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ON A SPECIAL CLASS OF RECURRENT EVENTS

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I. Introduction. Let F be the set of all finite sequences (*words*) in the symbols $x \varepsilon X$. According to W. Feller ([2], Chap. VIII), a recurrent event $\mathcal{E}$ is a pair (A, μ) where A is a subset of F and μ a probability measure fulfilling the conditions recalled below; one says that the event $\mathcal{E} = (A, \mu)$ occurs at the last letter x_{i_n} of a word $f = x_{i_1}x_{i_2} \cdots x_{i_n}$ if and only if f belongs to the set A; we shall call A the *support* of $\mathcal{E}$ and denote by $T(A, \mu)$ the mean recurrence time of the event $\mathcal{E}$.

If the pair (B, μ') defines another recurrent event on F, the pair $(A \cap B, \mu')$ defines also a recurrent event. It results from the general theory of Feller ([2], Chap. VIII) that, when $T(B, \mu')$ is finite, the ratio $\pi = T(B, \mu')/T(A \cap B, \mu')$ is, in a certain sense, the limit of the conditional probability that a random word $f \varepsilon F$ belongs to A when it is known to belong to B. For given arbitrary A, it is in general possible to find infinitely many (B, μ') having finite $T(B, \mu')$ which are such that $\pi = 0$.

The main point of this note is to verify several statements which, together, imply the following property:

Property 1. If the support A *is such that* $T(A \cap B, \mu')$ *is finite for* every recurrent event (B, μ') having *finite* $T(B, \mu')$, *then, for every such* (B, μ'), π^{-1} *is an integer at most equal to a certain finite number* δ^* *which depends only upon* A.

Classical examples of this occurrence are the return to the origin in random walks over a finite group [3] and, in particular, the recurrent event which occurs at the end of every word whose length is an integral multiple of a particular integer.

In Section II, we discuss some properties of a class of recurrent events which we shall call *birecurrent*; in Section III, we verify the statements mentioned above, and in Section IV we describe examples of birecurrent supports.

II. Preliminary remarks. We consider F as the free monoid ([1], Chap. 1) generated by X; the empty word e is the neutral element of F and the product ff' of the words f and f' is the word f'' made up of f followed by f'; $f(f')$ is called a *left* (*right*) *factor* of f''; a word is *proper* if it is different from e.

Feller's condition ([2], Chap. VIII) that the non empty subset A of F is the support of a recurrent event can be expressed as follows: U_r : *if* $a \varepsilon A$ *and* $f \varepsilon F$, *then*, $af \varepsilon A$ *if and only if* $f \varepsilon A$. This condition implies that A is a submonoid of F (i.e., that $e \varepsilon A$ and $A^2 \subset A$). We shall say that A is *birecurrent* if it satisfies U_r and the symmetric condition U_l, U_l : *if* $a \varepsilon A$ *and* $f \varepsilon F$, *then*, $fa \varepsilon A$ *if and only if* $f \varepsilon A$.

It follows immediately that, if $\{A_i\}$ is any collection of supports of recurrent

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(birecurrent) events, the same is true of the intersection C of the sets A_i; indeed, C is a submonoid because every A_i is a submonoid and, if, e.g., a, $af \varepsilon C$, the word f belongs to all the sets A_i (because of U_r) and consequently it belongs also to C.

Throughout this paper, A will denote a recurrent (or, eventually, birecurrent) support and we shall use the following notations:

A^* = the set of all the proper words at the end of which the event whose support is A occurs for the first time; for any recurrent support B, B^* is defined similarly.

$S = F - A^*F$ (= the complement in F of the right ideal A^*F);

$R = F - FA^*$.

We state explicitly the following well known facts:

II.1. Every $f \varepsilon F$ admits one and only one factorization $f = as$ with $a \varepsilon A$ and $s \varepsilon S$ and at least one factorization $f = ra'$ with $a' \varepsilon A$ and $r \varepsilon R$. If and only if A is birecurrent the second factorization is unique for all $f \varepsilon F$.

II.1′. Every proper a of A admits a unique factorization as a product of elements of A^*.

The two statements are quite intuitive but a formal proof of them has been given in ([5]); II.1′ shows that any bijection (i.e., one to one mapping onto) of A^* onto a set Y can be extended to an isomorphism of A onto the free monoid generated by Y.

The following remark will be used repeatedly in the course of this paper:

II.1″. When A is birecurrent, if $s, s' \varepsilon S$ ($r, r' \varepsilon R$) are such that s is a right factor of s' (r is a left factor of r') and that $sf, s'f \varepsilon A$ ($fr, fr' \varepsilon A$) for some $f \varepsilon F$, then $s = s'$ ($r = r'$). If, furthermore, $f \varepsilon R$ ($f \varepsilon S$), then $sf \varepsilon A^* \cup \{e\}$.

PROOF. Because of the perfect symmetry of U_r and U_l we can limit ourselves to the proof of the statement concerning s and s'. By hypothesis, $s' = f's$ for some $f' \varepsilon S$ and $sf, f'sf \varepsilon A$; because of U_l, this implies $f' \varepsilon A$. Because of $s' \varepsilon S = F - A^*F$ and II.1′, this, in turn, implies $f' = e$, and we have proved that $s' = es = s$. Let us assume now that $sr \varepsilon A$ with $s \varepsilon S$ and $r \varepsilon R$. If, in addition, $sr = e$, the result is proved. If $sr \varepsilon A - \{e\}$, II.1′ shows that $sr = aa'$ with $a \varepsilon A^*$ and $a' \varepsilon A$; as above, a cannot be a left factor of s and, consequently, a' is a right factor of r; but, by a symmetrical argument, this shows that $a' = e$ and that consequently $sr = a \varepsilon A^*$. This concludes the proof of II.1″.

Let us assume now that A is birecurrent; we denote by $\Delta Sf(\Delta Rf)$ the set of the right (left) factors of f that belong to $S(R)$ and by Δf the set of the triples (r, a, s) such that $f = ras$ and that $r \varepsilon R$, $a \varepsilon A$, $s \varepsilon S$; such a triple will be called an *A-factorization* of f and δf will denote the number of distinct triples in the set of the A-factorizations of f.

II.2. For any $f, f' \varepsilon F$, $\delta ff' \geqq \max(\delta f, \delta f')$ and $\delta ff' = \delta f$ ($= \delta f'$) if and only if for every left (right) factor f'' of f' (of f) the product ff'' ($f''f'$) has a factorization $ff'' = sa$ ($f''f' = ar'$) where $a \varepsilon A$ and where f'' is a right (left) factor of a.

PROOF. Let us consider any element $g \varepsilon F$ and prove that there exists a bijection $\sigma_g : \Delta Rg \rightarrow \Delta Sg$. Indeed, by II.1, to any $r \varepsilon \Delta Rg$ (i.e., to any $r \varepsilon R$ which is such that $g = rg'$ for some $g' \varepsilon F$) there corresponds a unique $s \varepsilon \Delta Sg$ (determined by the conditions $g' = as, a \varepsilon A, s \varepsilon S$) which we call $\sigma_g r$; because of the symmetry implied by the hypothesis that A is birecurrent we can construct in a similar manner a mapping $\Delta Sg \rightarrow \Delta Rg$ which we call σ_g^{-1}. Since, clearly, for any $r \varepsilon \Delta Rg$ we have $(\sigma_g^{-1} \circ \sigma_g)r = r$ and similarly for any $s \varepsilon Sg$, this shows that σ_g is a bijection and also that the A-factorizations of g are in a one-to-one correspondence with the elements of ΔRg.

We now revert to the proof of II.2. By the above construction we know that $\delta ff'$ is equal to δf (i.e., to the number of elements in ΔRf) plus the number of proper $r' \varepsilon \Delta Rf'$ such that $fr' \varepsilon R$. Thus, $\delta ff' \geqq \delta f$ with the equality sign if and only if we do not have $ff'' \varepsilon R - \Delta Rf$ for some left factor f'' of f', i.e., if and only if every such ff'' satisfies the condition stated in II.2. Because of the symmetry this concludes the proof.

For any $f \varepsilon F$, let us denote by αf the smallest positive integer for which $f^{\alpha f} \varepsilon A$; αf is infinite if the only finite power of f that belongs to A is f^0 ($= e$, by definition).

II.3. A sufficient condition that the recurrent support A is birecurrent is that αf is finite for all $f \varepsilon F$; reciprocally if A is a birecurrent support, then, for any $f \varepsilon F$, αf is at most equal to the supremum $\delta' f$ of δf^m over all the positive powers of f.

PROOF. By hypothesis, A satisfies U_r and, in order to show that it is birecurrent, it will be enough to show that if a and fa belong to A then f also belongs to A. Let us assume that $(af)^m \varepsilon A$ for some positive finite m; we have $(af)^m = a(fa)^{m-1}f \varepsilon A$ and, because of the fact that a, $(fa)^{m-1} \varepsilon A$ and U_r, this implies $f \varepsilon A$. This proves the first part of II.3.

Now let A be birecurrent and f such that $\delta' f$ is finite; by II.1, any $f^n (0 \leqq n \leqq \delta' f)$ admits an A-factorization (e, a_n, s_n) and, by II.2, to each such s_n there corresponds one A-factorization of $f^{\delta' f}$. Since, by definition, $\delta f^{\delta' f} \leqq \delta' f$, we must have $s_n = s_m$ ($= s$, say) with $0 \leqq m, n \leqq \delta' f$ and, e.g., $m < n$. Thus, $f^n = as$ and $f^m = a's$ with $a, a' \varepsilon A$ and, after cancelling s, we obtain $f^{n-m}a' = a$. Because of U_l, this last relation shows that f^{n-m} belongs to A and, since $0 < n - m \leqq \delta' f$, by construction, the result is entirely proved.

Let us assume now that A is birecurrent and that f is such that $\delta f = \delta f^2 < \infty$. We consider the set K (containing at least f^2) defined by $K = \{f' \varepsilon fFf : \delta f' = \delta f\}$.

II.4. There exists a group G, a subgroup H of G and a mapping $\sigma : K \rightarrow G$ that have the following properties: σ is an epimorphism (i.e., homomorphism onto) and G is finite; $\sigma^{-1}H = K \cap A$ and the index of H in G is at most δf.

PROOF. According to II.2, the hypothesis $\delta f = \delta f^2$ implies the existence of a bijection $\sigma^* : \Delta Sf \rightarrow \Delta Rf$ defined for each $s \varepsilon \Delta Sf$ by $\sigma^* s$, the unique $r \varepsilon \Delta Rf$ which is such that $sr \varepsilon A$; trivially, $\sigma^* e = e$. Also, by II.2 and the very definition of K, we have $\Delta Rk = \Delta Rf$ and $\Delta Sk = \Delta Sf$ for any $k \varepsilon K$; consequently, $K^2 \subset K$.

Thus, recalling the definition of σ_f given in the proof of II.2, we can associate to any $k \varepsilon K$ a bijection $\sigma_k^* : \Delta Rf = \Delta Rf$ defined by $\sigma_k^* = \sigma^* \circ \sigma_k$.

Let us now verify that for any $k, k' \varepsilon K$ we have $\sigma_{kk'}^* = \sigma_k^* \circ \sigma_{k'}^*$. Indeed, if $(r, a, s) \varepsilon \Delta k$ and $(r', a', s') \varepsilon \Delta k'$ we shall have $(r, a'', s') \varepsilon \Delta kk'$ for some $a'' \varepsilon A$ if and only if $sr' \varepsilon A$ and the identity is verified. Because of the hypothesis that δf is finite, this construction shows that the set $\{\sigma_k^*\}$ $(k \varepsilon K)$ is a group G and that the mapping σ which sends every $k \varepsilon K$ onto σ_k^* is an epimorphism.

Observe now that k belongs to A if and only if $(e, k, e) \varepsilon \Delta k$, that is, if and only if σ_k^* keeps e invariant. Again, because G is finite, the elements $k \varepsilon K$ which have this last property map onto a subgroup H of G and, clearly, $\sigma^{-1}H$ is contained in A. The fact that the index of H in G is at most equal to the number of elements in ΔRf (i.e., to the number δf) is a standard result from group theory. As a corollary of II.4 we state II.4′.

II.4′. If A is such that the supremum δ^* of $\delta f'$ over all $f' \varepsilon F$ is finite and if $\delta f = \delta^*$, then the representation $\{\sigma_k^*\}$ described in II.4 is isomorphic to the repsentation of G over the cosets of H.

Proof. The property stated amounts to the statement that the group $G = \{\sigma_k^*\}$ is transitive or, in an equivalent fashion, to the fact that for every $s \varepsilon \Delta Sf$ there exists at least one $k \varepsilon K$ such that $\sigma_k e = s$, i.e., such that $k = as$ with $a \varepsilon A$.

In order to prove this, let $(r, a', s) \varepsilon \Delta f$. By II.3 we know that there exist finite positive integers m and m' such that $f^m \varepsilon A$ and $r^{m'} \varepsilon A$. Thus the product $f^m f^{m'-1} f = f^m f^{m'} as$ admits the factorization $a''s$ with $a'' = f^m f^{m'} a' \varepsilon A$ and it belongs to K since, under the hypothesis that δf is maximal, K is identical to fFf.

The next statement is not needed for the verification of property 1. Its aim is to show that the representation described in Section IV below covers all the birecurrent supports with finite $\delta^* = \sup \delta f$.

II.5. If A is a birecurrent support with finite δ^* there exists a monoid M and an epimorphism (homomorphism onto) $\gamma: F \to M$ such that $\gamma^{-1}\gamma A = A$, and that M admits minimal ideals.

Proof. Let us consider any $f \varepsilon F$ and denote by $\{\gamma f\}$ the set of all $f' \varepsilon F$ which satisfy the following condition: for any $f_1, f_2 \varepsilon F$, $f_1 f f_2 \varepsilon A$ if and only if $f_1 f' f_2 \varepsilon A$. The relation $f' \varepsilon \{\gamma f\}$ is reflexive and transitive and it is well known that it is compatible with the multiplicative structure of F (i.e., it is a congruence); thus we can identify each set $\{\gamma f\}$ with an element γf of a certain quotient monoid M of F. Since $f \varepsilon A$ if and only if $f_1 f f_2 \varepsilon A$ with $f_1 = f_2 = e$, A is the union of the sets $\{\gamma a\}$ $(a \varepsilon A)$ and, trivially, $\gamma^{-1}\gamma A = A$.

Let us now take an element f such that $\delta f = \delta^*$, a finite quantity; according to II.2, the maximal character of δf implies that for every f_1 the product $f_1 f$ has a left factor $f_1 r \varepsilon A$ for some $r \varepsilon \Delta Rf$. Thus, because of the symmetry, any relation $f_1 f f_2 \varepsilon A$ implies $f_1 r$, $sf_2 \varepsilon A$ with $(r, a, s) \varepsilon \Delta f$.

It follows immediately that for any two $k, k' \varepsilon K(= fFf)$, the relation $\gamma k = \gamma k'$ is equivalent to the relation $\sigma k = \sigma k'$ in the notations of II.4. Thus, σK is isomorphic to a group and since K is the intersection of a right and of a left ideal of F, this shows that M admits minimal ideals.

We now revert to the preparation of the proof of the main property and we consider A, a birecurrent support, B a recurrent support and $C = A \cap B$; we assume that C does not reduce to $\{e\}$ and that consequently C^* (the set of the proper words at the end of which the events whose supports are A and B respectively occur together for the first time) is not empty.

II.6. Any element f from $F - C^*F$ has a unique factorization $f = f_1f_2$ with $f_1 \varepsilon B - C^*B$ and $f_2 \varepsilon F - B^*F$; conversely any such product f_1f_2 belongs to $F - C^*F$.

Proof. Because of II.1 any f has a unique factorization $f = f_1f_2$ with $f_1 \varepsilon B$ and $f_2 \varepsilon F - B^*F$. Since C is a recurrent support contained in B, any product $f_1'f_2'$ with $f_1' \varepsilon B$ and $f_2' \varepsilon F - B^*F$ belongs to $F - C^*F$ if and only if f_1' belongs to $B - C^*B$ and this concludes the proof.

As mentioned in II.1′, there exists an isomorphism $\beta: B \to Q$ where Q is the free monoid generated by $Q^* = \beta B^*$ and it is easily verified that the image P of C by β satisfies U_r and U_l when, according to our hypothesis, A is birecurrent. Indeed, P is surely a submonoid of Q and it is enough to verify that the relations $p, p', pqp' \varepsilon P$ imply $q \varepsilon Q$ (because $\beta^{-1}p, \beta^{-1}p', \beta^{-1}pqp' \varepsilon A$ imply, e.g., $\beta^{-1}qp \varepsilon A$, by U_r, then $\beta^{-1}q \varepsilon A$, by U_l and, finally $q \varepsilon P = \beta(A \cap B)$).

As before, we define a P-factorization of an element $q \varepsilon Q$ as a triple $(\bar{r}, p, \bar{s})$ such that $q = \bar{r}p\bar{s}$ and that $\bar{r} \varepsilon \bar{R} = Q - QP^*$, $p \varepsilon P$, $\bar{s} \varepsilon \bar{S} = Q - P^*Q$ with $P^* = \beta C^*$. All the remarks made in II.2 apply here since P is a birecurrent support in Q, and we define $\bar{\delta}q$ as the number of P-factorizations of q.

II.7. For any $b \varepsilon B$, $\bar{\delta}\beta b \leqq \delta b$.

Proof. Let $\bar{r}$ be any element of $\bar{R}$ and define $\beta^*\bar{r}$ as the (uniquely determined) element $r \varepsilon R$ such that $(r, a, e) \varepsilon \Delta b$ for some $a \varepsilon A$. We show that the restriction of the mapping β^* to any set $\Delta\bar{R}q$ $(q \varepsilon Q)$ is an injection (i.e., is one to one into). Indeed, if $\bar{r}, \bar{r}' \varepsilon \Delta\bar{R}q$ we have, e.g., $\bar{r}' = \bar{r}q'$ for some $q' \varepsilon Q$; thus, if $\beta^*\bar{r} = \beta^*\bar{r}'$ ($= r$, say), we have the following relations: $\beta^{-1}\bar{r} = ra \varepsilon B$ with $a \varepsilon A$; $\beta^{-1}\bar{r}' = ra' \varepsilon B$ with $a' \varepsilon A$; $ra' = rab'$ with $b' = \beta^{-1}q\beta \varepsilon B$. Consequently, $a' = ab'$ and, because of U_r, $b' \varepsilon A$. This shows that $q' = \beta b'$ belongs to P and that finally, $q' = e$ because of the relation $\bar{r}' = \bar{r}q' \varepsilon \bar{R}$. Thus, $\bar{r}' = \bar{r}$ and our contention is proved.

The remark II.7 is also proved since we have shown that for any $b \varepsilon B$ there exists an injection of $\Delta\bar{R}\beta b$ into ΔRb.

II.8. If δ^* ($= \sup \delta f$) is finite and if $\delta b = \delta^*$ for at least one $b \varepsilon B$, then $\bar{\delta}^*$ ($= \sup \bar{\delta}q$) is a divisor of δ^*.

Proof. Under these hypotheses, we may assume without loss of generality that B contains an element f such that $\delta f = \delta^*$ and $\bar{\delta}\beta f = \bar{\delta}^*$. We use the notations of II.4 and II.4′. By construction, the image G' by σ of $B \cap K$ is a subgroup of G and we have $B \cap \sigma^{-1}(H \cap G') = A \cap B \cap K$. Thus, by a standard result of group theory the index δ'^* of $H \cap G'$ in G' is a divisor of the index of H in G (i.e., of δ^*). We prove now that δ'^* is in fact equal to $\bar{\delta}^*$; for this we repeat the construction of II.4 and II.4′ with $\beta(B \cap K)$ in the role of K and we obtain an epimorphism $\bar{\sigma}: \beta(B \cap K) \to \bar{G}$ such that $\bar{\delta}^*$ is the index of the subgroup $\bar{H}$ of $\bar{G}$. We recall the definition of the mapping β^* used in II.7 and we observe that we can define a bijection β^{*-1}: $\Delta Rf \cap \beta^*\Delta\bar{R}\beta f \to \Delta\bar{R}\beta f$ such that $\beta^{*-1} \circ \beta^*$ is the identity mapping of $\Delta\bar{R}\beta f$ onto itself; β^{*-1} induces in a natural fashion an epimorphism β^{**}: $G' \to \bar{G}$ and, trivially, $H \cap G'$ is the inverse image of $\bar{H}$ by β^{**}. Thus $\bar{\delta}^*$ is equal to δ'^* and II.8 is proved.

III. Verification of property 1. We keep the notations already introduced and we assume that (A, μ) is a recurrent event. According to Feller, μ satisfies the two conditions:

M_0 : $\mu e = 1$ and for any $f \in F$, $\mu f = \sum (\mu fx: x \in X)$,

M_r : if $a \in A$ and $f \in F$ then $\mu af = \mu a \mu f$.

We shall say that μ is a *positive product measure* if $\mu ff' = \mu f \mu f' > 0$ for any f, $f' \in F$, and, in this case, M_r is trivially satisfied.

We denote by $|f|$ the length of the element f and for any subset F' of F we use the following notations: $F'_n = \{f \in F': |f| \leqq n\}$; $\mu F' = \lim_{n\to\infty} \sum \{\mu f: f \in F'_n\}$. It follows that $\mu F' \leqq 1$ if F' is such that any $f \in F$ has at most one left factor which belongs to F'; this condition is satisfied in particular by any subset of A^* and, according to Feller's definition, we shall say that (A, μ) is *persistent* if and only if $\mu A^* = 1$. The next two statements are verified by an imitation of Feller's proof procedure.

III.1. For any recurrent event (A, μ) we have $T(A, \mu) = \mu S$.

Proof. Let us introduce for any $s \in S$ the notation $S(s) = S \cap sF$. We verify the identities

(III.1). for all $m \geqq |s|$: $0 \leqq \mu s - \mu A^*_{m+1}(s) = \mu S_{m+1}(s) - \mu S_m(s)$;

(III.1′). for all $m \geqq 1$: $(1 - \mu A^*) + (\mu A^* - \mu A^*_m) = \mu S_m - \mu S_{m-1}$

Indeed, (III.1) is an immediate consequence of M_0 and of the fact that the sets $\{s\} \cup S_m(s)X$ and $S_{m+1}(s) \cup A^*_{m+1}(s)$ are identical for any $m \geqq |s|$. (III.1′) is the special case of (III.1) for $s = e$.

From this second identity we deduce that if $\mu A^* = 1$ we have $\lim_{m\to\infty} (\mu S_m - \mu S_{m-1}) = 0$. Thus, *a fortiori* (from the first identity) $\mu A^* = 1$ implies $\mu s = \mu A^*(s)$. We now sum the second identity from $m = 1$ to $m = n$. After rearranging terms, we obtain:

(III.1″). $\mu S_n = (n + 1)(1 - \mu A^*_n) + \sum \{|a| \mu a: a \in A^*_n\}$.

This shows that if (A, μ) is not persistent, μS is infinite and we assume now that $\mu A^* = 1$. Under this hypothesis, $T(A, \mu)$ is defined as $\lim_{n\to\infty} \sum \{|a| \mu a: a \in A^*_n\}$, and since $\mu A^* = 1$ implies that

$$(n + 1)(1 - \mu A^*_n) = \sum \{(n + 1)\mu a: a \in A^* - A^*_n\},$$

we can write for all n

$$\sum\{|a|\,\mu a: a \,\varepsilon\, A_n^*\} \leqq \mu S_n \leqq \sum\{|a|\,\mu a: a \,\varepsilon\, A^* - A_n^*\} + \sum\{|a|\,\mu a: a \,\varepsilon\, A_n^*\}.$$

This concludes the proof since it shows that $\mu S = T(A, \mu)$ if this last quantity is finite and that μS is infinite if $T(A, \mu)$ is so.

For any $s \,\varepsilon\, S$ let us define $R^*(s)$ as $\{e\}$ if $s = e$ and, as the set of those $f \,\varepsilon\, F$ such that $sf \,\varepsilon\, A^*$, if $s \neq e$.

III.2. If A is birecurrent, μ a product measure and (A, μ) persistent, we have $T(A, \mu) = \mu R$ and, for all $s \,\varepsilon\, S$, $1 = \mu R^*(s)$.

Proof. Under these hypotheses all the notions are perfectly symmetrical. Thus, the identity (III.1″) shows that $\mu R_n = \mu S_n$ and, as a special case, that $\mu R = T(A, \mu)$. Since any $a \,\varepsilon\, A^*(s)$ has a unique factorization $a = sf$ with $f \,\varepsilon\, R^*(s)$, and since μ is a product measure, we have for all $m \geqq |s|$ the identity

$$\text{(III.2)} \qquad \mu A_m^*(s) = \mu s \mu R_{m-|s|}^*(s).$$

Thus, we have in any case $\mu R(s) = \mu A^*(s)/\mu s \leqq 1$ because of the formula (III.1); with the equality sign when (A, μ) is persistent because as seen above $\mu s = \mu A^*(s)$.

III.3. If A is birecurrent and μ a product measure, $T(A, \mu) = \delta^*$.

Proof. We use the notations of Section II and we recall the following facts:

(1) According to II.1″, $R^*(s)$ is a subset of R;

(2) for the same reason, if $s, s' \,\varepsilon\, \Delta Sf$ for some $f \,\varepsilon\, F$, the sets $R^*(s)$ and $R^*(s')$ are disjoint.

(3) if δ^* is finite and $\delta f = \delta^*$ then, by II.2, to every $r \,\varepsilon\, R$ there corresponds one $s \,\varepsilon\, \Delta Sf$ such that $sr \,\varepsilon\, A^*$. Thus, in this case, the union of the sets $R^*(s)$ over all $s \,\varepsilon\, \Delta Sf$ is equal to R. Now to the proof! We shall show that if $\delta f = \delta^*$ we have the inequalities $\mu R \leqq \delta f \leqq \mu R$ and, trivially, the result will follow by III.2.

The second inequality is vacuously true when (A, μ) is not persistent since, then, μR is infinite. When (A, μ) is persistent we have for any $f' \,\varepsilon\, F$ the inequality $\delta f' = \sum\{\mu R^*(s): s \,\varepsilon\, \Delta Sf'\} \leqq \mu R$ since, then, $\mu R^*(s) = 1$ and since the sets $R^*(s)$ are pairwise disjoint. Thus the second inequality is always true. If now $\delta f = \delta^*$, we know by 3 above that $\sum\{\mu R^*(s): s \,\varepsilon\, \Delta Sf\} = \mu R$. Since in any case, as we have seen in the proof of III.2, we have $\mu R^*(s) \leqq 1$, it follows that $\mu R \leqq \delta^*$ and the result is proved.

III.4. If (B', μ) is a recurrent event and if A is birecurrent we have

$$T(A \cap B', \mu) = \bar{\delta}^* T(B', \mu)$$

where $\bar{\delta}^*$ is defined below.

Proof. Let $B = \{b \,\varepsilon\, B': \mu b > 0\}$ and $C = A \cap B$; it is easily verified that (B, μ) is again a recurrent event and that, according to III.1. we have

$$\begin{aligned} T(A \cap B', \mu) &= T(A \cap B, \mu) = \mu(F - C^*F) \\ T(B', \mu) &= T(B, \mu) \quad\;\; = \mu(F - B^*F). \end{aligned}$$

We keep the notations used in the proofs of II.6 and II.7 and we observe that, by taking into account II.6 and the condition M_r on μ, the remark III.4 is equivalent to the relation $\mu(B - C^*B) = \bar{\delta}^*$. In order to prove this identity we define a measure ν on Q by the relation $\nu\beta b = \mu b$, for all $b \varepsilon B$; because of M_r and of the definition of B, ν is a positive product measure and, since we know that $P = \beta C$ is birecurrent, (P, ν) is a recurrent event on Q. Because of III.1 and III.3 $T(P, \nu) = \nu(Q - P^*Q) = \bar{\delta}^*$. But, by definition, $\nu(Q - P^*Q) = \nu\beta(B - C^*B) = \mu(B - C^*B)$ and the result is proved.

III.5. If δ^* is finite, and (B', μ) persistent for some measure μ which satisfies the condition that for every $f \varepsilon F$ at least one element from FfF has positive measure, then $\bar{\delta}^*$ is a divisor of δ^*.

Proof. Because of the conditions satisfied by μ and δ^* we can find an element f such that $\delta f = \delta^*$ and that $\mu f > 0$; we have $f = b's'$ with $b' \varepsilon B$ and $s' \varepsilon F - B^*F$. Because (B, μ) is persistent, it follows from III.1 that $\mu(B^* - s'F) = \mu s'$. Since this last quantity is positive, there exists at least one element $b \varepsilon B^* \cap s'F$. Finally, because of II.2 we have $\delta b'b = \delta^*$ with $b'b \varepsilon B$. Thus, we can apply II.8 and the result is proved.

The next statement is intended to give a characterization of the birecurrent supports in terms of their intersection with other recurrent events; by E we mean any fixed birecurrent support such that $T(E, \mu)$ is finite for some positive product measure μ; E^* is defined as usual and we say that (E', μ') belongs to the family $((E))$ if the two following conditions are met:

(i). (E', μ') is a recurrent event on F;

(ii). there exists a finite integer m such that any element from E'^* is the product of m words from E^*. It is trivial that under these hypotheses E' is birecurrent. Since F itself is a birecurrent support (with $F^* = X$) a simple example of a family $((E))$ is the family of the birecurrent events $(F_{(m)}, \mu_m)$ where $F_{(m)}$ is the set of all words whose length is a multiple of m and where μ_m is a suitable measure.

III.6. If the recurrent support A is such that $(A \cap E', \mu')$ is persistent for every $(E', \mu') \varepsilon ((E))$, then, A is a birecurrent support.

Proof. This is a simple application of II.3 and we use the notations of this remark. If αf is finite for all f, then we know by II.3 that A is birecurrent. Thus we may suppose that A and f are such that αf is infinite and we show that $(A \cap E', \mu')$ is not persistent for some suitable (E', μ'). Indeed, by the second part of II.3 we know that $f^m \varepsilon E$ for some finite positive m. Thus f^m admits a factorization as a product of m' elements from E^*. We take E' defined by the condition $E'^* = E^{*m'}$ and μ' defined by the condition that $\mu' f^m = 1$ and $\mu' f' = 0$ for any other $f' \varepsilon E'^*$. The conditions M_0 and M_r recalled at the beginning of this section are obviously satisfied and $T(E', \mu')$ is finite. Finally, $(A \cap E', \mu')$ cannot be persistent since $A \cap E'$ reduces to $\{e\}$ and this ends the proof.

Clearly, the conditions of III.6 are satisfied if A is such that $T(A \cap B, \mu) < \infty$ for any (B, μ) with finite $T(B, \mu)$.

The next statement is a simple application of II.2.

III.7. If A is birecurrent and if δ^* is finite, then, for any product measure, μ, the distribution of the recurrence time of (A, μ) has moments of every order.

PROOF. Let $A' = \{a \varepsilon A: \mu a > 0\}$. Trivially, A' is birecurrent and, by II.7 we know that every $f \varepsilon F$ has at most δ^* A'-factorizations. Since the distribution of the recurrence times of (A, μ) and (A', μ) are the same, there is no loss of generality in assuming that $A = A'$, i.e., that μ is positive.

Since δ^* is finite there exists an element $f \varepsilon F$ which, because of II.2, has the property that for any proper $s \varepsilon S$ the product sf has a factorization $sf = ar$ with $a \varepsilon A^*A$. Thus, for any integer n, the definition $S = F - A^*F$ allows us to write the inequality

$$\Sigma\{\mu f': f' \varepsilon A^*, n|f| < |f'| \leqq (n+1)|f|\} = \mu A^*_{(n+1)}|f| - \mu A^*_n|f| \leqq (1 - \mu f)^{n+1}.$$

Consequently the distribution of the $|a|$ for $a \varepsilon A^*$, i.e., of the recurrence time of A^*, is dominated by an exponential distribution and this proves the result.

IV. Examples. We want to describe a class of monoids, V, which allows the construction of birecurrent supports. For this purpose, we consider a group G' (whose elements are identified with the corresponding elements of its Frobenius algebra) and a subgroup H' which contains no proper normal subgroup of G'; $I = \{i\}$ and $J = \{j\}$ are two sets of indices and w is a $I \times J$ matrix with entries w_{ij} in H'. Without loss of generality we can assume that there exists no pair of indices $j, j' \varepsilon J$ $(i, i' \varepsilon I)$ and no element $h \varepsilon H'$ such that $w_{ij}h = w_{ij'}$ $(hw_{ij} = w_{i'j})$ identically for all $i \varepsilon I$ $(j \varepsilon J)$.

We shall denote by V the set of all $I \times I$ matrices v with entries in $G' \cup \{0\}$ that have the following property: for each $j \varepsilon J$ there exists an index $j' \varepsilon J$ and an element $g_{jj'} \varepsilon G'$ which are such that the product $vw._j$ (with $w._j =$ the jth column vector of w) is equal to $w._{j'}g_{jj'}$ (i.e., to the vector whose ith entry is equal to $w_{ij'}g_{jj'}$). Trivially, this condition implies that v has one and only one non zero entry in each line; it also implies the existence of an isomorphism $v \rightarrow \bar{v}$ which sends V onto the monoid $\bar{V}$ of the $J \times J$ matrices defined by the symmetric condition and which is such that $vw = w\bar{v}$, identically; V is a monoid and it contains as minimal ideal the set V_0 of all matrices whose ith column vector is equal to $w._jg$ (with any $i \varepsilon I, j \varepsilon J, g \varepsilon G'$) and whose i'th column vector is zero for $i' \neq i$.

IV.1. The subset $L \subset V$ of the matrices of V which have at least one entry in H' satisfies U_r and U_l.

PROOF. L is not empty since it contains at least the neutral element of V. Let us assume that $v \varepsilon L$ and that $v_{ii'} \varepsilon H'$. Because of the hypothesis that all the entries of w belong to H', the ith coordinate of $vw._j$ for any $j \varepsilon J$, (that is, $v_{ii'}w_{i'j}$) belongs to H'. Thus, $vw._j = w._{j'}h$ for some $j' \varepsilon J$ and $h \varepsilon H$; it follows that all the non zero entries of v belong to H'. This shows that L is a monoid and, trivially, that it satisfies U_r and U_l.

IV.1′. If F is a free monoid and $\gamma': F \rightarrow V$ an homomorphism, then the subset $A = \gamma'^{-1}(L \cap \gamma'F)$ is a birecurrent support and the corresponding parameter, δ^*, is at most equal to the index of H' in G'.

Proof. The first part of the statement does not need a proof; we verify the second part by showing that for any $f \varepsilon F$ (with the notations of II.2) there exists an injection of ΔRf into the set of the left H'-cosets. Let $r, r' \varepsilon \Delta Rf$ with, e.g. $r' = rf'$; for any $i \varepsilon I$, the condition $(\gamma' r)_{ii'} \neq 0$ defines in a unique manner $i' \varepsilon I$ and $g = (\gamma' r)_{ii'} \varepsilon G'$. In a similar way, we define $i'' \varepsilon I$ and $g' \varepsilon G'$ by the condition $0 \neq (\gamma' f')_{i'i''} (= g')$. Since $(\gamma' r')_{ii''} = (\gamma' rf')_{ii''} = gg'$ we see that g and g' belong to the same H'-coset if and only if $g \varepsilon H'$, that is, if and only if $f' \varepsilon A$, that is, finally, if and only if $r = r'$ and this ends the proof.

Reciprocally, if A is a birecurrent support with finite δ^* we can take (with the notations of II.5) $G' = G$ and $H' = H$ and find, I, J and w such that $\gamma F = M$ is a submonoid of V. Then $V_0 \subset \gamma F$ and a sufficient condition that $\gamma f \varepsilon V_0$ is $\delta f = \delta^*$. We shall not prove these results here since they are a straightforward application of Clifford's theory [4].

IV.1″. If δ^* is finite and if for each $f \varepsilon F$ there exists a finite positive m such that $\gamma f^m \varepsilon V_0$, then the parameter $\bar{\delta}^*$ defined in II.7 is always a divisor of δ^*.

Proof. We consider the group G' defined in II.8. According to the general theory of monoids [4] the only groups contained in γF under the hypothesis of IV.1″ are in fact contained in V_0. Consequently, they are isomorphic to subgroups of G and this concludes the proof.

IV.2. If A is a birecurrent support such that A^* is a finite set then either there exists an $s \varepsilon S$ for which $sF \cap A = \phi$ (and then (A, μ) is not persistent for any positive product measure μ) or else, the conditions of IV.1″ are satisfied by A. In this second case, γF is a group if and only if A^* reduces to the set of all the words having some fixed finite length. [5].

Proof. We assume that A^* is finite and that $A \cap sF \neq \phi$ for all $s \varepsilon S$; then, by the very definition of γ the monoid γF is finite. By II.2 we see that if r, $r' \varepsilon \Delta Rf$ for some $f \varepsilon F$, then the equation $\gamma r = \gamma r'$ implies $r = r'$. Thus, the parameter δ^* is finite. Let us take any element $f \varepsilon F$; the hypothesis that $\delta f < \delta^*$ implies that for some pair (f', f'') one has $f'ff'' \varepsilon A^*$. Thus for all $f \varepsilon F$, $\delta f^m = \delta^*$ for large enough m since, otherwise, A^* would not be finite. This proves that A satisfies the conditions of IV.1″.

We now make the supplementary assumption that γF is a group G with $\gamma A \doteq H$, and we consider a, an element of maximal length of A^*. If $|a| = 1$ the result is vacuously true since, then, $A = F$. If $|a| \geqq 2$ we write $a = sxx'$ with $x, x' \varepsilon X$. Because of U_r, no left factor of a belongs to A^* and because of the maximality of $|a|$, we have $sxx'' \varepsilon A$ for all $x'' \varepsilon X$. Thus, all the generators of F belong to the same left H-coset. For this reason, we cannot have $sx'' \varepsilon A^*$ for any $x'' \varepsilon X$ and, because again of the maximal character of $|a|$ this implies that $sx''x''' \varepsilon A^*$ for any two $x'', x''' \varepsilon X$. Thus, for any two elements $x, x' \varepsilon X$, the left coset $xx'H$ does not depend upon the choice of x and x'. If $|a| = 2$, this proves the result. If $|a| \geqq 3$ we can write $s = s'y$ with $y \varepsilon X$ and by the same argument we prove that for any $x, x', x'' \varepsilon X$ the coset $xx'x''H$ does not depend upon the choice of these three elements. Since $|a|$ is finite, by hypothesis, a simple induction gives the result.

The next statements discuss the existence of birecurrent supports with finite δ^*. Without loss of generality, we shall assume from now on that X contains a *finite* number $\geqq 2$ of elements.

IV.3. For any finite $n \geqq 3$ there exist infinitely many different birecurrent supports with this value of δ^*.

PROOF. In the next section we shall show the existence of at least one birecurrent support with $\delta^* = 2$ and A^* infinite. In this section we show that to every birecurrent support A and element $u \varepsilon A^*$ we can associate one other birecurrent support B with $\delta_B^* = \delta^* + 1$ and B^* infinite and that, for the same A^* and different choice of $u \varepsilon A^*$, the two corresponding new supports are different. Thus IV.3. will be entirely proved with the help of IV.4.. Let us now take $u \varepsilon A^*$, a fixed element, and define: $J = (uF \cap Fu) - \{u\}$; $J^* = J - J^2$ (i.e., = the subset of those elements of J that cannot be written as the product of two elements of J). With the help of II.1″, it is easily verified that there exists a birecurrent support B which is such that $B^* = J^* \cup (A^* - \{u\})$ and we prove that for all $f \varepsilon F$ the number (say, $\delta(B, f)$) of its B-factorizations is at most equal to $\delta f + 1$. In order to do this, we slightly extend the notations of II.2, and for any subset F' of F we say that the triple (f'', f', f''') is a F'-factorization of f if $f' \varepsilon F'$ and $f''f'f''' = f$; also, we denote by $\delta(F', f)$ the number of distinct F'-factorizations of f and we observe that by induction on the length of f, the result of II.3 can be summarized by the identity $|f| + 1 = \delta(A, f) + \delta(A^*, f)$.

Here, we have

$$\delta(A^*, f) = \delta(A^* - \{u\}, f) + \delta(\{u\}, f),$$

$$\delta(B^*, f) = \delta(A^* - \{u\}, f) + \delta(J^*, f),$$

We want to show that $\delta(B^*, f) \leqq \delta(A^*, f) + 1$. If $\delta(\{u\}, f) = 0$ or 1, we have $\delta(J^*, f) = 0$ and the result is proved; consequently, we assume now that $\delta(\{u\}, f) \geqq 2$ and we consider two $\{u\}$-factorizations (f_1, u, f_1') and (f_2, u, f_2') with, e.g. $|f_1| \leqq |f_2|$. The element w determined by the equation $f = f_1 w f_2'$ belongs to J; it belongs to J^* if and only if there is no $\{u\}$-factorization (f_3, u, f_3') for which $|f_1| < |f_3| < |f_2|$; it follows instantly that $\delta(J^*, f) = \delta(\{u\}, f) - 1$ and the result is proved.

IV.4. For each finite $n \geqq 3$ there exist at least two different birecurrent supports with A^* finite and $\delta^* = n$.

PROOF. One of these supports has been described in IV.2; in order to produce the other one, we take a birecurrent support A, a fixed element $u \varepsilon (F - A^*F) \cap (F - FA^*)$ and we construct another birecurrent support B with $\delta_B^* = \delta^*$; in the last part of the proof we verify that by a proper choice of u and A^* we can make B^* finite.

Let the following sets be defined:

$$C^* = A^* - A^* \cap (uF \cup Fu),$$
$$Z = \{f: uf \in A^* - A^* \cap Fu\},$$
$$Z' = \{f: fu \in A^* - A^* \cap uF\},$$
$$J^* = A^* \cap uF \cap Fu,$$
$$P^* = \{f: fu \in A^* \cap uF\}.$$

Thus, A^* admits a partition into the sets C^*, uZ, $Z'u$ and J^*; by construction, there exists a recurrent support P such that $P^* = P^2 - P$ (with $P = \{e\}$ if P^* is empty) and one can verify that there exists a birecurrent support B such that B^* admits a partition into the sets C^*, $\{u\}$ and $Z'PuZ$.

In order to verify that $\delta_B^* = \delta^*$ we take an arbitrary positive product measure μ and, for any $F' \subset F$, we write $T(F')$ as an abbreviation for $\sum (|f| \, \mu f : f \in F')$. Thus, by III.3, we have, e.g., $\delta^* = T(A, \mu) = T(A^*)$.

By a simple computation, we obtain when δ^* is finite: $\delta^* = T(A^*) = T(C^*) + T(P^*) + |u|(\mu Z + \mu Z' + \mu P^*)\mu u + (T(Z) + T(Z') + T(P^*))\mu u$. Also, $\mu Z = \mu Z' = 1 - \mu P^*$; $\mu P = (1 - \mu P^*)^{-1}$; $T(P) = (1 - \mu P^*)^{-2} T(P^*)$. Now, $T(B^*)$ $(= \delta_B^*)$ is equal to the sum $T(C^*) + |u|\mu u + T(Z'PuZ)$; because of the above relations, we have $T(Z'PuZ) = |u|\mu u \mu Z + (T(Z) + T(Z') + T(P^*))\mu u$ and this concludes the second part of the proof.

Let us now observe that B^* is finite if and only if C^* is finite and $P = \{e\}$. The first condition is surely satisfied when A^* is finite and the second one is equivalent to $P^* = \phi$, that is, to $A^* \cap uF \cap Fu = \phi$.

Thus, if A^* is the set of all words of length $n > 2$ and if $x_1, x_2 \in X$, the word $u = x_1^{n-2} x_2$ belongs to $F - A^*F$ and to $F - FA^*$ and it satisfies our last condition; this ends the proof of IV.4.

If we take $n = 2$ and $u = x_1$ we find that $P^* = x_1$ and the corresponding B^* is infinite; this is the example needed for IV.3.

IV.5. For each finite n there exists only a finite number of birecurrent supports A with $\delta^* = n$ which satisfy one or the other of the two following supplementary conditions: that γF is a group or that A^* is finite.

PROOF. This is obvious for the first condition since, because of II.4′, it amounts to the fact that for any finite n there exist only finitely many groups of permutation on n symbols.

With respect to the second condition we first verify the following elementary remark: let $K_0 = F - \{e\}, K_1, K_2, \cdots$ be a decreasing sequence of subsets of F defined inductively by the relation $K_{i+1} = \{fFf : f \in K_i\}$. If X is finite there exists for every finite i a finite value $d(i)$ which is such that every word of length at least $d(i)$ has at least one factor belonging to K_i. Indeed, if $d(i)$ has already been defined, we take $d(i + 1)$ as $d(i)\ (1 + |X|^{d(i)})$ where $|X|$ denotes the number of elements of X. Then, every word of length $d(i + 1)$ contains at least two disjoint identical factors of length $d(i)$ and the result follows by induction.

We now observe that if $f \neq e$ the hypothesis that A^* is a finite set (with finite δ^*) implies that $\delta ff'f \geqq \inf(\delta^*, \delta f + 1)$. Indeed, this is surely true if $\delta f' > \delta f$ or if $\delta ff'f = \delta^*$; in the remaining case, i.e., in the case that $\delta f = \delta ff'f < \delta^*$, we would have according to II.2, for all finite m, $\delta(ff')^m f = \delta f < \delta^*$ and, according to the same remark, there would exist for all finite m at least one $a \varepsilon A^*$ admitting $(ff')^m f$ as a factor, which is impossible since A^* is assumed to be finite.

Thus, by induction, every word f of length $\geqq d(\delta^*)$ is such that $\delta f = \delta^*$ and, consequently, it cannot be a factor of a word $a \varepsilon A^*$. This proves that for given δ^* the hypothesis that A^* is finite imposes that the lengths of the words from A^* is bounded and it concludes the proof (cf.[6]).

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