REMARK ON A THEOREM OF DÉNES

by

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The aim of this note is to give a slightly more explicit form to the correspondence between labelled trees and cycles which was found by Dénes (1).

Let $X = \{x_i\}$ $(1 \leq i \leq n)$ be a set of vertices and

$T = \{t_j = (x_{i_j}, x_{i'_j}) = (x_{i'_j}, x_{i_j})\}$, $(1 \leq j \leq n-1)$ be a set of edges so that (X, T) is a tree.

Considering T as an abstract alphabet, we denote by G the set of all words in the letters $t \in T$ that contain each t at most once and we define a mapping from G into the symmetric group of permutations S_n on elements $1, 2, \ldots, n$, by associating the transposition $\bar{t}_j = (i_j, i'_j)$ with each $t_j = (x_{i_j}, x_{i'_j})$ and the product $\bar{g} = \bar{t}_{j_1} \bar{t}_{j_2} \ldots \bar{t}_{j_k}$ with each $g = t_{j_1} t_{j_2} \ldots t_{j_k}$. It is trivial that any two $\bar{t}_j$, s corresponding to disjoint edges commute. Thus, unless (X, T) is a „bush" not all the associated cycles are distinct.

For any permutation $s \in S_n$ and triple (a, b, c) of distinct elements, we define the indicator $\sigma(s; a, b, c)$ with value zero if a, b, and c do not belong to the same cycle of s and with value ± 1 depending upon whether b is or is not between a and c. Formally,

$\sigma(s; a, b, c) = \sigma(s; b, c, a) = -\sigma(s; a, c, b) = 1$ if $s^n a = b$, $s^{n'} a = c$ with $n \leq n'$ and $s^{n''} a \neq c$ for all $n'' < n'$.

Property 1. *For any $g = g't \in G$ and triple (a, b, c) if $\sigma(\bar{g}'; a, b, c) \neq 0$ then $\sigma(\bar{g}'; a, b, c) = \sigma(\bar{g}; a, b, c)$.*

Proof. The subgraph $(X, T') \subset (X, T)$ corresponding to the factors of $\bar{g}'$ is a disjoint union of trees $T' = (T''_1, T''_2, \ldots, T''_k)$ some of which may be reduced to a single vertex.

By DÉNES' Theorem there is a one to one correspondence between the trees T''_j and the cycles which constitute the factors of the permutation $\bar{g}'$.

Now $T' \cup \{t\} \subset T$ is also a disjoint union of trees. The new edge $t = (x_i, x_j)$ connects two disconnected components of the graph (T', X). Furthermore, i and j belong to two different cycles of $\bar{g}'$ say, $(i\, i_2\, i_3 \ldots i_m)$ and $(j\, j_2\, j_3 \ldots j_p)$. Now $\bar{g} = \bar{g}' i$ is obtained by replacing these two cycles by the single cycle $(i\, j_2\, j_3 \ldots j_p\, j\, i_2 \ldots i_m)$. The result follows immediately. As a consequence, we have:

Property 2. *If $t = (x_i, x_j)$ and $t' = (x_j, x_k)$ are two distinct factors of the word $g = g_1\, t\, g_2\, t'\, g_3$, then $\sigma(\bar{g}; i, j, k) = 1$.*

Proof. Using the construction which has been given for property 1, it follows that i and j are in one cycle and k in another cycle of $\bar{g}_1\, \bar{t}\, \bar{g}_2$, the particular cycles being $(il_2 \ldots l_m\, j l_{m+2} \ldots l_{m+m})$ and $(kk_2 \ldots k_{m''})$, say. Thus

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$\bar{g}_1 \bar{t} \bar{g}_2 \bar{t}'$ contains the cycle $(jk_2 \ldots k_{m''} kl_{m+2} \ldots l_{m+m'} il_2 \ldots l_m)$. Hence, $\sigma(\bar{g}_1 \bar{t} \bar{g}_2 \bar{t}'; i, j, k) = 1$ and the results follow from property 1.

Let $h_i \in G$ denote a product of the set of all edges t_{j_i} incident to x_i. Let us now consider $W = \{w = (h_1, h_2, \ldots, h_n)\}$ the set of all n-tuples of elements $h_1, \ldots, h_n$. Clearly the number of elements w in W is

$$\Delta(X, T) \overset{\text{def}}{=} \prod_{1 \leq i \leq n} (\deg x_i)\,!$$

For any given $w \in W$ if $g^* = \prod_{j=1}^{n-1} t_j$ is a product of maximal degree $n - 1$, and if for each i, the symbol t_j corresponding to edges incident to x_i appear in g^* in the same order as in h_i, then we shall say after LYNDON [2] that the word g^* is a *minimal infiltration* of $w(g^* \in ((w)))$. Clearly if $\deg x_i = 1$, h_i reduces to a single t and may as well be omitted from w.

For instance if $X = (x_1, x_2, x_3, x_4)$ and $T = \{t_1 = (x_1, x_2), t_2 = (x_2, x_3), t_3 = (x_3, x_4)\}$ we have

$$W = \{(t_1, t_1 t_2, t_2 t_3, t_3),\ (t_1, t_2 t_1, t_2 t_3, t_3),\ (t_1, t_2 t_1, t_3 t_2, t_3),\ (t_1, t_1 t_2, t_3 t_2, t_3)\} =$$
$$= \{(t_1 t_2, t_2 t_3),\ (t_2 t_1, t_2 t_3),\ (t_1 t_2, t_3 t_2),\ (t_2 t_4 t_3 t_2)\}$$

and

$$((t_2 t_1, t_2 t_3)) = \{t_2 t_1 t_3, t_2 t_3 t_1\}$$
$$((t_1 t_2, t_3 t_2)) = \{t_1 t_3 t_2, t_3 t_1 t_2\}$$
$$((t_2 t_1, t_3 t_2)) = \{t_3 t_2 t_1\}$$
$$((t_1 t_2, t_2 t_3)) = \{t_1 t_2 t_3\}.$$

We now wish to prove:

Property 3. *If $g \in ((w))$ and $g' \in ((w'))$ then $\bar{g} = \bar{g}'$* i. f. f. $w = w'$.

Proof. By 2) we know that $w \neq w'$ implies $\bar{g} \neq \bar{g}'$ for every $g \in ((w))$ $g' \in ((w'))$ since there must be at least one triple such that $\sigma(g; i, j, k) = -\sigma(g'; i, j, k)$.

To prove the backward implication let us now assume that $f, f' \in ((w))$ and $f \neq f'$.

According to 3) it may be that $f = f'$ i.e. $f = t_{i_1} t_{i_2} \ldots t_{i_{n-1}}$, $f' = t_{i'_1} t_{i'_2} \ldots t_{i'_{n-1}}$ but that one can be reduced to the other by a certain number of exchanges of adjacent t's corresponding to disjoint edges in which case $\bar{f} = \bar{f}'$.

Assume there is no such reduction f^* of f to f'. Thus there exists some minimal n such that the left factor of f^* of degree n is different from the left factor of f', i.e.

$$f^* = g_1(x_j, x_k)\, g_2\,;\ f' = g_1^*(x_1, x_m)\, g_2'(x_j, x_k)\, g_3'.$$

Now g_1' contains an edge with either x_j or x_k as end point since otherwise n is not minimal. Suppose this edge is (x_j, x_p). Then $f^* = g_1(x_j, x_k)\, g_2'(x_j, x_p)\, g_{3'}$ and $f' = g_1(x_1, x_m)\, g_{2'}'(x_j, x_p)\, g_{3'}'(x_j, x_k)\, g_{4'}'$ in which case by property 2, $\sigma(f^*; j, k, p) \neq \sigma(f'; j, k, p)$.

This completes the proof and it follows immediately that the number of distinct cycles associated to the tree (X, T) is simply $\Delta(X, T)$.

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REFERENCES

[1] DÉNES, J.: „The Representation of a Permutation as the Product of a Minimal Number of Transpositions and its Connection with the Theory of Graphs." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **4** (1959) 63—70.

[2] LYNDON, R. C.: „On Burnside's Problem. I." *Trans. Amer. Math. Soc.* 77 (1954) 207—215.

ЗАМЕЧАНИЕ ОБ ОДНОЙ ТЕОРЕМЕ DÉNES-A

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Резюме

Авторы устанавливают соответствие между числом деревьев с n нумерированными вершинами и числом разложений цикла степени $n-1$ на произведений транспозиций. Они доказывают что эти два числа равны.