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# On A Family of Formal Power-Series

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## 1. Introduction

In [6] we considered three modules $R_{pol}(X) \subset R_{rat}(X) \subset R_{alg}(X)$ of formal power-series (with coefficients in a unital ring R) in the non-commuting variables $x \in X$. These formal power-series are related to polynomials and to Taylor series expansions of rational and algebraic functions.

We recall that the family $\mathcal{R}$ of the so-called <u>regular events</u> consists of all subsets of a finitely generated free monoid F(Z) that are a finite union of sets of the form $\phi^{-1}h$ ($= \{g \in F(Z) : \phi g = h\}$ ) for some homomorphisms $\phi$ of F(Z) onto a <u>finite</u> quotient monoid $\phi F(Z) = \{h\}$ ([2],[7]). It is trivial that:

(I.rat). The generating function $c_{F'} = \Sigma \{g : g \in F'\}$ of any $F' \in \mathcal{R}$ belongs to $R_{rat}(X)$.

(I'.rat). Any $a \in R_{rat}(X)$ can be represented in the form $a = \theta c_{F'} = \lim_{n \to \infty} \Sigma \{\theta g : g \in F', \deg g < n\}$ for some suitable $F' \in \mathcal{R}$ and homomorphism $\theta : F(Z) \to R_{pol}(X)$.

However, if one replaces the condition that $\phi F(Z)$ is <u>finite</u> by the condition that $\phi F(Z)$ is <u>abelian</u> in the definition of $\mathcal{R}$, the generating function of $\phi^{-1}h$ does not necessarily belong to $R_{alg}(X)$ [5]. Then one may ask what type of monoids give a family $\mathcal{L}'$ of subsets of F(Z) having the properties (I.alg) and (I'.alg) derived from (I.rat) and (I'.rat) by substituting $R_{alg}$ to $R_{rat}$. We shall show that a partial answer is given by the <u>extensions of a free group by a finite monoid</u> [4]. This provides alternative proofs of some theorems of [1] and [3].

This note is part of common research with N. Chomsky.

## 2. <u>Preliminary definitions.</u>

Let there be: 1) Three finite sets Z, S and X; 2) a homomorphism $\psi$ of F(Z) onto a <u>finite</u> monoid K; 3) Three mappings $\alpha'$, $\sigma'$ and $\chi$ of (K,S,X) into F(Z), S and the family of all subsets of K, respectively. For our present purpose there is no loss of generality in assuming that $\psi g \neq \psi 1$ if $g \neq 1$ and that every $g \in F(Z)$ has at least one right factor in each $\chi(k,s,x)$.

For any $\bar{g} = (g,s) \in \bar{G} = (F(Z),S)$ and $x \in X$ : $\alpha(\bar{g},x) = \alpha(g,s,x) = g'$, the word of highest degree such that $g\,\alpha'(\psi g,s,x) = g'g''$, with $g'' \in \chi(\psi g,s,x)$. $\sigma(\bar{g},x) = \sigma(g,s,x) = \sigma'(\psi g,s,x)$, $\bar{g}.x = (\alpha(\bar{g},x),\ \sigma(\bar{g},x)) \in \bar{G}$.

In the usual fashion we extend this mapping $(\bar{G},X) \longrightarrow \bar{G}$ to a representation of F(X) (= the free monoid generated by X) by mappings of $\bar{G}$ into itself. If $f \in F(X)$, $\bar{g} \in \bar{G}$ and $\bar{g}' = \bar{g}.f$, we write $\bar{g}' = (\alpha(\bar{g},f),\ \sigma(\bar{g},f))$.

Let $f' < f$ denote that $f'$ is a proper (i.e., $\neq f$) left factor of $f$. For each 5-tuple $j = (j_i)$ with arbitrary $j_1, j_3 \in F(Z)$; $j_2, j_4 \in S$; $j_5 \in F(Z) \cup \{0\}$ we define:

$C(j) = \{f \in F(X) : f \neq 1, (j_1,j_2).f = (j_3,j_4)\}$, if $j_5 = 0$;

$= \{f \in F(X) : f \neq 1, (j_1,j_2).f = (j_3,j_4); j_5 < \alpha(j_1,j_2,f')$ for each $f'$, $1 < f' < f\}$, if $j_5 \in F(Z)$.

2.1. The generating function $c(j)$ of any $C(j)$ with $j_5 = 0$ belongs to $R_{alg}(X)$.

Proof. Let $J$ be the set of all $j$'s which satisfy any one of the conditions $j_3 \leq j_1 = j_5$, $j_1 < j_3 = j_5$, $j_3 \leq j_5 < j_1$, or $j_5 = 0$. Let $Y = \{y(j) : j \in J\}$ be a set of new variables. For each $j \in J$, $p(j) \in R^*_{pol}(X \cup Y)$ is defined as follows:

If $j_3 \leq j_1 = j_5$ or $j_1 < j_3 = j_5$

$p(j) = \Sigma\{x : x \in X \cap C(j)\} + \Sigma\{xy(\alpha(j_1,j_2,x), \sigma(j_1,j_2,x), j_3, j_4, j_5) :$

$x \in X, j_5 < \alpha(j_1,j_2,x)\}$.

If $j_3 \leq j_5 < j_1$ $g_2 \leq g_3 < g_1$

$p(j) = p(j_1,j_2,j_3,j_4,j_1) + \Sigma\{y(j_1,j_2,j_5g,s,j_5g)\ y(j_5g,s,j_3,j_4,j_5g') :$

$s \in S$, $g \in g'Z$, $j_5g \leq j_1\}$.

$y(q_1, q_3q, q_3q)\ y(q_3q, q_3, q_3q$

If $j_5 = 0$

$p(j) = p(j_1,j_2,j_3,j_4,j_3) + \Sigma \{y(j_1,j_2,g,s,0)\ y(g,s,j_3,j_4,j_3) : s \in S,\ g \leq j_3\}$.

Clearly, each equation expresses a unique factorisation property of the words of $C(j)$ as products of elements of $X$ and words from other sets $C(j')$. Hence, each equation is an identity if $p(j) = y(j) = c(j)$ for all $j \in J$. Let $J_d$ denote the subset of all 5-tuples such that $\deg j_1$, $\deg j_2$, $\leq d$. If $j \in J_d$ the right member of the equation which defines $p(j)$ contains only variables $y(j')$ with $j' \in J_d$ or with $j'$ of the form $j_3 \leq j_1 = j_5$. In this last case $\deg j_3 \leq d$ and $\deg j_1 - \deg j_3 \leq \max \{\deg \alpha(\bar{g},x) : \bar{g} \in \bar{G},\ x \in X\}$.

Now let $\psi_2 g$ denote, for any $g \in F(Z)$, the subset $\{(\psi g', \psi g'') : g'g'' = g\}$ of $(K, K)$. If $\psi_2 g_1 = \psi_2 g_4$, $\psi_2 g_2 = \psi_2 g_5$, $j = (g_1g_2g_3, s, g_1, s', g_1g_2)$, $f \in C(j)$, induction on $\deg f'$ shows that for each $f' < f$, $\alpha(g_1g_2g_3,s,f') = g_1g_2g'$, $\alpha(g_4g_5g_3,s,f') = g_4g_5g'$ with the same $g'$ and $\sigma(g_1g_2g_3,s,f') = \sigma(g_4g_5g_3,s,f')$. Thus, $C(j) = C(g_4g_5g_3,s,g_4,s',g_4g_5)$. It follows that there exists a finite $d^*$ such that for any fixed $d \geq d^*$ and $j \in J_d$, each $y(j')$ with $j' \notin J_d$ in the right member of $p(j)$ can be replaced by $y(j'')$ with $j'' \in J_d$ and $C(j') = C(j'')$. Making this substitution the set $(p(j))_{j \in J_d}$ becomes a proper system in the notation of [6] and 2.1. is verified.

3. <u>Verification of (I.alg)</u>.

(I.alg). If $\bar{\gamma}$ is a homomorphism of $F(X)$ into an extension $\bar{G} = \{\bar{g}\}$ of a free group $G$ by a finite monoid $H$, the generating function $c_{\bar{g}}$ of any $\bar{\gamma}^{-1}\bar{g}$ belongs to $R_{alg}(X)$.

<u>Proof</u>. Let $G$ be generated by $\{z_i\}$ $1 \leq i \leq m$; $Z = \{z_{i'}\}$ $i' = \pm i$ and $\gamma^*$ be the homomorphism of $F(Z)$ onto $G$ such that $(\gamma^* z_i)^{-1} = \gamma^* z_{-i}$ for all $z_i \in Z$.

(i) Let us consider the special case of $\bar{G} = G$. Then $\bar{\gamma}$ is given by a homomorphism $\gamma : F(X) \longrightarrow F(Z)$ and $\bar{\gamma} f = (\gamma^* \circ \gamma) f$. Since $\gamma$ itself is determined by its restriction to the finite set $X$ we can assume $m < \infty$. If $\rho : F(Z) \longrightarrow F(Z)$ is such that $\rho g z_i z_{-i} g' = \rho g g'$ and $\rho g = g$ for all $g$ having no factor of the form $z_i z_{-i}$, the word $\rho g$ is the so-called <u>reduced form</u> of $g$ and $\gamma^* \rho g = \gamma^* g$ with $\rho g' \neq \rho g$ i.f.f. $\gamma^* g' \neq \gamma^* g$.

We consider the following special case of the representation defined in the preceding section:

1) $K$ and $S$ are identified with $F' = \{g \in F(Z) : \deg g \leq d\}$ where $d = \max\{\deg \gamma x : x \in X\}$. 2) $\psi g = g$ if $g \in F'$, $\psi g$ = the right factor of degree $d$ of $g$ if $g \notin F'$. 3) $\alpha'(k,s,x) = \rho(s\gamma x)$; $\chi(k,s,x) = k\,\rho(s\gamma x) = \sigma'(k,s,x)$.

Thus, if $\bar{g} = (g,s) \in (F(Z),F)$ one computes successively $s\gamma x$, $\rho(s\gamma x)$, $g\,\rho(s\gamma x)$; $\alpha(g,s,x)$ and $\sigma(g,s,x)$ are determined by $\alpha(g,s,x)\ \sigma(g,s,x) = g\,\rho\ (s\gamma x)$ and $\sigma(g,s,x) = \psi(g\,\rho(s\gamma x))$. Induction on deg f shows that for each f the word $\alpha(1,1,f)\ \sigma(1,1,f)$ is precisely equal to $\rho f$ and the result is a consequence of 2.1.

(ii) In the general case [4], $\bar{\gamma}$ is given by a homomorphism $\phi : F(Z) \to H$, and a mapping $\gamma : (H,X,H) \to F(Z)$. Then $\bar{\gamma} f = (\gamma^{*} o \gamma(\phi 1, f, \phi 1), \phi f) \in (G,H)$ where $\gamma : (H, F(X), H) \to F(Z)$ is defined by the identities:

for all $h$, $h' \in H$, $f$, $f' \in F(X)$, $x \in X$

$\gamma(h,1,h') = 1$; $\gamma(h,fxf',h') = \gamma(h,f,(\phi x f')h')\ \gamma(h\phi f,x,(\phi f')h')\ \gamma(h\phi f x,f',h')$.

Let $X' = \{x'(h,x,h') : (h,xh') \in (H,X,H)\}$ be a set of new variables; $\xi$ and $\gamma'$ are the homomorphisms of $F(X')$ into $F(X)$ and $F(Z)$ induced by $\xi x'(h,x,h') = x$ and $\gamma' x'(h,x,h') = \gamma(h,x,h')$. If $\bar{H} = (H,H) \cup \{0\}$ we define a representation $(\bar{H},F(X')) \to \bar{H}$ of $F(X')$ by the identities:

for all $x' \in X'$, $0.x' = 0$; for all $x'(h,x,h') \in X'$ and $(h_1,h_2) \in \bar{H}$, $(h_1,h_2).x'(h,x,h') = (h_1\ \phi x,h')$ if $h = h_1$ and $h_2 = (\phi x)h'$; $(h_1,h_2).x'(h,x,h') = 0$, otherwise.

Thus, for any $h \in H$, the restriction of $\xi$ to $F'_h = \{f' \in F(X') : (\phi 1,h).f' = (h, \phi 1)\}$ is a 1-1 mapping onto $\{f \in F(X) : \phi f = h\}$ and for each $f' \in F'_h$,

$\gamma'f' = \gamma(\phi 1, f', \phi 1)$. It follows that $c_{\bar{g}} = \xi c'_{g,\bar{h},\bar{h}'} = \Sigma\{\xi f' : f' \in F(X') : \rho\gamma' f' = g, \bar{h}.f' = \bar{h}'\}$ for suitable $g \in F(Z)$, $\bar{h}$, $\bar{h}'$, $\in H$.

Now let $S, K, \psi, \alpha', \chi, \sigma'$ be the same as in (i), $\bar{S} = (S, \bar{H})$. For each $\bar{s} = (s, \bar{h}) \in \bar{S}$, $x' \in X'$, we define: $\bar{\alpha}'(k, \bar{s}, x') = \alpha'(k, s, \xi x')$; $\bar{\chi}(k, \bar{s}, x') = \{(\chi(k, s, \xi x'), \bar{h}') : \bar{h}' \in \bar{H}\}$; $\bar{\sigma}'(k, \bar{s}, x') = (\sigma'(k, s, \xi x'), \bar{h}.x')$. It is trivial that $(g, (s, \bar{h})).x' = (\alpha(g, s, \xi x'), (\sigma(g, s, \xi x'), \bar{h}.x'))$, identically. Hence, $c'_{g,\bar{h},\bar{h}'}$ (or $c'_{g,\bar{h},\bar{h}'} - 1$) is a component of the solution of a proper system $p' \in R^{*M}_{pol}(X' \cup Y)$. Clearly, if one extends $\xi$ to a homomorphism $R^{*M}_{pol}(X' \cup Y) \to R^{*M}_{pol}(X \cup Y)$ by $\xi y = y$ for all $y \in Y$, $p = \xi p'$ is again a proper system and (in the notation of [6]) $p(n) = \xi p'(n)$ for all $n$. This concludes the verification of (I.alg).

4. Verification of (I'.alg).

Let $Z, G, \gamma^*$ be the same as in section 3, $1 < m < \infty$; $\mathcal{R}' = \{F' \subset F(Z) : F' = (\gamma^{*-1} 1) \cap F'', F'' \in \mathcal{R}\}$.

(I'.alg). Any $a \in R^*_{alg}(X)$ can be represented in the form $a = \lim_{n \to \infty} \Sigma\{\theta g : g \in F', \deg g < n\}$ for some suitable $F' \in \mathcal{R}'$ and homomorphism $\theta : F(Z) \to R_{pol}(X)$.

Proof. (i) Let $a$ be a component of the solution of the proper system $(p_j) = p \in R^{*M}_{pol}(X \cup Y)$. The support, Supp. b, of any formal power-series $b$ is the set of all words having a non-zero coefficient in $b$. Since each $p_j$ belongs to $R_{pol}(X \cup Y)$ there exists $d^* < \infty$ such that any $f \in \{f' \in \text{Supp. } p_j, 1 \leq j \leq M\}$

either belongs to $F(X)$ or has a factorisation

$f = f_1 y_{i_1} f_2 y_{i_2} f_3 \dots f_d y_{i_d} f_{d+1}$ with $f_1, f_2, \dots f_{d+1} \in F(X)$, $y_{i_1}, y_{i_2}, \dots, y_{i_d} \in Y$, $1 \leq d = \deg_Y f < d^*$. We introduce a set $Z = \{z(j,f,d,\epsilon) : 1 \leq j \leq M,\ f \in \underline{\text{Supp}}.\ p_j,\ 1 \leq d \leq d^*,\ \epsilon = + \text{ or } -\}$ of new variables and make the definitions:

If $f \in F(X) \cap \underline{\text{Supp}}.\ p_j$, $\theta z(j,f,d,\epsilon) = \langle p_j, f \rangle f$ if $d = 1$ and $\epsilon = +$, $= 1$, otherwise; $\zeta_j f = z(j,f,1,+)\, z(j,f,1,-)\, z(j,f,2,+)\, z(j,f,2,-) \dots z(j,f,d^*,+)\, z(j,f,d^*,-)$.

If $f = f_1 y_{i_1} f_2 \dots y_{i_d} f_{d+1} \in \underline{\text{Supp}}.\ p_j$ as above, $\theta z(j,f,d',\epsilon) = \langle p_j, f \rangle f_1$ if $d' = 1$, and $\epsilon = +$; $= f_{d'}$ if $1 < d' \leq \deg_Y f + 1$ and $\epsilon = +$; $= 1$, otherwise; $\zeta_j f = z(j,f,1,+) y_{i_1}\, z(j,f,1,-)\, z(j,f,2,+) y_{i_2}\, z(j,f,2,-) \dots \dots z(j,f,d,+) y_{i_d}\, z(j,f,d,-)\, z(j,f,d+1,+)\, z(j,f,d+1,-) \dots z(j,f,d^*,+)\, z(j,f,d^*,-)$.

Thus, $\theta q_j = p_j$ where $q_j = \Sigma\{\zeta_j f : f \in \underline{\text{Supp}}.\ p_j\}$ and $q = (q_j) \in R^*_M(Z \cup Y)$ is a proper system such that (in the notation of [6]) $\lim_{n \to \infty} \theta q(n) = p(\infty)$, the solution of $p$. Moreover, if $Q_j = \underline{\text{Supp}}.\ q_j$, $P_j(n) = F(Z) \cap \underline{\text{Supp}}.\ q_j(n)$ and if $\eta_n$ is the homomorphism of $F(Z \cup Y)$ into $R_{pol}(Z)$ induced by $\eta_n z = z$, $\eta_n y_j = P_j(n)$ for all $z \in Z$, $y_j \in Y$, it follows from the definitions that, for all $n$, $q_j(n+1)$ is the generating function of $\Sigma\{\eta_n g : g \in Q_j\}$. Hence it suffices to show that the sets $P_j(\infty)$ have the desired form.

(ii) Let $V \subset F(Z)$ consist of:

all words $z(j,f,d,+)\ z(j',f',1,+)$ or $z(j',f',d^*,-)\ z(j,f,d+1,-)$

with $d \leq \deg_Y f$ and $j'$ equal to the index $i_d$ of the d-th factor

$y_{i_d}$ of $f$;

all words $z(j,f,d,+)\ z(j,f,d,-)$ with $d > \deg_Y f$;

all words $z(j,f,d,-)\ z(j,f,d+1,+)$.

We take $H$ in 1-1 correspondence with $\{0, 1, Z, (Z,Z')\}$ and define

the homomorphism $\phi : F(Z) \rightarrow H$ by

$\phi g = h_g$ if $\deg g < 2$;

$\phi g = 0$ if $g$ has at least one factor of degree two not belonging

to $V$;

$\phi g = h_{z,z'}$ if $\phi g \neq 0$ and $g \in z\,F(Z)z'$.

$H_j = \{h_{z,z'} : z = z(j,f,1,+),\ z' = z(j,f,d^*,-),\ f \in$ Supp. $p_j\}$.

The homomorphism $\gamma^* : F(Z) \rightarrow G$ is defined by $(\gamma^*\, z(j,f,d,+))^{-1} =$

$\gamma^*\, z(j,f,d,-)$ for all elements of $Z$.

Induction on $n$ shows that $P_j(\infty) \subset D = \{g \in F(Z) : \gamma^* g = 1,\ \phi g \neq 0\}$.

Let $\overline{P}(j,d,d') = \{g \in F(Z) : g \in$ Supp.$(\lim_{n \rightarrow \infty} \eta_n g') : g' \in Q'(j,d,d')\}$ where

$O'(j,d,d')$ denotes the set of all $g' \in F(Z \cup Y)$ of the form $z(j,f,d,+)g''\,z(j,f,d',-)$

that are a factor of some $g \in Q_j$; $\overline{P} = \bigcup\{P(j,d,d') : 1 \leq j \leq M,\ 1 \leq d < d' \leq d^*\}$.

Then $\overline{P}(j,d,d') \subset D$ and $P_j(\infty) = \overline{P}(j,1,d^*) = \overline{P} \cap \phi^{-1} H_j$.

Thus, it suffices to show that, conversely, every $g \in D$ belongs to $\overline{P}$. This is trivial if $\deg g \leq 2$. We assume the result proved for all words of degree < n and we consider $g \in D$ of degree $n > 2$.

Let the factorisation $g = z\, g'\, z'\, g''$ of $g$ be determined by the condition that $zg'z'$ is the left factor $\neq 1$ of lowest degree of $g$ that satisfies $\gamma^* zg'z' = \gamma^* 1$. Since $\gamma^*$ is a homomorphism into a free group, this implies $\gamma^* g'' = \gamma^* z\, z' = \gamma^* g' = \gamma^* 1$.

If $g'' \neq 1$, the induction hypothesis shows that $z = z(j,f,d,+)$, $z' = z(j,f,d',-)$, $g'' = z(j',f',d'',+)\; g'''\; z(j',f',d''',-)$ for some $j,j'$, $f,f'$, $d,d'$, $d'',d'''$ and $g''' \in F(Z)$. Because of $\phi g \neq 0$, we have $j = j'$, $f = f'$, $d'' = d'+1$ and the result is proved in this case.

If $g'' = 1$, the induction hypothesis shows that $1 \neq g' = z(j',f',d'',+)\; g'''\; z(j',f',d''',-)$. Because of $\phi g \neq 0$ and $\gamma^* z\, z' = \gamma^* 1$, we have $z = z(j,f,d,+)$, $d'' = 1$, $d''' = d^*$, (i.e. $g' \in \phi^{-1} H_{j'}$), $z' = z(j,f,d,-)$ and $z(j,f,d,+)\; y_{j'}\; z(j,f,d,-)$ is a factor of a word of $Q_j$. Thus, $g \in \overline{P}(j,d,d)$ and the verification of (I'.alg) is completed.

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References


1. Kesten, M. Symmetric random walks on groups. Trans. Am. Math. Soc. 92: 336-354, 1959.

2. Rabin, M. O., and Scott, D. Finite automata and their decision problems. I.B.M. Journal of Research. 3: 114-125, 1959.

3. Raney, G. N. Functional composition patterns and power-series reversion. Trans. Am. Math. Soc. 94: 441-451, 1960.

4. Redei, L. Die Verallgemeinerung der schreierschen Erweiterungs Theorie. Acta. Sc. Math. Szeged. 14: 252-273, 1952.

5. Schützenberger, M. P. Un probleme de la theorie des automates. Seminaire Dubreil Pisot, Paris, 13 eme annee, Nov. 1959, No. 3.

6. Schützenberger, M. P. On a theorem of R. Jungen. To appear in Proc. Am. Math. Soc.

7. Shepherdson, J. C. The reduction of two way automata. I.B.M. Journal of Research. 3: 198-200, 1959.