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Major Index and Inversion Number of Permutations

By Dominique Foata in Strasbourg
and Marcel-Paul Schützenberger in Paris

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1. Introduction

Consider the fixed finite chain $[n]=\{1<2<\ldots<n\}$. With each mapping s of $[n]$ into itself one associates its *inversion number* INV s defined as the number of pairs (i, j) such that $1\leqq i<j\leqq n$ and $s(i)>s(j)$. One also defines the *down set* of s by

$$\text{DOWN } s=\{i\colon 1\leqq i\leqq n-1,\quad s(i)>s\,(i+1)\}\ ,$$

and the *major index* MAJ s of s as the sum (possibly zero) of the elements in DOWN s.

When S is the set of all mappings s such that the sequence card $s^{-1}(j)$ $(1\leqq j\leqq n)$ has a fixed value, the generating function for the inversion number over S has a remarkably simple form (see [13] chap. 4 and [6] p. 108). Major MacMahon to whom we owe the consideration of the major index ([11] § 104) found that it has the same generating function ([10], [12]). A combinatorial proof of this theorem was obtained in [7]. Further results on these parameters are due to Carlitz ([1], [4]) and Stanley [18].

The case where S is the set of all the $n!$ permutations of $[n]$, enjoys special properties. In the present paper we restrict our attention to that case. We can then speak of the *idown set* of s, denoted by IDOWN s, and defined by

$$\text{IDOWN } s=\text{DOWN } s^{-1},$$

with s^{-1} the inverse of s in the group S. The notions of down and idown sets are classical. Carlitz [3] referred to "*patterns*" and Foulkes [8] to "*up-down*" and "*inversion sequences*". The pattern or updown sequence of s is a sequence of $(n-1)$ plus or minus signs whose i-th term is $+$ or $-$ according as $s(i)$ is greater than $s\,(i+1)$ or not. Our down set is simply the set of all indices i for which the i-th term of the pattern (or up-down sequence) is a plus. Clearly, the integer i of $[n]$ belongs to IDOWN s if and only if there exists a pair (j, k) such that $1\leqq j<k\leqq n$, $s(j)=i+1$ and $s(k)=i$, that is to say, if $i+1$ is to the left of i.

For instance, with

$$s=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\7&4&9&2&6&1&5&8&3\end{pmatrix},$$

one gets the pattern $+-+-+--+$, the down set $\{1, 3, 5, 8\}$, and the idown set $\{1, 3, 5, 6, 8\}$.

Our first result is the following theorem.

Theorem 1. *There exists a bijection* $\varphi: S \to S$ *preserving* IDOWN *and exchanging* INV *and* MAJ. *In other words, one has identically*

$$\text{IDOWN}\ \varphi(s) = \text{IDOWN}\ s \quad \textit{and} \quad \text{INV}\ \varphi(s) = \text{MAJ}\ s\ .$$

Theorem 1 has two corollaries which are easily stated. For each s in S let IMAJ s be the sum of the elements in IDOWN s, just as MAJ s was the sum of the elements in DOWN s.

Corollary 1. *The three pairs of parameters* (MAJ, INV), (IMAJ, INV) *and* (IMAJ, MAJ) *have the same bivariate distribution over the* $n!$ *elements of* S.

As the distribution of the pair (IMAJ, MAJ) is symmetric by definition, the same holds for (MAJ, INV). Professor Chung C. Wang, of the University of Kentucky, has published [19] tables of the distribution of (MAJ, INV) up to $n=7$ and so observed the symmetry. A sharper form of this property is expressed in the next corollary.

Corollary 2. *There exists an involution* ψ *on* S *with the property that* INV $\psi(s) =$ $=$ MAJ s *and* MAJ $\psi(s) =$ INV s *hold identically.*

It may be pointed out that this symmetry is not be observed in general for other sets S of mappings covered by MacMahon's theorem, which only asserts that the marginal distributions of (MAJ, INV) are equal.

Let us now state our second result.

Theorem 2. *There exists an involution* $\boldsymbol{j}$ *of* S *preserving* IDOWN *and exchanging* DOWN *with its complement to* n. *In other words one has identically*

$$\text{IDOWN}\ \boldsymbol{j}s = \text{IDOWN}\ s$$

and

$$\text{DOWN}\ \boldsymbol{j}s = \{n - x : x \in \text{DOWN}\ s\}\ .$$

The two bijections φ and $\boldsymbol{j}$ involved in theorems 1 and 2 were introduced in earlier papers ([7] and [17]). Hereafter their properties will be systematically explored. The construction of $\boldsymbol{j}$ given here involves the Robinson correspondence between permutations and ordered pairs of standard Young tableaux. It would be interesting to find a proof of theorem 2 that could avoid the use of that correspondence. The constructions of φ and φ^{-1} appear in section 2. Theorem 1 and its two corollaries are proved in section 3 that also contains the construction of the involution ψ. The proof of theorem 2 will be completed in section 6.

For each s in S let us define the numerical parameters

$$\text{DES}\ s = \text{card DOWN}\ s \quad \text{and} \quad \text{IDES}\ s = \text{card IDOWN}\ s\ .$$

Of course, DES s is the number of *descents* of s. Its generating function over S is the classical Eulerian polynomial (see [3] or [14] pp. 213–216). The joint distribution of (DES, MAJ) is its q-generalization, as was shown by CARLITZ [4]. Finally, the joint distribution of (DES, IDES) was studied by CARLITZ et al. [5] who obtained explicit formulas in connection with SIMON NEWCOMB's problem. This motivates the second part of our paper in which we examine the symmetries of the joint distribution of all those statistics.

For each s in S let $V(s)$ be the 4-vector (DES s, IDES s, MAJ s, IMAJ s) and for each 4-vector v let $N(v)$ denote the number of s in S for which $V(s)=v$. We suggest that the reader has a look at tables 2 where this function is displayed for $n=3$, 4, 5 and 6, since our theorem 3 is nothing but the proof that the regularities observed there still hold for arbitrary n. The display shows sublocks corresponding to fixed values (a, b) of (DES, IDES). There are symmetries within the whole table, symmetries within each subblock, and symmetries between subblocks. More precisely notice the following facts.

(i) The symmetry along the main diagonal. It results trivially from the definition of (IDES s, IMAJ s) as (DES s^{-1}, MAJ s^{-1}), where $s \to s^{-1}$ is an involution of S. This leads to the identity

$$N(a, b, x, y)=N(b, a, y, x) .$$

(ii) Consider the subblock corresponding to the value (a, b) of (DES, IDES). It has one horizontal (resp. vertical) axis of symmetry, with ordinate (resp. abscissa) MAJ $=na/2$ (resp. IMAJ $=nb/2$). This suggests introducing the notation

$$N'(a, b, x', y') \quad \text{for} \quad N(a, b, x'+(na)/2, y'+(nb)/2) ,$$

that is to say, replacing x and y by their distances $x'=x-(na)/2$, $y'=y-(nb)/2$ to the appropriate "central values" $(na)/2$ and $(nb)/2$. One then obtains that $N'(a, b, x', y')$ depends only on the absolute values of x' and y', i.e.

$$N(a, b, x, y)=N'(a, b, \pm x', \pm y') .$$

(iii) The subblocks corresponding to DES $=a$, IDES $=b$, and to DES $=n-1-a$, IDES $=n-1-b$ are equal. By using the same notations in terms of the "centralized" variables this gives

$$N'(a, b, x', y')=N'(n-1-a, n-1-b, x', y') .$$

These remarks are summarized in the next theorem.

Theorem 3. *The following identities hold*

(i) $N(a, b, x, y)=N(b, a, y, x)$;
(ii) $N(a, b, x, y)=N'(a, b, \pm x', \pm y')$;
(iii) $N'(a, b, x', y')=N'(n-1-a, n-1-b, x', y')$.

By combining these identities one finds that each $v=(a, b, x, y)$ belongs to a set of sixteen $(=2\times4\times2)$ vectors for which N takes the same value. The underlying group G is the direct product of the dihedral group D_4 of order 8 by a group of two elements. In section 4 we describe a version for the dihedral group D_4. Section 5 contains the construction of the group G by means of the ROBINSON correspondence. Finally, the proofs of theorems 2 and 3 are completed in section 6.

2. Construction of the bijection φ

For the construction of φ it will be convenient to regard each permutation s of $[n]$ as an associative monomial or word $w=s(1)\, s(2) \ldots s(n)$ in the n distinct letters $s(1), s(2), \ldots, s(n)$. In the same manner, let $1 \leqq m \leqq n$ and $v=s(1)\, s(2) \ldots s(m)$ be a word with m distinct letters taken out of $[n]$. Denote by $\{t(1)<t(2)<\ldots<t(m)\}$ the increasing chain made of the m elements of the set $\{s(1), s(2), \ldots, s(m)\}$. Then the word v will be regarded as the permutation

$$v: t(i) \rightarrow s(i) \quad (i=1, 2, \ldots, m)$$

of the set $\{t(1), t(2), \ldots, t(m)\}$. Let $p \geqq 1$ and $w_1, w_2, \ldots, w_p$ be p non-empty words. If w is the concatenation product of $w_1, w_2, \ldots, w_p$, in this order, i.e., if $w=w_1 w_2 \ldots w_p$, it is said that

$(w_1, w_2, \ldots, w_p)$ is a *factorization* of w.

Let x be an integer and v a non-empty word. If the last letter of v is greater (resp. smaller) than x, the word v admits a unique factorization

$$(v_1 y_1, v_2 y_2, \ldots, v_p y_p),$$

called its *x-factorization* having the following properties

(i) y_i is a letter satisfying $y_i>x$ (resp. $y_i<x$) for each $i=1, 2, \ldots, p$;

(ii) w_i is a word which is either empty, or has all its letters smaller (resp. greater) than x $(1 \leqq i \leqq p)$.

Put

$$\gamma_x(v)=y_1 v_1 y_2 v_2 \ldots y_p v_p .$$

(Note that $v=v_1 y_1 v_2 y_2 \ldots v_p y_p$.) The bijection φ will be defined by induction on the length of the words. If w has length one, let

$$\varphi(w)=w .$$

If w has length at least two, write $w=vx$ with x its last letter and put

$$\varphi(vx)=\gamma_x(\varphi(v))\, x .$$

In other words, define $\varphi(v)$ by induction, apply γ_x to the word $\varphi(v)$ and put the letter x at the end of the transformed word $\gamma_x(\varphi(v))$.

It was proved in [7] that φ was bijective. It seems convenient for further reference to describe the effective algorithms for both φ and its inverse φ^{-1}.

Algorithm for φ. Let $w=s(1)\, s(2) \ldots s(n)$ be a permutation.

(i) Define $w_1=s(1)$; assume that w_k has been defined for some k with $1 \leqq k<n$, then

(ii) if the last letter of w_k is greater (resp. smaller) than $s\,(k+1)$, split w_k *after* each letter greater (resp. smaller) than $s\,(k+1)$; then

(iii) in each compartment of w_k determined by the splits move the last letter to the beginning; for obtaining w_{k+1} put $s\ (k+1)$ at the end of the transformed word; replace k by $k+1$;
(iv) if $k=n$, then $\varphi(w)=w_k$; if not, return to (ii).

For instance, the image under φ of the word $w=7\ 4\ 9\ 2\ 6\ 1\ 5\ 8\ 3$ is obtained as follows

$$\begin{array}{rl}
w_1 = & 7 \mid \\
w_2 = & 7 \mid 4 \mid \\
w_3 = & 7 \mid 4 \mid 9 \mid \\
w_4 = & 7\ \ 4 \mid 9\ \ 2 \mid \\
w_5 = & 4 \mid 7 \mid 2 \mid 9 \mid 6 \mid \\
w_6 = & 4 \mid 7\ \ 2 \mid 9\ \ 6\ \ 1 \mid \\
w_7 = & 4 \mid 2 \mid 7 \mid 1 \mid 9\ \ 6 \mid 5 \mid \\
w_8 = & 4 \mid 2\ \ 7 \mid 1\ \ 6 \mid 9 \mid 5 \mid 8 \mid \\
\varphi(w)=w_9 = & 4\ \ 7\ \ 2\ \ 6\ \ 1\ \ 9\ \ 5\ \ 8\ \ 3
\end{array}$$

Algorithm for φ^{-1}. Let $v=t(1)\ t(2)\ \ldots\ t(n)$; for getting $w=s(1)\ s(2)\ \ldots\ s(n)=$ $=\varphi^{-1}(v)$ apply the following procedure to v;
(i) put $v_{n-1}=t(1)\ t(2)\ \ldots\ t\ (n-1)$ and $s(n)=t(n)$; assume that the word v_k and the integers $s\ (k+1),\ s\ (k+2), \ldots, s(n)$ have been defined for some k with $1 \leqq k < n$;
(ii) if the first letter of v_k is greater (resp. smaller) than $s\ (k+1)$, split v_k *before* each letter greater (resp. smaller) than $s\ (k+1)$;
(iii) in each compartment of v_k determined by the splits move the first letter to the end; for obtaining v_{k-1} delete the last letter of the transformed word; furthermore, put $s(k)$ equal to that deleted letter;
(iv) if $k=1$ then $\varphi^{-1}(v)=s(1)\ s(2)\ \ldots\ s(n)$; if not, replace k by $k-1$ and return to instruction (ii).
For instance the image of $v=6\ 4\ 9\ 7\ 2\ 5\ 8\ 1\ 3$ under φ^{-1} is

$$\begin{array}{rll}
v_8 = & 6 \mid 4 \mid 9 \mid 7\ \ 2 \mid 5 \mid 8\ \ 1\ .\ 3 & = s(9) \\
v_7 = & 6 \mid 4\ \ 9 \mid 2 \mid 7 \mid 5 \mid 1\ .\ 8 & = s(8) \\
v_6 = & 6 \mid 9 \mid 4 \mid 2 \mid 7 \mid 5\ .\ 1 & = s(7) \\
v_5 = & 6 \mid 9\ \ 4\ \ 2 \mid 7\ .\ 5 & = s(6) \\
v_4 = & 6 \mid 4 \mid 2\ \ 9\ .\ 7 & = s(5) \\
v_3 = & 6 \mid 4 \mid 9\ .\ 2 & = s(4) \\
v_2 = & 6\ \ 4\ .\ 9 & = s(3) \\
v_1 = & 6\ .\ 4 & = s(2) \\
 & 6 & = s(1) \\
w=\varphi^{-1}(v)= & 6\ \ 4\ \ 9\ \ 2\ \ 7\ \ 5\ \ 1\ \ 8\ \ 3 &
\end{array}$$

3. Symmetry of the distribution of the major index and inversion number

In [7] it was proved that φ was bijective and satisfied the identity

$$\text{INV } \varphi(s) = \text{MAJ } s$$

under very general conditions. Thus we only have to verify the further identity

$$\text{IDOWN } \varphi(s) = \text{IDOWN } s\ ,$$

that holds only for permutations. Let us first establish the following lemma.

Lemma 3.1. *Let $m \geqq 1$ and $w = s(1)\ s(2) \dots s\ (m+1)$ be a word with $(m+1)$ distinct letters. Put $v = s(1)\ s(2) \dots s(m)$ and $x = s\ (m+1)$. Then*

(i) $\text{IDOWN } vx = \text{IDOWN } v$ *if* $x = \max \{s(1), s(2), \dots, s\ (m+1)\}$
$= \text{IDOWN } v \cup \{x\}$ *otherwise*;

(ii) $\text{IDOWN } \gamma_x(v) = \text{IDOWN } v$.

Proof. Assertion (i) is straightforward, for x belongs to IDOWN vx if and only if $x+1$ occurs in v, i.e. if x is not the maximum letter of vx.

Let $t = t(1)\ t(2) \dots t\ (m+1)$ be the increasing rearrangement of the word $w = vx$. There so exists a unique integer l with $1 \leqq l \leqq m+1$ and $t(l) = x$. If $l = 1$, i.e. $x = \min \{s(1), s(2), \dots, s\ (m+1)\}$ (resp. $l = m+1$, i.e. $x = \max \{s(1), s(2), \dots, s\ (m+1)\}$), the x-factorization of v is simply $(s(1), s(2), \dots, s(m))$. With the notations of the preceding section the v_i's are empty, $p = m$ and $y_i = s(i)$ for $i = 1, 2, \dots, p$. Hence

$$\gamma_x(v) = v\ ,$$

and

$$\text{IDOWN } \gamma_x(v) = \text{IDOWN } v\ .$$

Assume $2 \leqq l \leqq m$. The integer $t(i)$ $(1 \leqq i \leqq m;\ i \neq l)$ belongs to IDOWN v if and only if $t\ (i+1)$ is to the left of $t(i)$ in v. Note that $t\ (l-1)$ is in neither IDOWN v, nor IDOWN $\gamma_x(v)$. Assume that $1 \leqq i \leqq m$ and $i \neq l-1, l$. If $t(i)$ and $t\ (i+1)$ are letters of two different factors of the x-factorization $(v_1 y_1, v_2 y_2, \dots, v_p y_p)$ of v, say $v_j y_j$ and $v_k y_k$, they are also letters of $y_j v_j$ and $y_k v_k$. Hence $t(i)$ is in IDOWN $\gamma_x(v)$ if and only if $t(i)$ belongs to IDOWN v. If $t(i)$ and $t(i+1)$ are letters of the same factor, say $v_j y_j$, of the x-factorization of v, neither one can be the letter y_j, for either $1 \leqq t(i) < t\ (i+1) < x$, or $x < t(i) < t(i+1) \leqq m+1$ must hold. Thus the mutual order of $t(i)$ and $t(i+1)$ remains the same in both v and $\gamma_x(v)$,

q. e. d.

The proof of theorem 1 is completed as follows. Let $w = vx$ be a word with final letter x. Then

$$\begin{aligned}\text{IDOWN } \varphi(w) &= \text{IDOWN } \varphi(vx) = \text{IDOWN } \gamma_x(\varphi(v))\ x \text{ (by definition of } \varphi)\\ &= \text{IDOWN } \gamma_x(\varphi(v)) \text{ or IDOWN } \gamma_x(\varphi(v)) \cup \{x\} \text{ (by lemma1 (i))}\\ &= \text{IDOWN } \varphi(v) \text{ or IDOWN } \varphi(v) \cup \{x\} \text{ (by lemma 1(ii))}\\ &= \text{IDOWN } v \text{ or IDOWN } v \cup \{x\} \text{ (by induction)},\end{aligned}$$

according as x is the maximum letter of vx or not.

Thus

$$\begin{aligned}\text{IDOWN}\ \varphi(w) &= \text{IDOWN}\ vx\ \text{(by lemma 1}(i))\\ &= \text{IDOWN}\ w,\end{aligned}$$

q. e. d.

Let us turn our attention to the two corollaries of theorem 1. Denote by $\boldsymbol{i}$ the involution of S that maps each s in S onto its inverse $s^{-1}=\boldsymbol{i}s$. By the very definition of INV one has

(1) $\quad \text{INV}\ \boldsymbol{i}s = \text{INV}\ s\ .$

On the other hand, as IMAJ $s=$ card IDOWN s, theorem 1 implies that

(2) $\quad \text{IMAJ}\ \varphi(s) = \text{IMAJ}\ s\ .$

Consider the sequence

(3) $\quad s \xrightarrow{\ \boldsymbol{i}\ } s_1 \xrightarrow{\ \varphi^{-1}\ } s_2 \xrightarrow{\ \boldsymbol{i}\ } s_3 \xrightarrow{\ \varphi\ } s_4 \xrightarrow{\ \boldsymbol{i}\ } s_5\ .$

From theorem 1, (1) and (2) it follows that

$$\text{MAJ}\ s = \text{IMAJ}\ s_1 = \text{IMAJ}\ s_2 = \text{MAJ}\ s_3 = \text{INV}\ s_4 = \text{INV}\ s_5$$

$$\text{INV}\ s = \text{INV}\ s_1 = \text{MAJ}\ s_2 = \text{IMAJ}\ s_3 = \text{IMAJ}\ s_4 = \text{MAJ}\ s_5\ .$$

As every mapping occurring in (3) is bijective, the pairs (MAJ, INV), (IMAJ, INV) and (IMAJ, MAJ) are identically distributed. This proves corollary 1.

Next form the composition product $\psi=\boldsymbol{i}\varphi\boldsymbol{i}\varphi^{-1}\boldsymbol{i}$ that maps s onto s_5, as shown in (3). Direct computation shows that $\psi\psi$ is the identity map. Thus ψ is an involution of S. Furthermore

$$\text{MAJ}\ s = \text{INV}\ s_5 = \text{INV}\ \psi(s)\ .$$

$$\text{INV}\ s = \text{MAJ}\ s_5 = \text{MAJ}\ \psi(s)\ .$$

This establishes corollary 2.

4. The dihedral group D_4

Denote by $\sum$ the group of all the permutations of S. Three elements of $\sum$ are now defined. First $\boldsymbol{i}$ is the *inverse* operation already introduced

$$\boldsymbol{i}: s \to s^{-1}\ .$$

Second $\boldsymbol{c}$ is the *complement* to $(n+1)$. If $s=s(1)\,s(2)\dots s(n)$, then

$$\boldsymbol{c}s = (n+1-s(1))\,(n+1-s(2))\dots(n+1-s(n))\ .$$

Finally, $\boldsymbol{r}$ sends each $s=s(1)\,s(2)\dots s(n)$ onto its *reversal* $\boldsymbol{r}s=s(n)\dots s(2)\,s(1)$. Direct computation shows that $\boldsymbol{r}=\boldsymbol{ici}$. The next property is stated for the sake of completeness.

Property 4.1. *The subgroup of* $\sum$ *generated by* $\{\boldsymbol{i}, \boldsymbol{c}\}$ *is isomorphic to the dihedral group* D_4 *of order* 8.

Proof. Consider the product $[n]\times[n]$, regarded as a square with the four vertices $(1, 1)$, $(1, n)$, (n, n), $(n, 1)$. Let Γ be the graph of a permutation s. It consists of a set of n points $(1, s(1))$, $(2, s(2)), \ldots, (n, s(n))$ contained in the square. When the reflection about the horizontal axis of ordinate $(n+1)/2$ (resp. about the major diagonal) is performed, the graph Γ is transformed into the graph of the permutation $\boldsymbol{c}s$ (resp. $\boldsymbol{i}s$). As those two reflections generate all the symmetries of the square and the correspondence between graphs and mappings is one-to-one, the proof of the lemma is completed, q. e. d.

Note that the following relations hold $\boldsymbol{rc}=\boldsymbol{cr}$, $\boldsymbol{ir}=\boldsymbol{ci}$, $\boldsymbol{irc}=\boldsymbol{rci}$.

Property 4.2. *For each s in S one has*

$$\mathrm{DOWN}\ \boldsymbol{c}s=[n-1]\backslash \mathrm{DOWN}\ s;$$

$$\mathrm{DOWN}\ \boldsymbol{rc}s=n-\mathrm{DOWN}\ s=\{n-i: i\in \mathrm{DOWN}\ s\}\ .$$

Proof. Let $s=s(1)\,s(2)\ldots s(n)$, $\boldsymbol{c}s=s'(1)\,s'(2)\ldots s'(n)$ and $\boldsymbol{rc}s=s''(1)\,s''(2)\ldots s''(n)$, where by definition $s'(j)=n+1-s(j)$ and $s''(j)=n+1-s\,(n+1-j)$. Suppose j in DOWN s. This is equivalent with $j\in[n-1]$ and $s(j)>s\,(j+1)$, hence with $s'(j)<s'\,(j+1)$ and with $s''(j'')<s''\,(j''+1)$ where $j''=n+1-j-1$. It follows immediately that j belongs to DOWN s if and only if j belongs to $[n-1]$ and, in equivalent fashion, $j\notin$DOWN $\boldsymbol{r}s$ or $n-j\in$DOWN $\boldsymbol{rc}s$,

q. e. d.

5. The Robinson correspondence

In what follows we have to rely upon the Robinson correspondence, that establishes a bijection between our set S and a new set, say $\mathfrak{T}^{(2)}$, of the pairs of standard Young tableaux of the same shape. The reader is referred to the excellent exposition of the relevant material given in ([9], pp. 48–72) by Donald E. Knuth, of Stanford University. However our treatment will be axiomatic in the sense that nothing will be used that is not stated in the following theorem.

Theorem 4. *There exists a surjection* ROB: $S\to\mathfrak{T}$ *onto a set* $\mathfrak{T}$ *having the following properties*

(i) $s\to$(ROB s, ROB $\boldsymbol{i}s$) *is injective*;

(ii) *if* $s, s'\in S$ *and* ROB $s=$ROB s',
then ROB $\boldsymbol{r}s=$ROB $\boldsymbol{r}s'$ *and* ROB $\boldsymbol{c}s=$ROB $\boldsymbol{c}s'$;

(iii) *if* $s, s'\in S$ *and* ROB $\boldsymbol{i}s=$ROB $\boldsymbol{i}s'$, *then* DOWN $s=$DOWN s';

(iv) *for each s in S there exists an element s' of S satisfying*

$$(\mathrm{ROB}\ s', \mathrm{ROB}\ \boldsymbol{i}s')=(\mathrm{ROB}\ \boldsymbol{r}s, \mathrm{ROB}\ \boldsymbol{ri}s)\ .$$

Of course, theorem 4 does not say the full truth: $\mathfrak{T}$ is indeed the set of all standard Young tableaux of order n. On $\mathfrak{T}$ there is the equivalence "to have the same shape", which is such that the mapping $s\to$(ROB s, ROB $\boldsymbol{i}s$) is bijective upon the pairs of equivalent tableaux. Furthermore, the operation $P \to P^T$ below is the *transposition*. The algorithm called "S" by KNUTH ([9], pp. 57–59) transforms each standard YOUNG tableau P into a tableau P'. Replacing each integer i in P' by $n+1-i$ yields a new tableau denoted by P^J. The transposed tableau of P^J is precisely P^F that is further introduced. The fundamental discovery that there exists a surjection ROB: $S \to \mathfrak{T}$ having property (ii) was made by ROBINSON [15]. SCHENSTED [16] proved the part of the above property concerning T, namely the first part of (ii). The remaining proofs were given in [17]. A numerical example is given at the end of section 6.

As ROB is surjective, each element of $\mathfrak{T}$ can be written as ROB s with s in S. From (ii) it follows that we may define the two mappings $P \to P^T$ and $P \to P^F$ of $\mathfrak{T}$ into itself by

$$(\mathrm{ROB}\, s)^T = \mathrm{ROB}\, \boldsymbol{r}s \quad \text{and} \quad (\mathrm{ROB}\, s)^F = \mathrm{ROB}\, \boldsymbol{c}s\,.$$

Property 5.1. *The operations T and F are involutions of $\mathfrak{T}$ that commute with each other, i.e.*

$$T^2 = F^2 = 1 \quad and \quad TF = FT\,.$$

Proof. From $\boldsymbol{r}^2 = 1$ we deduce that

$$\mathrm{ROB}\, s = \mathrm{ROB}\, \boldsymbol{r}^2 s = (\mathrm{ROB}\, \boldsymbol{r}s)^T = (\mathrm{ROB}\, s)^{T^2}.$$

Thus $T^2 = 1$. In the same manner

$$\mathrm{ROB}\, s = \mathrm{ROB}\, \boldsymbol{c}^2 s = (\mathrm{ROB}\, \boldsymbol{c}s)^F = (\mathrm{ROB}\, s)^{F^2},$$

showing that $F^2 = 1$. Finally, from $\boldsymbol{cr} = \boldsymbol{rc}$ we get

$$\begin{aligned}(\mathrm{ROB}\, s)^{TF} &= ((\mathrm{ROB}\, s)^T)^F = (\mathrm{ROB}\, \boldsymbol{r}s)^F = \mathrm{ROB}\, \boldsymbol{cr}s \\ &= \mathrm{ROB}\, \boldsymbol{rc}s = (\mathrm{ROB}\, \boldsymbol{c}s)^T = ((\mathrm{ROB}\, s)^F)^T = (\mathrm{ROB}\, s)^{FT}.\end{aligned}$$

Thus $TF = FT$, q.e.d.

Next put $J = FT$. Clearly J is involutive and commutes with T. Let $\mathfrak{T}^{(2)}$ be the set of all ordered pairs (ROB s, ROB $\boldsymbol{i}s$) where s runs over all of S.

Property 5.2. *If (P, Q) belongs to $\mathfrak{T}^{(2)}$ then the following three pairs*

$$(Q, P), \quad (P, Q^J), \quad (P^T, Q^T)$$

also belong to $\mathfrak{T}^{(2)}$.

Proof. Let s be the element of S with $= (P, Q) = (\mathrm{ROB}\, s, \mathrm{ROB}\, \boldsymbol{i}s)$. Then $(Q, P) = (\mathrm{ROB}\, \boldsymbol{i}s, \mathrm{ROB}\, \boldsymbol{ii}s)$ also belongs to $\mathfrak{T}^{(2)}$ according to theorem 4 (i). Next consider the pair (ROB s, (ROB $\boldsymbol{i}s)^J$). As $\boldsymbol{rci} = \boldsymbol{rir}$, we get

$$(\mathrm{ROB}\, \boldsymbol{i}s)^J = (\mathrm{ROB}\, \boldsymbol{i}s)^{FT} = \mathrm{ROB}\, \boldsymbol{rci}s = \mathrm{ROB}\, \boldsymbol{rir}s\,.$$

Hence (ROB s, (ROB $\boldsymbol{i}s)^J$) = (ROB $\boldsymbol{rr}s$, ROB $\boldsymbol{rir}s$). From theorem 4 (iv) there exists an element s' of S with the property that

$$(\mathrm{ROB}\, \boldsymbol{rr}s, \mathrm{ROB}\, \boldsymbol{rir}s) = (\mathrm{ROB}\, s', \mathrm{ROB}\, \boldsymbol{i}s')\,,$$

that is, (ROB s, (ROB $\boldsymbol{i}s)^J$) belongs to $\mathfrak{T}^{(2)}$. Finally $(P^T, Q^T) = (\mathrm{ROB}\, \boldsymbol{r}s, \mathrm{ROB}\, \boldsymbol{ri}s)$ is also in $\mathfrak{T}^{(2)}$ according to theorem 4 (iv), q.e.d.

We can then define the following operations on $\mathfrak{T}^{(2)}$

$$\boldsymbol{i}'(P,Q)=(Q,P);\quad \boldsymbol{j}'(P,Q)=(P,Q^J);\quad \boldsymbol{t}'(P,Q)=(P^T,Q^T)\,.$$

Let G' be the subgroup of the permutation group acting on $\mathfrak{T}^{(2)}$ that is generated by $\{\boldsymbol{i}',\boldsymbol{j}',\boldsymbol{t}'\}$. The relations

$$\boldsymbol{i}'^2=\boldsymbol{j}'^2=\boldsymbol{t}'^2=(\boldsymbol{i}'\boldsymbol{j}')^4=1,\quad \boldsymbol{i}'\boldsymbol{t}'=\boldsymbol{t}'\boldsymbol{i}',\quad \boldsymbol{j}'\boldsymbol{t}'=\boldsymbol{t}'\boldsymbol{j}'$$

follow immediately from the above definition for $\boldsymbol{i}',\boldsymbol{j}',\boldsymbol{t}'$ and property 5.1. The Cayley diagram of the group G' is shown in figure 1.

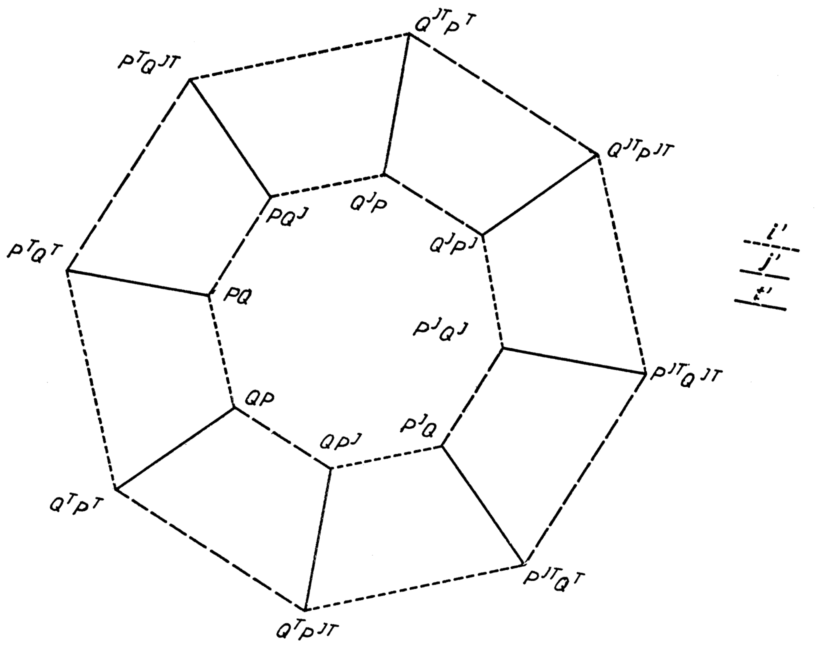

fig. 1

Clearly G' is the direct product of the dihedral group D_4 (generated by $\{\boldsymbol{i}',\boldsymbol{j}'\}$) by the group of two elements $\{1,\boldsymbol{t}'\}$.

From theorem 4 (iii) it also follows that we may define a mapping $\triangle$ of $\mathfrak{T}$ into $2^{[n-1]}$ by

$$\varDelta(\mathrm{ROB}\ \boldsymbol{i}s)=\mathrm{DOWN}\ s\,.$$

Hence, we get

$$\varDelta(\mathrm{ROB}\ s)=\mathrm{IDOWN}\ s\,.$$

Property 5.3. *For each Q in $\mathfrak{T}$ one has*

$$\varDelta(Q^J)=n-\varDelta(Q)$$

$$\varDelta(Q^T)=[n-1]\backslash\varDelta(Q)\,.$$

Proof. Let s be such that ROB $\boldsymbol{i}s = Q$. Then $(\text{ROB}\,\boldsymbol{i}s)^J = (\text{ROB}\,\boldsymbol{i}s)^{FT} = \text{ROB}\,\boldsymbol{rci}s = \text{ROB}\,\boldsymbol{irc}s$. Hence $\Delta(Q^J) = \text{DOWN}\,\boldsymbol{rc}s = n - \text{DOWN}\,s$ according to property 4.2. Thus $\Delta(Q^J) = n - \Delta(Q)$.

In the same manner

$$(\text{ROB}\,\boldsymbol{i}s)^T = \text{ROB}\,\boldsymbol{ri}s = \text{ROB}\,\boldsymbol{ic}s\ .$$

Again, from property 4.2

$$\begin{aligned}\Delta(Q^T) &= \Delta(\text{ROB}\,\boldsymbol{ic}s) = \text{DOWN}\,\boldsymbol{c}s = [n-1]\backslash\text{DOWN}\,s\\ &= [n-1]\backslash\Delta(Q)\,,\end{aligned}$$

q.e.d.

6. Proofs of theorems 2 and 3

From theorem (i) and the very definition of $\mathfrak{T}^{(2)}$ the mapping

$$\varrho: s \to (\text{ROB}\,s, \text{ROB}\,\boldsymbol{i}s)$$

is a bijection of S onto $\mathfrak{T}^{(2)}$. Let

$$\boldsymbol{j} = \varrho^{-1}\boldsymbol{j}'\varrho \quad \text{and} \quad \boldsymbol{t} = \varrho^{-1}\boldsymbol{t}'\varrho\ .$$

As $\boldsymbol{i} = \varrho^{-1}\boldsymbol{i}'\varrho$, we see that the subgroup G of Σ generated by $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{t}\}$ is isomorphic to G'. In particular, the following relations hold

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{t}^2 = (\boldsymbol{ij})^4 = 1\,, \quad \boldsymbol{it} = \boldsymbol{ti}\,, \quad \boldsymbol{jt} = \boldsymbol{tj}\ .$$

Also the group G contains the dihedral group D_4 generated by $\{\boldsymbol{i}, \boldsymbol{c}\}$, since we can easily verify the following relations

$$\boldsymbol{r} = \boldsymbol{tj} = \boldsymbol{jt} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{ijti}\ .$$

In fact, if $\varrho(s) = (P, Q)$, we have the relations

$$\varrho(\boldsymbol{r}s) = (P^T, Q^{JT}) \quad \text{and} \quad \varrho(\boldsymbol{c}s) = (P^{JT}, Q^T)\ .$$

Let us now complete the proof of theorem 2. With s in S we get

$$\varrho(\boldsymbol{j}s) = (\text{ROB}\,s, (\text{ROB}\,\boldsymbol{i}s)^J)\ .$$

Hence

$$\text{IDOWN}\,\boldsymbol{j}s = \Delta(\text{ROB}\,\boldsymbol{j}s) = \Delta(\text{ROB}\,s) = \text{IDOWN}\,s\ .$$

Thus the involution $\boldsymbol{j}$ preserves IDOWN. Furthermore

$$\text{DOWN}\,\boldsymbol{j}s = \Delta(\text{ROB}\,\boldsymbol{ij}s) = \Delta((\text{ROB}\,\boldsymbol{i}s)^J)\ .$$

From property 5.3 it then follows that

$$\text{DOWN}\,\boldsymbol{j}s = n - \Delta(\text{ROB}\,\boldsymbol{i}s) = n - \text{DOWN}\,s\ .$$

This completes the proof of theorem 2.

Property 6.1. *For each s in S the following identities hold*

$$\begin{aligned}&\text{DOWN}\,\boldsymbol{t}s = [n-1]\backslash\text{DOWN}\,s\\ &\text{IDOWN}\,\boldsymbol{t}s = [n-1]\backslash\text{IDOWN}\,s\ .\end{aligned}$$

Proof. Again property 5.3 implies that

$$\text{DOWN } \boldsymbol{t}s = \Delta(\text{ROB } \boldsymbol{it}s) = \Delta((\text{ROB } \boldsymbol{i}s)^T = [n-1] \backslash \Delta(\text{ROB } \boldsymbol{i}s)$$
$$= [n-1] \backslash \text{DOWN } s\,.$$

Also IDOWN $\boldsymbol{t}s$ = DOWN $\boldsymbol{it}s$ = DOWN $\boldsymbol{ti}s = [n-1]\backslash$DOWN $\boldsymbol{i}s = [n-1]\backslash$IDOWN s q.e.d

We are now ready to prove theorem 3. Recall that $N(a, b, x, y)$ is the set of all s in S with DES $s=a$, IDES $s=b$, MAJ $s=x$, IMAJ $s=y$. Clearly the involution $\boldsymbol{i}\colon s \to s^{-1}$ of S maps in a one-to-one manner each set

$$\{s \in S\colon \text{DES } s=a, \text{IDES } s=b, \text{MAJ } s=x, \text{IMAJ } s=y\}$$

onto the set

$$\{s \in S\colon \text{DES } s=b, \text{IDES } s=a, \text{MAJ } s=y, \text{IMAJ } s=x\}\,.$$

This proves the first identity $N(a, b, x, y) = N(b, a, y, x)$.

Now remember that DES s (resp. IDES s) is the number of elements in DOWN s, while MAJ s (resp. IMAJ s) is the sum of the elements in DOWN s (resp. IDOWN s). It then follows from theorem 2 that

$$\text{IDES } \boldsymbol{j}s = \text{IDES } s \quad \text{and} \quad \text{IMAJ } \boldsymbol{j}s = \text{IMAJ } s\,.$$

Also

$$\text{DES } \boldsymbol{j}s = \text{DES } s$$

and

$$\text{MAJ } \boldsymbol{j}s = \Sigma\,\{n-x\colon x \in \text{DOWN } s\} = n \text{ DES } s - \text{MAJ } s\,.$$

Thus the involution $\boldsymbol{j}$ maps each set

$$\{s \in S\colon \text{DES } s=a, \text{IDES } s=b, \text{MAJ } s=x, \text{IMAJ } s=y\}$$

onto

$$\{s \in S\colon \text{DES } s=a, \text{IDES } s=b, \text{MAJ } s=na-x, \text{IMAJ } s=y\}\,,$$

which establishes the identity

$$N(a, b, x, y) = N(a, b, na-x, y).$$

Hence

$$N'(a, b, x', y') = N'(a, b, -x', y').$$

Combining with the first identity of theorem 3 gives

$$N'(a, b, x', y') = N'(a, b, \pm x', \pm y').$$

The last identity of theorem 3 is a consequence of property 6.2. We have

$$\text{DES } \boldsymbol{t}s = n-1-\text{DES } s; \quad \text{IDES } \boldsymbol{t}s = n-1-\text{IDES } s\,.$$

Also, as the sum of the elements in $[n-1]$ is $n\,(n-1)/2$ we deduce

$$\text{MAJ } \boldsymbol{t}s = n\,(n-1)/2 - \text{MAJ } s \quad \text{and} \quad \text{IMAJ } \boldsymbol{t}s = n\,(n-1)/2 - \text{IMAJ } s\,.$$

Thus the identity

$$N(a, b, x, y) = N(n-1-a,\ n-1-b,\ n\,(n-1)/2-x,\ n\,(n-1)/2-y)$$

holds, as well as the identities

$$N'(a, b, x', y') = N'(n-1-a,\ n-1-b,\ -x',\ -y')$$

and

$$N'(a, b, x', y') = N'(n-1-a,\ n-1-b,\ x',\ y')$$

because of theorem 3 (ii).

Example 6.2, Consider the two standard YOUNG tableaux of order 5

$$P = \begin{array}{|c|c|c|} \hline 3 & 4 & \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array}$$

As mentioned in the beginning of section 5 the two tableaux P^J and Q^J are obtained by first applying algorithm "S" (as described in [9], pp. 57–59) to P and Q, then replacing each integer i by $6-i$:

$$P^J = \begin{array}{|c|c|c|} \hline 4 & 5 & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad Q^J = \begin{array}{|c|c|c|} \hline 3 & 5 & \\ \hline 1 & 2 & 4 \\ \hline \end{array}$$

Hence

$$P^T = \begin{array}{|c|c|} \hline 5 & \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \qquad Q^T = \begin{array}{|c|c|} \hline 5 & \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \qquad P^{JT} = \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array} \qquad Q^{JT} = \begin{array}{|c|c|} \hline 4 & \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array}$$

When the group G' acts on the above pair (P, Q) we get the sixteen pairs of tableaux of figure 1. Each of these pairs is associated under the inverse ϱ^{-1} of the Robinson correspondence (see [9], p. 52) with a permutation of $\{1, 2, 3, 4, 5\}$, as shown in the next table.

Table 1.

tableaux	permutations	tableaux	permutations
$PQ = \varrho(s)$	$s = 3\ 1\ 4\ 2\ 5$	P^TQ^T	$ts = 2\ 5\ 1\ 4\ 3$
PQ^J	$js = 3\ 4\ 1\ 5\ 2$	P^TQ^{JT}	$rs = 5\ 2\ 4\ 1\ 3$
P^JQ	4 1 5 2 3	$P^{JT}Q^T$	$cs = 3\ 5\ 2\ 4\ 1$
P^JQ^J	1 4 2 5 3	$P^{JT}Q^{JT}$	3 2 5 1 4
QP	$is = 2\ 4\ 1\ 3\ 5$	Q^TP^T	3 1 5 4 2
Q^JP	3 5 1 2 4	$Q^{JT}P^T$	4 2 5 3 1
QP^J	2 4 5 1 3	Q^TP^{JT}	5 3 1 4 2
Q^JP^J	1 3 5 2 4	$Q^{JT}P^{JT}$	4 2 1 5 3

Note that DOWN $s = \{1, 3\}$

$$\text{DOWN}\, js = \{2, 4\} = 5 - \{1, 3\}$$

and

$$\text{IDOWN}\ s = \text{IDOWN}\, js = \{2\}\,.$$

Tables 2 show the distribution of the vector $V = (\mathrm{DES}, \mathrm{MAJ}, \mathrm{IDES}, \mathrm{IMAJ})$ over the $n!$ permutations of $[n]$ for $n = 3, 4, 5, 6$.

Note that the last two columns show the q-EULERian numbers $A_{n,k}(q)$ (see [1] p. 336) and the EULERian numbers $A_{n,k}$.

Tables 2.

DES ↓	IDES→ / IMAJ→ / MAJ ↓	0 / 0	1 / 12	2 / 3	$n=3$: $A_{3,k}(q)$	$A_{3,k}$
0	0	1			1	1
1	1		11		2	
	2		11		2	4
2				1	1	1

DES ↓	IDES→ / IMAJ→ / MAJ ↓	0 / 0	1 / 123	2 / 345	3 / 6	$n=4$: $A_{4,k}(q)$	$A_{4,k}$
0	0	1				1	1
	1		111			3	
1	2		121	1		5	11
	3		111			3	
	3			111		3	
2	4		1	121		5	11
	5			111		3	
3	6				1	1	1

DES ↓	IDES→ / IMAJ→ / MAJ ↓	0 / 0	1 / 1234	2 / 34567	3 / 6789	4 / 10	$n=5$: $A_{5,k}(q)$	$A_{5,k}$
0	0	1					1	1
	1		1111				4	
1	2		1221	111			9	26
	3		1221	111			9	
	4		1111				4	
	3			11211			6	
	4		11	13431	11		16	
2	5		11	24642	11		22	66
	6		11	13431	11		16	
	7			11211			6	
	6				1111		4	
	7			111	1221		9	
3	8			111	1221		9	26
	9				1111		4	
4	10					1	1	1

	IDES→	0	1					2							3							4					5	$n=6$	
DES ↓	IMAJ→ MAJ ↓	0	1	2	3	4	5	3	4	5	6	7	8	9	6	7	8	9	10	11	12	10	11	12	13	14	15	$A_{6,k}(q)$	$A_{6,k}$
0	0	1																										1	1
1	1		1	1	1	1	1																					5	
	2		1	2	2	2	1		1	1	2	1	1															14	
	3		1	2	3	2	1		1	2	3	2	1					1										19	57
	4		1	2	2	2	1		1	1	2	1	1															14	
	5		1	1	1	1	1																					5	
2	3							1	1	2	2	2	1	1														10	
	4			1	1	1		1	3	5	6	5	3	1		1	2	2	2	1								35	
	5			1	2	1		2	5	10	11	10	5	2		2	4	5	4	2								66	
	6			2	3	2		2	6	11	14	11	6	2		2	5	6	5	2				1				80	302
	7			1	2	1		2	5	10	11	10	5	2		2	4	5	4	2								66	
	8			1	1	1		1	3	5	6	5	3	1		1	2	2	2	1								35	
	9							1	1	2	2	2	1	1														10	
3	6														1	1	2	2	2	1	1							10	
	7								1	2	2	2	1		1	3	5	6	5	3	1		1	1	1			35	
	8								2	4	5	4	2		2	5	10	11	10	5	2		1	2	1			66	
	9				1				2	5	6	5	2		2	6	11	14	11	6	2		2	3	2			80	302
4	10								2	4	5	4	2		2	5	10	11	10	5	2		1	2	1			66	
	11								1	2	2	2	1		1	3	5	6	5	3	1		1	1	1			35	
	12														1	1	2	2	2	1	1							10	
	10																					1	1	1	1	1		5	
	11															1	1	2	1	1		1	2	2	2	1		14	
	12										1					1	2	3	2	1		1	2	3	2	1		19	57
	13															1	1	2	1	1		1	2	2	2	1		14	
	14																					1	1	1	1	1		5	
5	15																										1		1

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Note added in proof. Since the paper has been submitted for publication, several results related to the distribution of the five-vector (DES, IDES, MAJ, IMAJ, INV) have been published. Stanley [20] found the bivariate generating function for the pair (DES, INV), that appears to be a second q-analog for the Eulerian numbers, the first one being the generating function for (DES, MAJ) obtained by Carlitz ([1], [4]). Then, Gessel [21] developed an original combinatorial theory of q-series, that enabled him to get the three-variate distribution for (DES, MAJ, INV). On the other hand, by extending the results of the present paper Foata [22] showed that the ten marginal bivariate distributions of the above five-vector were known and reduced to four different analytical expressions. Finally, Garsia [23] has investigated the relations between the two q-analogs of the Eulerian numbers and obtained new formulas for those two q-extensions.

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Département de Mathématique
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg, France

Département de Mathématique
Université de Paris VII
2, Place Jussieu
75005 Paris, France