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CERTAIN INFINITE FORMAL PRODUCTS AND THEIR COMBINATORIAL APPLICATIONS

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I. Introduction.

This note is concerned with certain relations between properties of factorization of a free monoid (in a manner similar to that considered in group theory) and properties of words modulo a cyclic permutation of their letters.

From these we deduce identities involving infinite formal product in non-commuting variables that are related to combinatorial questions.

II. Notations.

1) For any set X we denote by $\mathcal{F}_X$ the free monoide generated by X (with neutral element 1) and $\mathcal{F}_X^+ = \{ f \in \mathcal{F}_X : f \neq 1 \}$.

A factorization $\mathcal{F}_X = \prod \{ F_i : i \in I \}$ of $\mathcal{F}_X$ is a collection of submonoids $\mathcal{F}_i \subset \mathcal{F}_X$ indexed by the elements of a totally ordered set I such that the following is true.

To any $f \in \mathcal{F}_X$ there corresponds a unique finite subset $I_f \subset I$ and to each $i \in I_f$ a unique element, say, $\alpha_i f \in F_i^+$ such that $f = \alpha_{i_1} f \alpha_{i_2} f \ldots \alpha_{i_m} f$ where $I_1 = \{ i_1 < i_2 < \ldots < i_m \}$.

For instance, if $I = \{1,2\}$, $\{F_1, F_2\}$ form a factorization of $\mathcal{F}_X$ if every $f \in \mathcal{F}_X$ has one and only one factorization $f = f_1 f_2$ with $f_1 \in F_1$, $f_2 \in F_2$. The less straight forward definition above is needed for covering the case where I is not finite.

2) Let $f\sim f'$ (f and f' are conjugate) if and only if there exist $f'',f''' \in \mathcal{F}_X$ such that $f = f''f'''$, $f' = f'''f''$. In fact, since $\mathcal{F}_X$ is a submonoid of the free group generated by X , $f\sim f'$ if and only if they are conjugate in the usual sense because then, $f \equiv f''f'f''^{-1}$.

3) Let X and Y be two sets and $\varphi : \mathcal{F}_Y \to \mathcal{F}_X$ be a homomorphism. We shall say that φ is a c-homomorphism if and only if

(i) $\varphi^{-1}1 = 1$

(ii) for all $g,g' \in \mathcal{F}_Y$ and $f \in \mathcal{F}_X \setminus \varphi\mathcal{F}_Y$ $\varphi g\, f \neq f \varphi g'$

Then the subset φY of $\mathcal{F}_X$ will be called "c-free". It is clear that a c-free set $A \subset \mathcal{F}_X$ generates a free submonoid (noted $\{1\} \cup A^*$) or in equivalent fashion that a c-homomorphism is a monomorphism because the condition (ii) above is stronger than the condition

U_d) For all $f \in \mathcal{F}_Y \setminus \mathcal{F}_X$

$$(\varphi\mathcal{F}_Y)f \cap f(\varphi\mathcal{F}_Y) \cap \varphi\mathcal{F}_Y = \emptyset$$

which, as it is well known, insures that φ is 1-1 (into) .

III. A preliminary result.

Let us denote by $\tilde{A}^*$ for any $A \subset \mathcal{F}_X$ the set of all conjugates of the words belonging to the least stable subset A^* that contains A . We have

Property 1. If the three submonoids F_1,F_2,F_3 of $\mathcal{F}_X$ form a factorization of $\mathcal{F}_X$ and are generated by A_1,A_2, and A_3 respectively, then

(i) A_1,A_2 and A_3 are c-free

(ii) $\{\tilde{A}_1^*,\tilde{A}_2^*,\tilde{A}_3^*\}$ is a partition of $\mathcal{F}_X^+$

The proof which is not difficult is based upon the remark that (i) and (ii) are trivially satisfied when $\{A_1 \cap X,\ A_2 \cap X,\ A_3 \cap X\}$ is not a proper partition of X . On the contrary when, e.g.

$A_1 \cap X = X'$ with $\emptyset \neq X' \neq X$ there exists a factorization of F into the three monoids $\{1\} \cup X'^*$, $\{1\}$, $\{1\} \cup X'' \mathcal{F}_X$ $(X'' = X \setminus X')$ where the last one is generated by the c-free set $X'' \cup X'' X'^*$. Observe that the uniqueness of the factorization implies that no word of $A_1 \setminus X' \cup A_2 \cup A_3$ begins with a letter from X' . This allows to "eliminate" X' , by considering a set Y and a monomorphism $\psi : \mathcal{F}_Y \to \{1\} \cup X'' \mathcal{F}_X$. Then $\mathcal{F}_Y$ has a factorization $\{G_1, G_2, G_3\}$ such that $\psi G_2 = F_2$, $\psi G_3 = F_3$ and $\psi G_1 = F_1 \setminus X'^*$. Furthermore one can show that ψ is a c-homomorphism and that the truth of (i) and (ii) for $\{G_1, G_2, G_3\}$ implies the truth of (i) and (ii) for $\{F_1, F_2, F_3\}$. Iterating 2n times this construction gives a free monoid $\mathcal{F}_Z$, with a factorization $\{H_1, H_2, H_3\}$ and a c-homomorphism $\bar{\psi} : \mathcal{F}_Z \to \mathcal{F}_X$ such that $\bar{\psi} H_2 = F_2$ and that all the words of degree less than n in $\bar{\psi} \mathcal{F}_Z$ belong to $\bar{\psi} H_2$. Since n is arbitrarily large, this gives the possibility of proving that A_2 is c-free and the rest of the proof is rather straight forward.

The method is essentially that of Lazard [2] . The same technique shows:

Property 2. To any partition $\{F_1', F_2', F_3'\}$ of $\mathcal{F}_X^+$, there corresponds one and only one triple of subsets $A_1 \subset F_1'$, $A_2 \subset F_2'$, $A_3 \subset F_3'$ satisfying the hypothesis of Property 1. Finally let $\{F_i\}$ $(i \in I)$ be a factorization of F such that the sets A_i $(i \in I)$ generating the submonoids F_i have the two properties

(i)' The sets A_i $(i \in I)$ are c-free

(ii)' The sets A_i^* $(i \in I)$ form a partition of $\mathcal{F}_X^+$.

Taking any one of the F_i's, say F_j we can construct a factorization $F_{j_1}, F_{j_2}, F_{j_3}$ of this monoid satisfying the hypothesis of Property 1.

The same argument shows that the collection $\{F_{i'}\}$ $i \in I'$ with I' obtained by replacing in I the element j by the triple $(j,1)$, $(j,2)$, $(j,3)$ still satisfies (i)' and (ii)'.

Now let us consider an injective mapping μ of $\mathcal{F}_X^+$ into the interval $[0,1]$, each number of this interval being represented by its ternary expansion

$$r = \sum \{r_n 3^{-n},\ n > 0\} \quad r_n = 0, 1, 2 \quad .$$

The first digit of μf gives a partition $\mathcal{F}'_j = \{f \in \mathcal{F}^+_X : (\mu f)_1 = j\}$ (j=0,1,2) from which we can derive a factorization F_1, F_2, F_3 of $\mathcal{F}_X$ by Property 2. Then, using the second digit in an obvious fashion we obtain a **factorization** of each of the three monoids F_1, F_2, F_3 .

Passing to the limit we obtain a subset $H = \{h\}$ of elements of $\mathcal{F}_X$ and a total order $<$ on H having the following properties:

1) $h < h'$ if and only if $\mu h < \mu h'$

2) The collection $\mathcal{H}$ of all the monoids $\{1\} \cup h^*$ form a factorization of F .

3) Every $f \in \mathcal{F}^+_X$ is conjugate to some power of one and only one $h \in \mathcal{H}$.

Because of the property expressed by 3 we shall say that H is a "cyclic transversal" of $\mathcal{F}_X$.

IV Formal products.

Let us now consider $\mathfrak{A}_X$ the large algebra of $\mathcal{F}_X$ over $\underline{Z}$. Any subset $A \subset \mathcal{F}_X$ has a (non-commutative) generating function $\bar{A} = \{f : f \in A\} \in \mathfrak{A}_X$.

As is well known the group $\mathcal{G}_X$ of invertible elements of $\mathfrak{A}_X$ consists of the elements of the form $1-a$ when a belongs to $\mathfrak{A}^+_X$, the module spanned by $\mathcal{F}^+_X$. Further, if this is so $(1-a)^{-1} = 1+\sum\{a^n ; n>0\}$, so that if $\bar{A}$ is the generating function of a subset A of $\mathcal{F}^+_X$, $(1-\bar{A})^{-1}$ is the generating function of the submonoid generated by A <u>if and only if</u> A is free.

Thus with these new notations the hypothesis of property 1 take the form of the identity $(1-\bar{X})^{-1} = (1-\bar{A}_1)^{-1}(1-\bar{A}_2)^{-1}(1-\bar{A}_3)^{-1}$

In fact, if $X' = A_1 \cap X$ and $A'_1 = A_1 \setminus X'$, $X'' = X \setminus X'$. The "elimination" of X' is expressed by the formal computation

$$(1-\bar{X}'-\bar{X}'')^{-1} = (1-\bar{X}'-\bar{A}'_1)^{-1}(1-\bar{A}_2)^{-1}(1-\bar{A}_3)^{-1}$$

$$(1-\bar{X}''(1-\bar{X}')^{-1})^{-1} = (1-\bar{A}'_1(1-\bar{X}')^{-1})^{-1}(1-\bar{A}_2)^{-1}(1-\bar{A}_3)^{-1}$$

or $$(1-\bar{X})^{-1} = (1-\bar{X}')^{-1}(1-\bar{A}'_1(1-\bar{X}')^{-1})^{-1}(1-\bar{A}_2)^{-1}(1-\bar{A}_3)^{-1}$$

Let us now define infinite formal products. Given a collection $\{a_i, i \in I\}$ of elements of $\mathcal{O}\!\mathcal{l}_X^+$ totally ordered by a relation $<$ we assume that for each $f \in \mathcal{F}_X$ there exists only a finite number of elements a_i such that f has a non zero coefficient $\langle a_i, f\rangle$ in them. Then for each f we define $\langle p, f\rangle$, the coefficient of f in p as the sum

$$\sum \langle a_{i_1}, f_{j_1}\rangle \langle a_{i_2}, f_{j_2}\rangle \ldots \langle a_{i_m}, f_{j_m}\rangle$$

extended to all factorizations $f = f_{j_1} f_{j_2} \ldots f_{j_m}$ into an arbitrary number $m>0$ of factors and for each such factorization to all m-tuples $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ such that $a_{i_1} < a_{i_2} < \ldots < a_{i_m}$ and $\langle a_{i_1}, f_{j_1}\rangle, \langle a_{i_2}, f_{j_2}\rangle, \ldots, \langle a_{i_m}, f_{j_m}\rangle \neq 0$

Clearly $1 + \sum\{\langle p, f\rangle f : f \in \mathcal{F}_X\} = p$ is a well defined element of $\mathcal{O}\!\mathcal{l}_X$ which we can consider as the infinite formal product of the elements $1+a_i$ with respect to $<$. Simple computation shows that p^{-1} is the infinite product of the elements $(1+a_i)^{-1}$ <u>with respect to the opposite order</u> $>$.

Applying our last remark of the previous section we obtain thus for each μ the identities

$$(*) \qquad (1-\bar{X})^{-1} = \prod_{<}\{(1-h)^{-1} : h \in H\} \qquad \text{or}$$

$$(**) \qquad 1-\bar{X} = \prod_{>}\{1-h : h \in H\}$$

where H is a cyclic tranversal. A special case of this construction has been given in [3] . A slight modification of the argument gives an identity of [5,1] . A commutative version of (**) has been used by Sherman [4] .

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