattice and its simplest definition is the following one.

Definition. The submonoid A of a monoid F belongs to $\mathfrak{R}(F)$ if and only if it satisfies the following three conditions :

1. There exists at least one homomorphism γ of F, compatible with A (i. e. $\gamma^{-1} \gamma A = A$) which is such that γA is isomorphic to a monoid admitting minimal left and right ideals ;

2. (N_k) : A intersects every right and every left ideal of F ;

3. A is maximal among the submonoids of F that have the same intersectoin with an arbitrarily small two-sided ideal of F.

Let us abbreviate by $N_d(N_r, N_l, N_k)$ the condition that A intersects every two sided (right, left, right and left) ideal of F (i. e. that A is "net" in P. Dubreil's theory [5]), by $M_d(M_r, M_l, M_k)$ the condition that γF admits minimal two-sided (right, left, right and left) ideals for some homomorphism γ compatible with A.

We shall verify that $\mathfrak{R}(F)$ can also be defined by the following set of three conditions : A satisfies

1′. M_r ;

2′. N_l;

3′. There exists some right representation of F by mappings of a set into itself that is such that A is submonoid which lets invariant one element from the set.

[1] Cambridge (Mass).

Let us recall that LEVY's condition [8] that a stable subset A of a free monoid F is isomorphic to a free monoid can be expressed in the form (Cf. [12]).

$$(U_d): \quad fA \cap Af \cap A \neq \varnothing \quad \text{only if} \quad f \in A$$

which remains meaningful even when F is not a free monoid.

We shall also verify that $\mathfrak{K}(F)$ is characterized by the following set of conditions on A :

1″. M_k ;
2″. N_k ;
3″. U_d .

When F is finite, the conditions 1, 1′ or 1″ become vacuous. Then $\mathfrak{K}(F)$ can be characterized by 3, 3′ or 3″ and the requirement that A contains at least one positive power of each element from F.

In § 2, as a preliminary step, we apply the classical theory of SUSCHKEWITSCH [18] and REES [11] for obtaining a direct characterization of $\mathfrak{K}(F)$ when F admits minimal left and right ideals. In §§ 3 and 4 respectively we discuss the sets of conditions (1″, 2″, 3″) and (1′, 2′, 3′). In order to make the paper self contained several results which are special cases of theorems due to other authors are given complete proofs.

Applications of the remarks developed here to the less restricted family of the submonoids which satisfy U_d only will be considered in another paper.

§ 2. A direct definition of $\mathfrak{K}(F)$

Let us verify first the following

Remark 2.1. If the stable subset A of a monoid F satisfies N_r and admits minimal right ideals, then, F also admits minimal right ideals.

Proof. Let us consider any $a \in A$ such that aA is a minimal right ideal of A; by definition this is equivalent to the statement that, for any $a' \in A$, there exists at least one $a'' \in A$ which is such that $aa'a'' = a$ since, unless, the right ideal $aa'A$ would be a proper subset of aA.

Trivially, if aA is minimal, the same is true of any $a'''A$ where $a''' \in A\,a\,A$.

Let us show that if A satisfies N_r, a^2F is a minimal right ideal of F. Indeed, for any $f \in F$, N_r implies that $A \cap afF \neq \varnothing$, i. e. that $aff' = a_1 \in A$ for some $f' \in F$; multiplying on the left by a, we obtain $a^2ff' = aa_1$. By our previous remark, there exists at least one $a_1' \in A$ which satisfies $aa_1a_1' = a$. Thus, $a^2ff'a_1'a = a^2$ and the result is verified.

We observe that when the homomorphism γ is compatible with A any of the conditions M_x, N_x or $U_x (x = d, r, l, k)$ defined in the introduction (or later) is true for γA in γF if and only if it is true for A in F. Since we have seen that when A satisfies N_k the condition 1 implies M_k, there will be no loss in generality for the description of a given A from $\mathfrak{K}(F)$ in assuming that F itself admits minimal ideals.

This convention will be kept in the §§ 2 and 3 and we shall use the following standing notations :

The monoid F admits the minimal right ideals R_i ($i \in I$) and the minimal left ideals L_j ($j \in J$). The minimal two-sided ideal of F is denoted by D and the following facts are classical (Cf. [18], [3], [16])

1. $$D = \bigcup_{i \in I} R_i = \bigcup_{j \in J} L_j$$

2. every quasi ideal $K_{i,j} = R_i \cap L_j$ is isomorphic to a certain group G, the SUSCHKEWITSCH *group* of F. (A *quasi ideal* is the intersection of a left and of a right ideal [16]).

3. The idempotent $e_{i,j}$ of $K_{i,j}$ is such that $de_{i,j} = d$ and $e_{i,j}\, d' = d'$ for any $d \in L_j$ and $d' \in R_i$; thus, identically, $K_{i,j} = e_{i,j}\, F\, e_{i,j}$.

We select a fixed arbitrary quasi ideal $K_{1,1}$ and isomorphism $\sigma : K_{1,1} \to G$ and we introduce the following standing notations : $g_{j,i} = \sigma(e_{1,j}\, e_{i,1})$ ($= e_G$, the neutral element of G when i or j is equal to 1 since $e_{1,j}\, e_{1,1}\, e_{i,1} = e_{1,1}$ identically).

G_0 = the subgroup of G generated by the elements $g_{j,i}$.

σ' = the mapping $D \to G$ which is defined by $\sigma' d = \sigma(e_{1,j}\, de_{1,1})$ where j is the index of the left ideal L_j containing d.

$\tau_{i,j}$ = the mapping $G \to K_{i,j}$ which is defined by $\tau_{i,j} g = e_{i,1} \cdot \sigma^{-1}(g_{j,i}^{-1} g) \cdot e_{1,j}$.

It is classical that $\tau_{i,j}$ and the restriction of σ' to $K_{i,j}$ are mutually inverse isomorphisms (onto) (Cf. [11], [2], [10]). Indeed, $\tau_{i,j}$ is a homomorphism because of the following more general formula valid for any $g, g' \in G$

$$(\tau_{i,j}\, g)\,(\tau_{i',j'}\, g') = e_{i,1} \cdot \sigma^{-1}(g_{j,i}^{-1}\, g) \cdot e_{1,j} \cdot e_{i',1} \cdot \sigma^{-1}(g_{j',i'}^{-1}\, g') \cdot e_{1,j'} =$$
$$= e_{i,1} \cdot \sigma^{-1}(g_{j',i}^{-1} g'') \cdot e_{1,j'} = \tau_{i,j'}\, g''$$

where

$g'' = g_{j',i}\, g_{j,i}^{-1}\, g\, g_{j,i'}\, g_{j',i'}^{-1}\, g'$; thus, when $i = i'$ and $j = j'$, we have simply

$$(\tau_{i,j} g)(\tau_{i',j} g') = \tau_{i,j}(gg').$$

Because of the formula

$$\sigma'\, \tau_{i,j}\, g = \sigma\left(e_{1,j}\left(e_{i,1}\, \sigma^{-1}\, (g_{j,i}^{-1}\, g)\, e_{1,j}\right) e_{1,1}\right) =$$
$$= \sigma(e_{1,j}\, e_{i,1}) \cdot g_{j,i}^{-1}\, g \cdot \sigma(e_{1,j}\, e_{1,1}) = g,$$

we see that $\tau_{i,j}$ is a monomorphism (i. e. isomorphism into). Finally, it is proved that $\tau_{i,j}$ (and consequently the restriction of σ') is an isomorphism (onto) by the formula valid for any $d \in K_{i,j}$

$$\tau_{i,j}\, \sigma'\, d = e_{i,1} \cdot \sigma^{-1}\left(g_{j,i}^{-1}\, \sigma\, (e_{1,j}\, d\, e_{1,1})\right) \cdot e_{1,j} =$$
$$= \left(e_{i,1} \cdot \sigma^{-1}\, (g_{j,i}^{-1}\, e_G) \cdot e_{1,j}\right) \cdot d \cdot e_{1,j} = e_{i,j}\, de_{1,j} = d.$$

We still need to recall the following simple statement. (Cf. [15], [17]).

Theorem 2.2. *For any non empty stable subset B of D the three following conditions are equivalent*

(i) *For at least one $K_{i,j}$ having a non empty intersection Q with B the subset $\sigma' Q$ of G contains the inverse of each of its elements;*

(ii) *There exist nonempty subsets I_B of I and J_B of J and a subgroup G' of G that have the following properties: G' contains every $g_{j,i}$ $\{(i,j) \in I_B \times J_B\}$, $B = \{d \in D: \sigma' d \in G'$ and $d \in K_{i,j}$ $[(i,j) \in I_B \times J_B]\}$*

(iii) *B admits minimal right (and left) ideals.*

Proof. (i) *implies* (ii). Because of the fact that the restriction of σ' to $K_{i,j}$ is an isomorphism, there is no loss in generality in taking $(i,j) = (1,1)$ in the condition (i) which then, (because B is stable) becomes equivalent to the condition that $G' = \sigma Q$ is a subgroup of G. Thus $e_{1,1} = \sigma^{-1} e_G$ belongs to B.

Trivially, if $b \in R_i \cap B$ and $b' \in L_j \cap B$, we have $bb' \in K_{i,j} \cap B$ and, thus, $K_{i,j} \cap B \neq \varnothing$ if and only if $(i,j) \in I_B \times J_B$ where I_B and J_B are subsets of I and J respectively.

Let b be any element from $K_{i,j} \cap B$; we have $\sigma(e_{1,1} b^3 e_{1\,1}) = g' \in G'$ and, since G' is a group, $b' = b\sigma^{-1}(g'^{-1})\, b$ belongs to $K_{i,j} \cap B$. A straightforward computation shows that $bb' = e_{i,j}$ and, thus, we have $e_{i,j} \in B$ and $g_{j,i} \in G'$ for all $(i,j) \in I_B \times J_B$. Consequently, for any such pair (i, j), the mappings $\tau_{i,j}$ and σ' can be carried out by using multiplications by elements from B only. It follows instantly that for any such (i, j) and $g \in G$ (rspectively, $d \in K_{i,j}$) one has $\tau_{i,j} g \in B$ (resp. $\sigma' d \in G'$) if and only if $g \in G'$ (resp. $d \in B$) and this is precisely the formula given in (ii).

(ii) *implies* (iii). Let I_B and J_B be any non empty subsets of I and J and G' any subgroup of G containing all the elements $g_{j,i}$ $(i, j) \in I_B \times J_B$. In order to prove that B as defined in (ii) admits right and left ideals it is enough to show that for any $(i, j) \in (I_B \times J_B)$ one has

$$(\tau_{i,j} G')\, B(\tau_{i,j} G') = \tau_{i,j} G'.$$

This again is a straightforward computation, which also shows that B^2 is contained in B, i. e. that B is stable.

(iii) *implies* (i). Let us assume only that the stable subset B admits minimal right ideals and, for simplicity, that $b \in K_{1,1} \cap B$ is such that bB is minimal. This implies in particular that, to any $b' \in K_{1,1} \cap B$, there corresponds at least one b'' in some suitable $K_{i,1}$ that is such that $bb'b'' = b$; writing $g = \sigma b$, $g' = \sigma b'$, $g'' = \sigma' b''$, it follows that $gg'g'' = g$, i. e. that $g'' = g'^{-1}$. Thus, since $b'b''b'' \in K_{1,1}$, the set $G' = \sigma(K_{1,1} \cap B)$ contains $\sigma(b'b''b'') = g'^{-1}$ whenever it contains g'. Consequently, G' is a subgroup of G and the proof is concluded.

It is useful to observe that the apparently weaker conditions (iii)′ below is in fact equivalent to (iii).

(iii)′. *There exists a homomorphism γ of F which is such that $D \cup \gamma^{-1} \gamma B = B$ and that γB admits minimal right ideals.*

Indeed, since $K_{1,1} F K_{1,1} = K_{1,1}$, any homomorphism γ of F sends $K_{1,1}$ onto a minimal quasi-ideal of $\gamma F = \overline{F}$ and, consequently, γ induces an epimorphism γ' (homomorphism onto) of G onto the Suschkewitsch group $\overline{G}$ of $\overline{F}$.

Let us assume now that γB admits minimal right ideals; because of theorem 2.2, γB admits minimal quasi-ideals and, since $(K_{1,1} \cap B)\, B(K_{1,1} \cap B)$ is contained in $K_{1,1} \cap B$, at least one of these minimal quasi ideals, $Q_{1,1}$ say, is contained in $\overline{K}_{1,1} = \gamma K_{1,1}$. Thus $\gamma' Q_{1,1}$ is a subgroup $\overline{G}'$ of $\overline{G}$ and the stable subset $\bar{G}'' = \gamma' \gamma(K_{1,1} \cap B)$ of $\bar{G}$ satisfies the conditions $\bar{G}' \bar{G}'' \bar{G}' = \bar{G}'$ and $\bar{G}' \subset \bar{G}''$.

From this we conclude that $\bar{G}' = \bar{G}''$, that is, $Q_{1,1} = \gamma(K_{1,1} \cap B)$.

This ends the proof because, when $D \cap \gamma^{-1} \gamma B = B$, it shows that the stable subset $K_{1,1} \cap B = D \cap \gamma^{-1} Q_{1,1}$ is equal to $\sigma^{-1} \gamma'^{-1} G'$ where $\gamma'^{-1} \bar{G}'$ is a subgroup of G, and that, consequently, the condition (i) is satisfied.

Let us define a mapping χ from F to the set of right cosets of G over G_0 by the rule

$$\chi f = G_0 \sigma(e_{1,i} f e_{1,1}).$$

We have

Remark 2.3. (i) If $f \in D$, $\chi f = G_0 \sigma' d$;

(ii) for any $f, f' \in F$, $\chi(ff') \subset (\chi f)(\chi f')$.

Proof. We verify first that for any $f \in F$ and $j \in J$, $\sigma(e_{1,j} f e_{1,1})$ belongs to χf. Indeed, $f e_{1,1}$ belongs to a well defined $K_{i,1}$ and, using $\tau_{i,1}$ we obtain

$$f e_{1,1} = e_{i,j} \cdot \sigma^{-1}(g_{1,i}^{-1} \sigma'(f e_{1,1})) \cdot e_{1,1} = e_{i,1} \cdot \sigma^{-1}(\sigma'(f e_{1,1})).$$

Thus, for any $e_{1,j}$,

$$\sigma(e_{1,j} f e_{1,1}) = \sigma(e_{1,j} e_{i,1}) \sigma'(f e_{1,1}) \in G_0 \chi f.$$

This proves the statement (i).

Let now $f, f' \in F$. The product $e_{1,1} f$ belongs to a well defined $K_{1,j}$ and we have

$$\sigma(e_{1,1} f f' e_{1,1}) = \sigma(e_{1,1} f e_{1,j} f' e_{1,1}) = \sigma(e_{1,1} f e_{1,1}) \sigma(e_{1,j} f' e_{1,1}),$$

that is,

$\chi(ff') = (\chi f) \sigma(e_{1,j} f' e_{1,1})$ and the statement (ii) follows from our initial remark.

Theorem 2.4. *A necessary and sufficient condition that A belongs to $\mathfrak{K}(F)$ is that*

$$A = \{f \in F : \quad \chi f \subset G'\}$$

where G' is any subgroup of G that contains G_0.

Proof. The condition is necessary because, if A belongs to $\mathfrak{K}(F)$, its intersection B with any $K_{i,j}$ is not empty (condition 2) and, according to the condition 1, it satisfies the condition (iii)' of theorem 2.2. Thus, by theorem 2.2 and remark 2.3 (i), we have $B = A \cap D = \{d \in D : \chi d \subset G'\}$ where G' is a subgroup containing G_0. Since remark 2.3 (ii) shows trivially that BfB is contained in B if and only if χf is contained in G' the condition 3 of the introduction implies that A is precisely the set of those elements from F.

The condition is sufficient because, if $A = \{f : \chi f \subset G'\}$, remark 2.3 (i) and (ii) show that $A^2 \subset A$, and that $A \cap D = B$ is a stable subset which satisfies the conditions of theorem 2.2 and $BAB = B$. Thus the conditions 1 and 2 are satisfied and since, as above, BfB is contained in B only if $\chi f \subset G'$, the maximality condition 3 is also verified.

As a consequence we have

Corollary 2.5. If A belongs to $\mathfrak{K}(F)$ and if the index m in G of the subgroup G' defined above is finite, at least one positive power $f^{m'}$, $m' \leq m$ of each f belongs to A.

Proof. Let us observe that, for any f, $f' \in F$, one has $G'\chi(ff') = G'\chi f'$ if and only if $G'\chi f' = G'$, that is, by theorem 2.4, if and only if f belongs to A.

Since, by hypothesis, not all the $m+1$ cosets

$$G',\ G'\chi f,\ G'\chi f^2,\ \ldots,\ G'\chi f^m$$

are distinct, one must have $G'\chi f^{m''} = G'\chi f^{m'+m''}$, i. e. $f^{m'} \in A$ for some positive m' at the most equal to m.

§ 3. The conditions U_x.

In this § we use the following conditions U_x ($x = d, r, f, k$) for characterizing $\mathfrak{K}(F)$. We recall that U_d is defined by

$$(U_d):\ fA \cap Af \cap A \neq \emptyset \quad \text{only if } f \in A.$$

Thus, if A is a nonempty stable subset satisfying U_d, it is a submonoid (i. e. it contains the neutral element e of F) because $eA \cap Ae \cap A \neq \emptyset$. It is readily verified that, when A is stable, equivalent forms of U_d are

$$fA \cap A \neq \emptyset \text{ and } Af \cap A \neq \emptyset \text{ only if } f \in A\ ;$$

(because a, $af = a_1 \in A$, and a', $fa' = a_1' \in A$ imply $(a_1' a) f = f(a' a_1) = a_1 a_1' \in A$)
and, also
a, af, $fa \in A$ only if $f \in A$.
We define U_r by

$$(U_r):\quad Af \cap A \neq \emptyset \text{ only if } f \in A.$$

Then, U_r (or the symmetric condition U_l, $fA \cap A \neq \emptyset$ only if $f \in A$) implies U_d. As it is easily checked (Cf. the beginning of 4 below), U_r is equivalent to the condition $2'$ of the introduction.

When A is a submonoid, the conjunction U_k of the conditions U_r and U_l is more expeditiously written as

$$(U_k):\quad A \cap AfA \neq \emptyset \quad \text{only if } f \in A.$$

A theory of the subsets, which satisfy U_x ($x = r, l, k$) ("les complexes unitaires") is due to P. Dubreil [6].

We first verify the following

Remark 3.1. When the submonoid A of F satisfies M_k, N_d and U_d, the condition N_r (respectively N_l) is a necessary and sufficient condition that it satisfies U_r (respectively U_l).

Proof. Because of M_k we can assume without loss of generality that F itself admits minimal right and left ideals and we use freely the notations of § 2. The condition N_d can be taken as the hypothesis that $A \cap K_{1,1}$ is not empty.

Let us first verify that $B = A \cap D$ satisfies the condition (i) of theorem 2.2 (i. e. $G' = \sigma(A \cap K_{1,1})$ is a subgroup of G). Indeed, if $g = \sigma a \in G'$, for some

$a \in A \cap K_{1,1}$, the element $b = \sigma^{-1}g^{-1}$ satisfies the relation $ba^2 = a^2b = a$, that is, $bA \cap Ab \cap A \neq \varnothing$. Thus, by U_d, $b \in A$ and, finally, $g^{-1} = \sigma b \in G'$.

(Reciprocally if F reduces to the union of D and a neutral element e, it is easily checked that for any B satisfying the conditions of theorem 2.2 the submonoid union of B and e, satisfies the condition U_d. Indeed, for any $d \in D$,

$$dB \cap B \neq \varnothing \quad \text{and} \quad Bd \cap B \neq \varnothing \quad \text{imply} \quad d \in K_{i,j} \text{ with } (i, j) \in I_B \times J_B$$

and, then b, $db \in B$ implies $\sigma' d \in G'$).

Now we have :

N_r implies U_r.

Because of our hypothesis, N_r is equivalent to the requirement that every $e_{i,1}$ $(i \in I)$ belongs to A, or, in the notations of theorem 2.2, that $I = I_B$. It follows that for any $d \in D$, if $bd \in A$ for some $b \in B$, then d belongs to A ; indeed, $bd \in A$ implies $d \in L_j$, where $j \in J_B$ and $\sigma' d \in G'$ since G' is a subgroup which contains all the elements $g_{j,i}$ with $(i, j) \in I \times J_B$.

This practically ends the proof because if a, $af \in A$, the element $d = fe_{1,1}$ from D satisfies the condition $bd \in A$ with $b = e_{1,1}a \in B$. Thus we have, a, af, $e_{1,1}$, $fe_{1,1} \in A$ and, by U_d, we conclude that a, $af \in A$ only if $f \in A$, that is, U_r.

The reciprocal statement (U_r implies N_r) is contained in the following slightly less special implication which will be needed later :

When M_r, N_d and U_r imply N_r.

We assume that F itself contains an element r which is such that the ideal rF is minimal ; thus, because of N_d, A contains at least one element $b \in FrF \cap A$ which is such that bF is a minimal right ideal.

Let us show that $A \cap fF \neq \varnothing$ for all $f \in F$, (i. e., N_r) ; indeed since bF is minimal, there exists at least one f' which is such that $b = bff'$. Because of U_r the product ff' belongs to A and this concluedes the proof.

Theorem 3.2. *If the submonoid A satisfies M_k, necessary and sufficient conditions that it belongs to $\mathfrak{K}(F)$ are U_d and N_k or U_r and N_l or U_k and N_d.*

Proof. Let us assume that A belongs to $\mathfrak{K}(F)$ and use the notations of theorem 2.4 ; by corollary 2.5 every idempotent of F belongs to A, and consequently A satisfies N_k ; the fact that $AfA \cap A \neq \varnothing$ only if $f \in A$ (i. e. U_k) has already been verified in the proof of theorem 2.4.

Reciprocally, we observe that, according to remark 3.1, the three conditions "U_x and N_x," are equivalent to "U_k and N_k" when A satisfies M_k. Using the notations of remark 3.1, the condition N_k imposes that $B = A \cap D$ intersects every $K_{i,j}$ and consequently $B = \{d \in D : \chi d \subset G'\}$ where G' is a subgroup containing G_0 ; once more, since $BfB \subset B$ only if χf is contained in G' we finally obtain $A = \{f : \chi f \not\cap G'\}$ and the result is entirely proved.

§ 4. The set of conditions (1′, 2′, 3′)

In order to make the proof clearer we recall first the following well known result (Cf. [19]) :

Theorem 4.1. *To any nonempty subset X of F there corresponds one quotient monoid $\gamma_X F$ which is characterized by the following properties*

(i) *The homomorphism γ_X is compatible with X ;*

(ii) *If γ' is any homomorphism of F compatible with $X, \gamma_X F$ is a homomorphic image of $\gamma' F$.*

Proof. Let us consider the mapping λ_X of F to the subsets of F that is defined by

$$\lambda_X f = \{f' \in F : ff' \in X\}$$

(Cf. [5]).

We have

1. if $x \in X$, $\lambda_X f = \lambda_X x$ only if $f \in X$ (because $\lambda_X f$ contains e if and only if $f \in X$) ;

2. if $\lambda_X f = \lambda_X f'$, then $\lambda_X (ff'') = \lambda_X(f'f'')$ for all $f'' \in F$.

Consequently, if S denotes the set of all $\lambda_X f (f \in F)$, we can define a representation $(S, F) \to S$ by

$(\lambda_X f) f' = \lambda_X(ff')$ for any $\lambda_X f \in S$ and $f' \in F$.

We denote the corresponding homomorphism of F by γ_X and we observe that the congruence relation $\gamma_X f = \gamma_X f'$ (i. e. $\lambda_X(f''f) = \lambda_X f(f''f')$ for all $f'' \in F$) can be expressed in the symmetrical form:

for all $f_1, f_2 \in F$, $f_1 f f_2 \in X$ if and only if $f_1 f' f_2 \in X$.

This shows instantly that γ_X is compatible with X since $efe \in X$ if and only if $f \in X$.

Let now $\gamma' : F \to \bar{F}$ be any homomorphism and define $\bar{X} = \gamma' X$; we can construct in the same manner as above a quotient monoid $\gamma_{\bar{X}} \bar{F}$ and for any $f, f' \in F$ we have $\gamma_{\bar{X}} \gamma' f = \gamma_{\bar{X}} \gamma' f'$ only when for all $f_1, f_2 \in \bar{F}$

$\gamma' f_1 f f_2 \in \gamma' X$ if and only if $\gamma' f_1 f' f_2 \in \gamma' X$.

Consequently, when $\gamma'^{-1} \gamma' X = X$, we have $\gamma_{\bar{X}} \gamma' f = \gamma_{\bar{X}} \gamma' f'$ only if $\gamma_X f = \gamma_X f'$ and the result is proved.

Incidentally, the notations introduced provide the formal verification that U_r is equivalent to the condition $2'$ of the introduction, because on the one hand, if A is stable and if it satisfies U_r, we have $e \in A$ and $\lambda_A a = A$ for any $a \in A$; thus $\lambda_A e = \lambda_A f$ if and only if $f \in A$ and A is precisely the submonoid which lets $\lambda_A e$ invariant in the representation $(S, F) \to S$ described above. On the other hand, if S' is any set and $(S', F) \to S'$ a representation, for any given $s \in S'$, the submonoid $A' = \{f \in F : sf = s\}$ satisfies U_r because of the associativity.

Theorem 4.2. *If the stable subset A of F satisfies M_r, N_l and U_r, it belongs to $\mathfrak{K}(F)$.*

Proof. Since N_l is stronger than N_d, we already know by the last part of the proof of remark 3.1 that A satisfies N_r and we shall repeatedly use this fact.

Without loss of generality we shall assume that $F = \gamma_A F$; consequently, because of theorem 2.1 and M_r, the monoid F itself admits minimal right ideals ; it will be enough to verify that it admits also minimal left ideals, because, then, by remark 3.1, U_l is a simple consequence of M_k, N_k and U_r.

The verification involves three steps.

i. Let $b \in A$ be a fixed element such that bF is a minimal right ideal (such an element exists because of N_d). We verify that for any $f \in F$ there exists at least one $f' \in F$ which is such that $\lambda_A fb = \lambda_A f'b^2$.

Indeed, by N_r, $fbf_1 \in A$ for some f_1; by N_l, $f'b^2f_1 \in A$ for some f'; by the hypothesis that bF is minimal, $bf_1f_1' = b$ for some f_1'. Thus

$$\lambda_A fbf_1 = \lambda_A f' b^2 f_1 = A$$

because of the hypothesis that A satisfies U_r. Finally, multiplying by f_1' we get

$$\lambda_A fb = \lambda_A fbf_1 f_1' = \lambda_A f' b^2 f_1 f_1' = \lambda_A f' b^2$$

and our remark is proved.

ii. Let us keep the same notations and define $\bar{b}$ by the condition that $b^2 \bar{b} = b$.

From the relation $\lambda_A fb = \lambda_A f'b^2$, we deduce by multiplication by $\bar{b} b$ that

$$\lambda_A fb\bar{b}b = \lambda_A f' b^2 \bar{b} b = \lambda_A f' b^2 = \lambda_A f b .$$

Since this holds for each $f \in F$, it follows from the hypothesis $\gamma_A F = F$ that $b \bar{b} b = b$. Consequently $b \bar{b}$ is an idempotent.

The last step is classical (cf. [3], [15], [16]) but we include its proof here for the sake of completeness :

iii. If F contains an idempotent c which is such that cF is a minimal right ideal, then, Fc is a minimal left ideal.

Indeed, for any $f_1 \in F$, we have $cf_1cf_2 = c$ for some f_2 and $cf_2cf_3 = c$ for some f_3 because of the minimal character of cF. Multiplying the last equality by cf_1 we get

$$cf_1 cf_2 cf_3 = cf_1c, \quad \text{that is} \quad ccf_3 c = c f_1 c.$$

Consequently, $cf_2cf_1c = c^2 = c$ and the result is proved since we have shown that c belongs to any left ideal Ff_1c.

Remark. Counter examples (cf. [13]) show that it is not possible to dispense entirely with some requirement on the minimal ideals in the various implications between the conditions N_x and U_x, described here.

For example, let F be the monoid of permutations of the set of integers generated by the translation $n \to n + 1$, and $n \to n - 1$ and the permutation which lets invariant the negative integers and which consists of the cycles

$$(1,2)\,(3,4,5)\,(6,7,8,9)\,\ldots\,\left(\frac{n\cdot n-1}{2}, \frac{n\cdot n-1}{2}+1, \ldots, \frac{n\cdot n+1}{2}-1\right)\ldots$$

Let A be the submonoid of F that lets 0 invariant. It is easily checked that $\gamma_A F = F$, that F has no minimal ideals and that A satisfies N_k and U_k.

We conclude by giving a simple characterization of $\gamma_A F$ for any A from $\mathfrak{K}(F)$ (cf. [14]).

The notations are that of §§ 2 and 3.

Remark. 4.3. If A belongs to $\mathfrak{K}(F)$, a necessary and sufficient condition that $\gamma_A F = F$ is that $f = f'$ if and only if

$$\pi\sigma(e_{1,j}\, fe_{i,1}) = \pi\sigma(e_{1,j}\, f'\, e_{i,1})$$

for all $(i, j) \in I \times J$ where π is a homomorphism of G whose kernel, E, is the largest normal subgroup of G contained in G'.

Proof. Let us observe that because of U_k and N_k, the relation $f_1 f f_2 \in A$ is equivalent to $e_{1,1} f_1 f f_2 e_{1,1} \in A$ for any three elements f_1, f and f_2 of F. With the help of the mapping $\tau_{i,j}$ (cf. § 1) can we write $e_{1,1} f_1$ and $f_2 e_{1,1}$ as $(\sigma^{-1} g_1) e_{1,i}$ and $e_{j,1}(\sigma^{-1} g_2)$ respectively, for suitable $g_1, g_2 \in G$ and idempotents $e_{1,i}$ and $e_{j,1}$.

Thus, $f_1 f f_2 \in A$ is equivalent to $g_1 \sigma(e_{1,i} f e_{j,1}) g_2 \in G'$ where g_1, g_2 and (i, j) do not depend upon f. It follows from the definitions of γ_A that $\gamma_A f'$ if and only if for each $(i,j) \in I \times J$ and, then, for all $g_1, g_2 \in G$, one has $g_1 \sigma(e_{1,i} f e_{j,1}) g_2 \in G'$ when and only when $g_1 \sigma(e_{1,i} f' e_{j,1}) g_2 \in G'$. Since for each (i,j) this relation between $g = \sigma(e_{1,i} f e_{j,1})$ and $g' = \sigma(e_{1,i} f' e_{j,1})$ is precisely $\gamma_{G'} g = \gamma_{G'} g'$ and since E is, trivially, the kernel of $\gamma_{G'}$ the result is proved.

It follows that a set of necessary and sufficient conditions that $D = \gamma_{G'} D$ is :

i. the only normal subgroup of G contained in G' is $\{e_G\}$;

ii. the $J \times I$ matrix $(g_{j,i})$ has all its rows and columns distinct.

As an application we can display the following example which shows that, even if F is finitely generated, the condition that for some fixed finite m, f^m belongs to A for all $f \in F$ does not insure that $\gamma_A F$ has only finitely many minimal quasi ideals.

Example. Let F consist of e, all the powers a^m of a certain element a and of a minimal two-sided ideal D of the type described in § 1. The group G will be the symmetric group on three elements generated by α and β satisfying the relations $\alpha^2 = \beta^3 = (\alpha\beta)^2 = e_G$; I, and J will be the set of positive integers.

The element a is entirely defined by the rules :

$$e_{1,j}\, a = \begin{cases} \tau_{1,j+1}(e_G) & \text{if } j \text{ is not a power of 2,} \\ \tau_{1,j+1}(\alpha) & \text{if } j \text{ is a power of 2.} \end{cases}$$

We define the right ideals R_i by $R_1 = e_{1,1} F$, $R_{i+1} = a R_i$ and, accordingly, the matrix $(g_{j,i})$ has all its entries in the subgroup $G_0 = \{e_G, \alpha\}$.

Finally, $A = \{f \in F : \chi f = G_0\}$ contains the sixth power of every element of F and it belongs to $\mathfrak{K}(F)$.

By considering for each value of m ≥ 0 the sub-block of the matrix $(g_{j,i})$ determined by $1 \leq i \leq 2^m$, $1 + 2^m \leq j \leq 2^{m+1}$, one easily checks that no two rows of this matrix are the same and that consequently, it also contains infinitely many distinct columns.

Thus $\gamma_A F$ is not finite and, since F is generated by a and $b = \tau_{1,1}(\beta)$, the example has all the properties stated.

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ОБ ОДНОМ СЕМЕЙСТВЕ ПОДМОНОИДОВ

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Резюме

В этой заметке описывается некоторое семейство $K(\boldsymbol{F})$ подмоноидов моноида $\boldsymbol{F}$, имеющих свойства, возможно близкие к свойствам подгрупп некоторой группы. Если $\boldsymbol{F}$ — свободный моноид, тогда подмоноиды семейства $K(\boldsymbol{F})$ имеют приложения к некоторым вопросам кодирования как особому случаю свободных подмоноидов моноида $\boldsymbol{F}$. Характерно, что если A принадлежит $K(\boldsymbol{F})$, то для каждого $f \in \boldsymbol{F}$ найдется хотя бы один f' такой, что $ff'f \in A$ (существование слабого обратного элемента) и, наоборот, если f и $ff'f$ принадлежат A, то f' тоже принадлежит A (каждый слабый обратный некоторого элемента подмоноида $A \in K(\boldsymbol{F})$ принадлежит A).

Большая часть статьи посвящена дисскуссии того заслуживающего внимания факта, что при обычных ограничениях относительно существования минимальных идеалов эти двухсторонние условия содержатся в еще более слабых аналогичных односторонних условиях.