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On an Enumeration Problem*

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ABSTRACT

We present an answer to a question raised by J. Riordan on the relationship between two families of maps of finite sets.

The following problem has been kindly communicated to me by Dr. J. Riordan.

Let $[n] = \{1, 2,..., n\}$ and define B_n as the set of all maps $\beta : [n] \to [n]$ such that there exists a permutation β^* of $[n]$ satisfying the condition:

For $j = 1, 2,..., n$, β^*j is the least integer $\geqslant \beta j$ not already contained in $\{\beta^*1, \beta^*2,..., \beta^*(j-1)\}$.

For instance B_2 consists of the three maps $(\beta 1 = \beta 2 = 1)$, $(\beta 1 = 1; \beta 2 = 2)$, $(\beta 1 = 2; \beta 2 = 1)$, the associated β^* being the identity map for the first two and the inversion $(\beta^* 1 = 2, \beta^* 2 = 1)$ for the last one. More generally one finds that

$$\text{Card } B_n = (n+1)^{n-1}.$$

As it is well known $(n+1)^{n-1}$ is also the cardinality of the set A_n of all acyclic maps $\alpha: [n] \to [n]$ (i.e., of the $\alpha: [n] \to [n]$ such that $\alpha^{n-1} = \alpha^n$), and it is asked to exhibit a 1-1 correspondence $\beta \to \bar{\beta}$ between B_n and A_n. This we do by induction on n, starting with $n = 2$, where we associate, respectively, the three members of B_2 listed above with the following three maps of A_2:

$$(\alpha 1 = \alpha 2 = 1), \quad (\alpha 1 = \alpha 2 = 2), \quad (\alpha 1 = 1; \alpha 2 = 2)$$

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For $n > 2$ we distinguish cases depending upon $\beta^*n = n, = 1$, or $=$ any other member of $[n]$.

CASE 1:

$$\beta^*n = n.$$

Assuming $\beta \in B_n$, the value of β^*n is the only remaining member of $[n]$ once β^*j has been constructed for $j \in 1, 2,..., n-1$. Thus $\beta^*n = n$ implies $\beta j < n$ for every $j \in [n-1]$. Reciprocally, if this condition is met by some map β: $[n] \to [n]$ we can always define the permutation β^* and we shall have $\beta^*n = n$ whatever the value of βn. Thus our hypothesis amounts to the single requirement that the restriction β_1 of β to $[n-1]$ is a member of B_{n-1} and by the induction hypothesis we have a well-defined $\bar{\beta}_1 \in A_{n-1}$ associated with β_1.

SUBCASE 1.1:

$$\beta n = n.$$

We set

$\bar{\beta} n = n$;

$\bar{\beta} j = n$ if $j \in [n-1]$ and $\bar{\beta}_1^{n-2} j = \bar{\beta}_1^{n-1} j$;

$\bar{\beta} j = \bar{\beta}_1 j$ otherwise.

SUBCASE 1.2:

$$\beta n = m < n.$$

We set

$\bar{\beta} n = m$;

$\bar{\beta} j = n$ if $j \in [n-1]$ and $\bar{\beta}_1^{n-2} j = \bar{\beta}_1^{n-1} j \neq \bar{\beta}_1^{n-1} m$;

$\bar{\beta} j = \bar{\beta}_1 j$ otherwise.

It is clear that $\bar{\beta} \in A_n$ because, for every $j \in [n]$, $\bar{\beta}^{n-1} j = \bar{\beta}^n j = n$ in Subcase 1.1 and $\bar{\beta}^{n-1} j = \bar{\beta}^n j = \bar{\beta}^n m$ in Subcase 1.2.

Further, the correspondence $\beta \to \bar{\beta}$ is a 1-1 application of the maps $\beta \in B_n$ satisfying $\beta^*n = n$ onto the maps $\bar{\beta} \in A_n$ having a single fixed point.

CASE 2:

$$\beta^*n = 1.$$

This implies $\beta n = 1$ and $\beta j > 1$ for every $j \in [n-1]$. In fact a map β: $[n] \to [n]$ belongs to B_n and satisfies $\beta^*n = 1$ iff $\beta 1 = 1$ and there exists a map $\beta_2 \in B_{n-1}$ such that $\beta(j+1) = 1 + \beta_2 j$ for every $j \in [n-1]$. Then clearly $\beta^*(j+1) = 1 + \beta_2^* j$.

As above, we derive $\bar{\beta}$ from $\bar{\beta}_2$ by setting simply $\bar{\beta}n = n$ and $\bar{\beta}j = \bar{\beta}_2 j$ for $j \in [n-1]$. Thus $\bar{\beta} \in A_n$ because the restrictions of $\bar{\beta}$ to $[n-1]$ and to $\{n\}$ are two acyclic maps of these sets onto themselves and the correspondence $\beta \to \bar{\beta}$ is a 1-1 application of the maps $\beta \in B_n$ satisfying $\beta^*1 = n$ onto the maps $\bar{\beta} \in A_n$ such that $\bar{\beta}^{-1}n = \{n\}$.

CASE 3:

$$1 < \beta^* n = m < n.$$

We define:

$$I_1 = \{j \in [n-1]: \beta^* j < m\},$$
$$I_2 = \{j \in [n-1]: \beta^* j > m\}.$$

By hypothesis the restriction of β^* to $I_1 \cup I_2 = [n-1]$ is a bijection onto $[n] \backslash \{m\}$ and it implies $\beta j < m$ (resp. $> m$) for every $j \in I_1$ (resp. I_2). More accurately the present hypothesis is equivalent to the existence of th feollowing objects:

(i) a map $\beta_1 \in B_{m-1}$ and a non-decreasing surjection $\lambda_1 : [m-1] \to I_1$ such that $\beta_1 j = \beta \lambda_1 j$ for each $j \in [m-1]$ (then $\beta_1^* j = \beta^* \lambda_1 j$).

(ii) a map $\beta_2 \in B_p (p = n - m)$ and a non-decreasing surjection $\lambda_2 : [p] \to I_2$ such that $m + \beta_2 j = \beta \lambda_2 j$ for each $j \in [p]$ (then $m + \beta_2^* j = \beta^* \lambda_2 j$).

Reciprocally, if this is the case, we have $\beta \in B_n$ (with $\beta^* n = m$, automatically) iff $\beta n \in [m]$.

Thus letting $I_1' = I_1 \cup \{n\}, \lambda_1' j = \lambda_1 j$ or $= m$ depending upon $j \in [m-1]$ or $= m$ and $\beta_1' j = \beta_1 j$ or $= \beta m$ depending on the same condition, we have $\beta_1' \in B_m$ satisfying $\beta_1'^* m = m$ and we can combine the two constructions already introduced in the definition of $\bar{\beta} \in A_n$:

$$\bar{\beta} \lambda_1' j = \lambda_1' \bar{\beta}_1' j \qquad \text{for each} \quad j \in [m];$$

$$\bar{\beta} \lambda_2 j = \lambda_2 \bar{\beta}_2 j \qquad \text{for each} \quad j \in [p].$$

By construction the restriction of $\bar{\beta}$ to I_1' (resp. I_2) is a map of this set into itself and this map is acyclic by the induction hypothesis. Further by the discussion of Case 1, we know that this restriction has a single fixed point, hence that I_1' can be retrieved from $\bar{\beta}$ as being the set of all $j \in [n]$ for which $\bar{\beta}^n j = \bar{\beta}^n n$. This shows the 1-1 character of our application $\beta \to \bar{\beta}$ and it ends the verification of the validity of the construction.

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