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On a Question of Eggan

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An elementary answer is given to a question raised by Eggan

In (Eggan, 1963), it is asked among other questions whether a free monoid X^* generated by a set X consisting of two distinct elements x and y contains subsets of arbitrary star height. McNaughton (unpublished Lecture Notes M.I.T., 1963–1964) has proved, as an application of very powerful and general methods, that it is so. The present note is devoted to the following more elementary example which answers directly Eggan's question.

EXAMPLE. *Let q be any fixed positive natural number and let γ be a homomorphism of X^* into the additive group $\{0, 1, \cdots, 2^q - 1\}$ of the integers modulo 2^q that satisfies $\gamma x = -\gamma y = 1$. The set*

$$\gamma^{-1}0 \ (= \{f \in X^* : \gamma f = 0\})$$

has star height q.

It is clear that $\gamma^{-1}0$ is a submonoid of X^* that is freely generated by $K = (\gamma^{-1}0 \cap XX^*) \setminus (\gamma^{-1}0 \cap XX^*)^2$ $(= \{f \in XX^* : \gamma f = 0; f \notin (\gamma^{-1}0 \cap XX^*)XX^*\})$. For $q = 1$, one has $K = X^2$, $\gamma^{-1}0(= K^*) = (X^2)^*$ and the example is trivial. Thus we can assume henceforth that $q > 1$. For each natural number m we set:

$$\bar{L}_m = \{f \in XX^* : \gamma f = \pm 2^{m+1}\};$$

$$L_m = (\bar{L}_m \cap xX^*) \setminus (\bar{L}_m XX^* \cup KXX^*);$$

$$L_m' = (\bar{L}_m \cap yX^*) \setminus (\bar{L}_m XX^* \cup KXX^*);$$

$$K_m' = K \setminus \bar{L}_m XX^*.$$

Thus, e.g., $L_0 = \{x^2\}$; $L_0' = \{y^2\}$; $K_0' = \{xy \cup yx\}$; $L_1 = x^2(xy \cup yx)^*x^2$; $L_1' = y^2(xy \cup yx)^*y^2$; $K_1' = K_0' \cup x^2(xy \cup yx)^*y^2 \cup y^2(xy \cup yx)^*x^2$; $\cdots \bar{L}_{q-1} = \gamma^{-1}0 \cap XX^*$; $K_{q-1}' = K$.

Let P_m denote any of the sets L_m, L_m' or $K_m' \setminus K_{m-1}'$ where $m \in [1, q-2]$ and let $f \in P_m$. Considering the left factor f_1 (resp. $f_1 f_2$) of minimal degree (or "length") (resp. of maximal degree) of f that satisfies

γf_1 (resp. $\gamma f_1 f_2$) $= \pm 2^m$ shows that f has one and only one factorization of the form $f = f_1 f_2 f_3$ where, on the one hand, $f_2 \in K'^*_{m-1}$ and on the other hand:
if $P_m = L_m$ (resp. L_m'), both f_1 and f_3 belong to L_{m-1} (resp. to L'_{m-1});
if $P_m = K_m' \setminus K'_{m-1}$, either $f_1 \in L_{m-1}$ and $f_3 \in L'_{m-1}$ or $f_1 \in L'_{m-1}$ and $f_3 \in L_{m-1}$.
Accordingly, one has the recurrence relations:

$$L_m = L_{m-1} K'^*_{m-1} L_{m-1}\,; \qquad L_m' = L'_{m-1} K'^*_{m-1} L'_{m-1}\,;$$

$$K_m' = K'_{m-1} \cup L_{m-1} K'^*_{m-1} L'_{m-1} \cup L'_{m-1} K'^*_{m-1} L_{m-1}$$

which show that P_m has star height at most m. Thus $K = K'_{q-1}$ has star height at most $q - 1$.

This concludes the verification that $\gamma^{-1}0 = K^*$ has star height at most q and we proceed to the verification that $\gamma^{-1}0$ has star height at least q.

Consider for each natural number $n > 1$ the sequence of words $w_{0,n}$, $w_{1,n}$, $w_{2,n}$, $\cdots$, $w_{q-1,n} \in \gamma^{-1}0$ defined recursively by the equations:

$$w_{0,n} = xy;\, w_{1,n} = x^2(xy)^n y^2 (xy)^n;\, \cdots$$

$$w_{k,n} = x^{2^k}(w_{k-1,n})^n y^{2^k}(w_{k-1,n})^n;\, \cdots$$

$$w_{q-1,n} = x^{2^{q-1}}(w_{q-2,n})^n y^{2^{q-1}}(w_{q-2,n})^n.$$

For $k = 0, 1, \cdots, q - 1$, let $\mathfrak{W}_k$ denote the family of all subsets F of X^* of minimal star height that satisfy the following two conditions:

(γ). *$\gamma f = \gamma f'$ for any two $f, f' \in F$.*

(ω_k). *There exist infinitely many values of n such that $(w_{k,n})^n$ is a factor of at least one word of F.*

If h_k is the common value of the star height of the members of $\mathfrak{W}_k$, it is clear that

$$0 < h_0 \leqq h_1 \leqq \cdots \leqq h_{q-1} \leqq q$$

the first (resp. last) of these inequalities resulting trivially from the fact that any member of $\mathfrak{W}_0$ contains an infinity of words (resp. that $\gamma^{-1}0$ itself satisfies (γ) and (ω_{q-1})). Thus, to prove that $\gamma^{-1}0$ has exactly star height q we have only to show that $h_{k-1} < h_k$ for $k = 1, 2, \cdots, q - 1$. To do this, consider any $F \in \mathfrak{W}_k$. By definition, the hypothesis that F has star height at most h_k is equivalent to the hypothesis that F is the union of a finite number of sets F_j each of which is a finite product of the form $A_1 A_2^* A_3 A_4^* \cdots A_{2i'-1} A_{2i'}^* \cdots A_{2m-1} A_{2m}^*$ where all the A_i's ($i = 1, 2, \cdots, 2m$) are subsets of X^* having star height at most $h_k - 1$ (A_{2m} may be empty). The hypothesis that F satisfies (γ) and (ω_k) implies that all the F_j's satisfy (γ) and that at least one of them satisfies (ω_k). Thus we can assume henceforth that F itself is the finite product $A_1A_2^* \cdots A_{2m-1}A_{2m}^*$. Now for each $i \in [1, 2m]$, there exist at least two words $f_1, f_2 \in X^*$ such that $f_1ff_2 \in F$ for all $f \in A_i$ (or, even, $\in A_i^*$ when i is even). Thus the hypothesis that F satisfies (γ) implies that each of the A_i's satisfies the same condition and in fact, since any submonoid of X^* contains the neutral element of X^*, we know that $A_i \subset \gamma^{-1}0$ when i is even. Owing to the hypothesis that F has minimal star height, this remark implies in turn that none of the sets A_i ($i \in [1, 2m]$) satisfies (ω_k).

However, since X is finite and since F has finite star height, Kleene's theorem asserts the existence of a homomorphism μ of X^* into a finite monoid M and of a subset M' of M such that $F = \{f \in X^* : \mu f \in M'\}$. Because of the finiteness of M, there exists a natural number p such that $u^{p'} = u^{p'+p}$ for all $p' \geqq p$ and all $u \in M$, and accordingly, the hypothesis that F satisfies (ω_k) implies here the formally stronger condition that there exist infinitely many values of n such that for infinitely many values of n', $(w_{k,n})^{n'}$ is a factor o. at least one word of F. Since none of the A_i's satisfies (ω_k), this shows that at least one of the submonoids $A_{2i'}^*$ ($i' = 1, 2, \cdots, m$), say $A_{2i_1}^*$, satisfies (ω_k). From this remark let us deduce that A_{2i_1} satisfies (ω_{k-1}) (which, together with the fact that $A_{2i_1} \subset \gamma^{-1}0$ has star height at most $h_k - 1$, will show that $h_k - 1 \geqq h_{k-1}$).

To see this, write $(w_{k,n})^2$ in the form $z_1w_1z_2w_2z_3w_3z_4w_4$ where $w_i = (w_{k-1,n})^n$ for $i = 1, 2, 3, 4$, and $z_i = x^{2^k}$ or y^{2^k} depending upon $i = 1, 3$ or 2, 4. Taking into account that γf (resp. $-\gamma f$) is contained in $\{0, 1, \cdots, 2^k - 1\}$ for any left (resp. right) factor f of $(w_{k-1,n})^n$, direct examination shows that any factor $a \in K$ of $(w_{k,n})^2$ which is not a factor of one of the w_i's ($i = 1, 2, 3, 4$) has the form $z_i'w_iz_{i+1}''$ where $i = 1, 2, 3$, $\gamma z_i' = -\gamma z_{i+1}''$ and where z_i' (resp. z_{i+1}'') is a nonempty right (resp. left) factor of z_i (resp. of z_{i+1}). Since $A_{2i_1} \subset \gamma^{-1}0 = K^*$, it follows that for infinitely many values of n, A_{2i_1} contains at least one word admitting a factor of the form $z_i'(w_{k-1,n})^nz_{i+1}''$. The verification that $\gamma^{-1}0$ has exactly star height q is concluded.

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