On a Family of Sets Related to McNaughton's L-Language

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I. Introduction

Let F be the free monoid generated by a fixed set X containing at least two elements and let Q_1 be the least family Q of subsets of F that satisfies the conditions (K1) and (K2) below where, as always in this paper, e denotes the neutral element of F.

(K1). $F \in Q$; $\{e\} \in Q$; $X' \in Q$ for any subset X' of X.

(K2). If Q contains A_1 and A_2, it also contains $A_1 \cup A_2$ $A_1 \backslash A_2$ $(=\{f \in F: f \in A_1, f \notin A_2\})$ and $A_1 \cdot A_2$ $(=\{ff' \in F: f \in A_1, f' \in A_2\})$.

The study of Q_1 is motivated by the fact (discussed in [5]) that Q_1 is closely related to the family of the subsets of F that can be described within the "L-language" of McNaughton ([3)]. The object of the present paper is to verify the *main property* below, which gives for certain subsets of F the possibility of deciding if they belong to Q_1. Finally, as a direct application of Eggan's theory ([1]), we show that for suitable X, Q_1 contains sets of arbitrarily large "star height."

For each positive natural number n, let $M_1(n)$ denote the family of all monoids having at most n elements and admitting only trivial subgroups ([4]); that is, let the monoid M belong to $M_1(n)$ if and only if it has $n' \leqslant n$ elements and if $m^n = m^{n+1}$ for each $m \in M$. Further, for $A \subset F$, let $A \subset Q_1'$ if and only if there exist a monoid $M \in \cup M_1(n)$, a subset M' of M and a homomorphism γ of F into M that satisfy $A = \{f \in F: \gamma f \in M'\}$. We have

MAIN PROPERTY

The families Q_1 and Q_1' of subsets of F are identical.

As an illustration, let us consider two disjoint subsets A_1 and A_2 of F and assume that we know three elements f, f', and f'' of F for which both $A_1 \cap \{f'f^nf'': n \in N\}$ and $A_2 \cap \{f'f^nf'': n \in N\}$ are infinite sets. Using the relation $Q_1 \subset Q'_1$, we can conclude that it is impossible to find a set $B \in Q_1$ satisfying $A_1 \subset B$ and $A_2 \subset F \setminus B$. Indeed, according to the definition of $Q'_1, B \in Q'_1$ would imply the existence of a finite integer n such that the set $\{f'f^{n'}f'' : n' \in N, n' > n\}$ is entirely contained in B or in $F \setminus B$.

II. Verification of $Q_1 \subset Q'_1$

Since Q_1 is defined as the least family which satisfies (K1) and (K2), $Q_1 \subset Q'_1$ follows instantly from the following two remarks from ([5]), which are reproduced here for the sake of completeness.

Remark 1. Q'_1 satisfies condition (K1).

Verification. Let the monoid $M = \{e', x', 0\} \in M_1(3)$ and the map $\gamma : F \to M$ be defined as follows:

$$\gamma e = e' = e'^2 \begin{cases} \text{for each } x \in X', \gamma x = x' = e'x' = x'e' \\ \text{for each } f \in F \setminus (\{e\} \cup X'), \gamma f = 0 = e'0 = 0e' = x'^2 = x'0 \\ \qquad\qquad\qquad\qquad = 0x' = 0^2. \end{cases}$$

Thus $F = \gamma^{-1}M$, $X' = \gamma^{-1}x'$, $\{e\} = \gamma^{-1}e'$. It is clear that γ is a homomorphism and Remark 1 is verified.

Remark 2. Q'_1 satisfies condition (K2).

Verification. Let for $j = 1, 2$ the homomorphism $\gamma_j : F \to M_j$, the monoid M_j, and the subset M'_j of M_j satisfy $M_j \in M_1(n_j)$ and $A_j = \{f \in F; \gamma_j f \in M'_j\}$.

We consider the family R of all sets of pairs $(m_1, m_2) \in M_1 \times M_2$ and for $m_1 \in M_1$, $m_2 \in M_2$, $r = \{(m_{1,i}, m_{2,i}) : i \in I_r\} \in R$, we let

$$m_1 r = \{(m_1 m_{1,i}, m_{2,i}) : i \in I_r\} \qquad r m_2 = \{(m_{1,i}, m_{2,i} m_2) : i \in I_r\}$$

Further, denoting by $\bar{M}$ the direct product (of sets) $M_1 \times R \times M_2$, we define the product for any two elements (m_1, r, m_2) and (m'_1, r', m'_2) of $\bar{M}$ by the formula

$$(m_1, r, m_2)(m'_1, r', m'_2) = (m_1 m'_1, m_1 r' \cup r m'_2, m_2 m'_2) \in \bar{M}$$

Finally for $f \in F$, we let

$$\gamma f = (\gamma_1 f, \{(\gamma_1 f', \gamma_2 f'') : f', f'' \in F; f = f'f''\}, \gamma_2 f) \in \bar{M}$$

The verification that we have defined an associative product and a homomorphism γ of F onto a finite monoid $M \subset \bar{M}$ is straightforward and it is omitted. The same applies to the verification that $A_1 \cup A_2$, $A_1 \backslash A_2$, and $A_1 \cdot A_2$ are images by γ^{-1} of suitable subsets of M. Thus the remark will follow from the fact that any subgroup $G = \{(m_{1,i}, r_i, m_{2,i}) : i \in I_G\}$ of M is isomorphic to a direct product $G_1 \times G_2$, where G_j is a subgroup of M_j $(j = 1, 2)$.

Indeed, by construction, $\{m_{j,i} : i \in I_G\} \subset M_j$ is a homomorphic image of G, hence a group G_j. Let e_j be its neutral element and let N be the intersection of G with the subset $\{(e_1, r, e_2) : r \in R\}$ of $\bar{M}$; N is a normal subgroup of G and G/N is isomorphic to a submonoid of $G_1 \times G_2$.

Therefore, for verifying $M \in \bigcup_{n>0} M_1(n)$, it suffices to show that N reduces to the neutral element $e'(=(e_1, r, e_2))$ of G. To see this, let $g(=(e_1, s, e_2))$ and $g(=(e_1, \bar{s}, e_2))$ be inverse elements of N. The equation $e' = e'^2$ gives $r = e_1 r \cup r e_2$ and the equation $e' = g\bar{g}$ gives $r = e_1 \bar{s} \cup s e_2$. Therefore, $e_1 r = e_1 \bar{s} \cup e_1 s e_2$ and, since $e_1 r \subset r$, we have $e_1 s e_2 \subset r$. However, the equation $g = e' g e'$ gives $s = e_1 r \cup e_1 s e_2 \cup r e_2$; that is, $s = r \cup e_1 s e_2$ and therefore, $s = r$. This shows that $e' = g$, hence that $N = \{e\}'$, and the verification is concluded.

III. Verification of $Q_1' \subset Q_1$

For each positive natural number n let $Q_1(n)$ denote the least family of subsets of F that satisfies the conditions (K1) and (K2) and that contains every set of the form $\gamma^{-1} M'$ if $M' \subset \gamma F$ and if $\gamma : F \to \gamma F$ is a homomorphism of F onto a member of $M_1(n)$. Thus $Q_1(1) = Q_1$, since for $M' \subset \gamma F$ and $\gamma F \in M_1(1)$, we have either $\gamma^{-1} M' = \phi$ or $\gamma^{-1} M' = F$. Thus the relation $Q_1' \subset Q_1$ will follow instantly from Remarks 3, 5, and 6, which show that for each $n > 0$ one has $Q_1(n+1) \subset Q_1(n)$ and, therefore, that $Q_1' = \bigcup_{n>0} Q_1(n)$ is a subfamily of Q_1. The cores of the arguments below are elementary special cases of well-known theorems of Green ([2]) and of Miller and Clifford ([4]) concerning the $\mathscr{D}$-classes and the $\mathscr{H}$-classes of monoids.

To simplify notations, we assume henceforth that $M = \gamma F \in M_1(n+1)$.

Remark 3. To show $\gamma^{-1} M' \in Q_1(n)$ for all subsets M' of M, it suffices to verify the same property for $M' = MmM$, $M' = Mm$, and $M' = mM$, where m is an arbitrary element of M.

Verification. Consider $a_1, a_2, a_3, a_4, b, b' \in M$ and assume that $b = a_1 b' a_2$, $b' = a_3 b$. This implies $b' = a_3 a_1 b' a_2 = (a_3 a_1)^n b' a_2^n$ for all positive n. Since M has only trivial subgroups we can take n so large that $a_2^n = a_2^{n+1}$. Then

$b' = (a_3a_1)^n b' a_2^n = (a_3a_1)^n b' a_2^{n+1} = b' a_2$. From this we conclude that $b = a_1 b' a_2 = a_1 b'$. Assume further that $b' = ba_4$. By a symmetric argument we obtain $b' = a_1 b'$ (and $b = b' a_2$), showing that $b = b'$ under this supplementary condition.

For any $m \in M$, let $W_m = \{m' \in M : m \in M \setminus Mm'M\}$ and $H_m = (mM \setminus W_m) \cap (Mm \setminus W_m)$. It is clear that W_m is a finite union of sets having the form $Mm''M$ and that $\gamma^{-1} H_m \in Q_1(n)$ if the same is true for Mm, mM, and W_m. We show that in fact $H_m = \{m\}$. Indeed, let $m' \in H_m$. We must have $m = a_1 m' a_2$ (since $m' \notin W_m$), $m' = a_3 m$, and $m' = m a_4$ for some elements $a_i \in M$. The computations made above show that $m = m'$, and Remark 3 is verified.

Remark 4. If $m \in M$ is such that W_m has two elements or more, then $A = \gamma^{-1} m$ belongs to $Q_1(n)$.

Verification. Let $\beta : M \to \bar{M}$ be a surjection of M onto a set $\bar{M}$ that has the following properties: for each $m' \in W_m$, $\beta m'$ is a distinguished element 0, of $\bar{M}$; the restriction of β to $M \setminus W_m$ is a bijection of this set onto $\bar{M} \setminus \{0\}$. Since $M . W_m . M = W_m$, we can define a structure of monoid on $\bar{M}$ by letting $(\beta m')(\beta m'') = \beta(m'm'')$ if $m'm'' \in M \setminus W_m$ and $= 0$ if $m'm'' \in W_m$. It is clear that $\bar{M}$ has only trivial subgroups and $\bar{M} \in M_1(n)$ follows from the hypothesis that W_m has two elements or more. Since $A = \{f \in F : \beta\gamma f = \beta m\}$ the remark is verified.

Remark 5. If $m \in M$, $M' = MmM$, and $A = \gamma^{-1} M'$, then $A \in Q_1(n)$.

Verification. Since $\gamma e \in M'$ implies $M' = M$ and $A = F$, we can assume $\gamma e \notin M'$. Let $X' = X \cap \gamma^{-1} m$. We have $F . X' . F \subset A$ and $F . X' . F \in Q_1(1)$. Thus, either $\gamma^{-1} m \subset F . X' . F$ and the result is already proved, or there exists at least one $f \subset \gamma^{-1} m \setminus F . X . F$. We consider this last case. The element f admits at least one minimal factor f'' such that $M . \gamma f'' . M = MmM$, that is, $f = gxf'x'g'$ $(g, f', g' \in F;\ x, x' \in X;\ f'' = xf'x')$, where letting $m_1 = x$, $m' = f'$, $m_2 = x'$, we have $Mm_1m'm_2M = MmM$, $MmM \neq Mm_1m'M$, $MmM \neq Mm'm_2M$. Thus A contains $F . X_1 . A' . X_2 . F$, where $X_1 = X \cap \gamma^{-1} m_1$, $A' = \gamma^{-1} m'$, $X_2 = X \cap \gamma^{-1} m_2$, and, since M is finite, it is clear that $A \setminus F . X' . F$ is a finite union of such sets. Therefore, using Remark 4, the result will follow from the verification that W_m contains at least two distinct elements.

To see this, assume for the sake of contradiction that $m_1 m'$ does not belong to $W_{m'}$, that is, assume that $m' = a_1 m_1 m' a_2$ for some $a_1, a_2 \in M$. According to the computations made at the beginning of the verification of

Remark 3, this implies $m' = a_1 m_1 m'$, hence $Mm'm_2 = Ma_1 m_1 m' m_2 M \subset Mm_1 m' m_2 M = MmM$. Since by construction $Mm_1 m' m_2 M \subset Mm' m_2 M$, this relation is excluded by the hypothesis $MmM \neq Mm' m_2 M$. Thus $m_1 m' \subset W_{m'}$, and by a symmetric argument, $m' m_2 \subset W_{m'}$, are proved. This implies $m_1 m' m_2 \subset w_{m'}$. Since it is clear that $m_1 m' m_2 = m_1 m' = m' m_2$ is impossible, the verification is concluded.

Remark 6. If $m \in M$, $M' = Mm$ or $= mM$ and $A = \gamma^{-1} M'$, then $A \subset Q_1(n)$.

Verification. It suffices to consider the case of $M' = Mm$. Moreover, because of Remark 5, we can assume $Mm \neq MmM$, that is $Mm \neq F$ and $m_0 \subset MmM \setminus Mm$ for at least one $m_0 \in M$.

Let $f \in \gamma^{-1} m$. As above, f has a minimal right factor $f'' = xf' \in A$ ($x \in X$, $f' \in F$), that is, letting $m_1 = \gamma x$, $m' = \gamma f'$, $Mm = Mm_1 m'$ and $Mm \neq Mm'$. We have $F.(X \cap \gamma^{-1} m_1)\ .\gamma^{-1} m' \subset A$ and A is a finite union of such sets. As above, we have only to show that W_m contains at least two elements. That $m \in W_{m'}$ follows from the argument developed in the verification of the last remark, and if $m_0 \in W_m$ we conclude that $W_{m'}$ contains m and m_0. If $m_0 \notin W_m$, we have $m \in Mm_0 M$, hence $MmM = Mm_0 M$ (since $m_0 \in MmM$) and therefore also $m_0 \in W_{m'}$. This concludes the verification.

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