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Note

A Conjecture on Sets of Differences of Integer Pairs

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We propose a conjecture on the set of differences of integer pairs taken out of a sufficiently dense subset of the plane.

Consider a finite set X of pairs of positive integers (i, j), with $i, j \geqslant 1$, and let $d = \max\{i + j \mid (i, j) \in X\}$.

Associate to X the vertex set $V = \{0, 1, ..., d-2\}$ and construct on V a directed graph G having a directed edge from v to w whenever there exist two distinct pairs (i, j) and (i', j') in X such that

$$v = |i - i'|, \qquad w = |j - j'|.$$

The following conjecture is suggested by a problem in coding theory:

CONJECTURE. *If* $\operatorname{Card}(X) \geqslant d$, *the graph* G *has a circuit from* 0 *to* 0.

The bound d is certainly the best one since, for $X = \{(i, d-i) \mid 1 \leqslant i \leqslant d-1\}$, the graph G is reduced to the loops (x, x), for $x = 1, 2, ..., d-2$.

The following result conforts the conjecture:

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PROPOSITION 1. *Let H be the graph obtained by adding to G the edges*

$$(i, d-i-1)$$

for all i in $V\backslash\{0\}$.

If $\mathrm{Card}(X) \geqslant d$, *then H has a cycle from O to O.*

Proof. Consider the set X^r of all sequences of r elements of X; if $c = \mathrm{Card}(X)$, then $\mathrm{Card}(X^r) = c^r$.

Now, to each $y = (y_1, ..., y_r)$ in X^r, we associate the sequence $s(y)$ of residuals mod $(d-1)$ of the numbers

$$j_k + i_{k+1}, \qquad k = 0, 1, ..., r,$$

for $y_k = (i_k, j_k)$ and $j_0 = i_{r+1} = 0$.

Let us suppose that $c \geqslant d$; then for all large enough r, one has

$$c^r > (d-1)^{r+1}.$$

One may then find at least two elements $y, z \in X^r$ such that

$$s(y) = s(z).$$

This provides a circuit from 0 to 0 using successively the edges

$$e_k = (v_k, w_k), \qquad k = 1, 2, ..., r,$$

where

$$v_k = i_k - m_k, \qquad w_k = j_k - n_k$$

and

$$y_k = (i_k, j_k), \qquad z_k = (m_k, n_k).$$

In fact, either $j_k + i_{k+1} = n_k + m_{k+1}$ and $w_k = v_{k+1}$, or

$$j_k + i_{k+1} = n_k + m_{k+1} \pm (d-1)$$

and w_k may be connected to v_{k+1} using one of the edges added to G. ∎

We also prove the following result which is closely related to the conjecture:

PROPOSITION 2. *Suppose that the projections of the set X on the two components are both equal to the set* $\{1, 2, ..., e\}$ *then the graph G has a circuit from* 0 *to* 0 *whenever* $\mathrm{Card}(X) \geqslant e + 1$.

Proof. If X satisfies the hypothesis and $\text{Card}(X) \geqslant e + 1$, we may find a permutation σ of the set $\{1, 2, \ldots, e\}$ such that X contains all the elements

$$(i, i\sigma), \qquad i = 1, 2, \ldots, e$$

and an extra element (m, n) with $n \neq m\sigma$.

If r is the order of the permutation τ defined by

$$i\tau = e - i\sigma + 1, \qquad i = 1, 2, \ldots, e,$$

then the two sequences of elements of X

$$y_k = (m\tau^k, m\tau^k\sigma), \qquad k = 0, \ldots, r - 1; \quad y_r = (m, n),$$
$$z_k = (n\sigma^{-1}\tau^k, n\sigma^{-1}\tau^k\sigma), \qquad k = 1, \ldots, r; \quad z_0 = (m, n)$$

provide a circuit from 0 to 0 by taking their differences, since for $k = 1, 2, \ldots, r - 1$, one has

$$m\tau^k\sigma + m\tau^{K+1} = n\sigma^{-1}\tau^k\sigma + n\sigma^{-1}\tau^{k+1} = e + 1. \quad \blacksquare$$

In the last proposition, it is not possible to suppose only that the projection of X on the first component is equal to $\{1, 2, \ldots, e\}$ and the projection on the second one is included in it. In fact, for

$$X = \{(1, 1), (2, 1), (2, 3), (3, 3)\}$$

the graph G reduces to $G = \{(0, 2), (1, 2), (1, 0)\}$, which has no circuit from 0 to 0.

NOTE

G. Hansel (private communication) showed that the number of distinct sequences $s'(y)$ taken as in the proof of Proposition 1, but in $\mathbb{N}$ instead of $\mathbb{Z}/(d-1)$, is a polynomial of degree $r + 1$ in d whose leading term is

$$[(d-1)/2^{1/2}]^{r+1}.$$

This entails the validity of the conjecture under the stronger hypothesis $\text{Card}(X) \geqslant (2^{1/2}/2)d$.

Other partial results towards the conjecture have been obtained by Imre Simon (São Paulo), J. E. Pin (Paris) and J. P. Duval (Rouen).

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