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SYMMETRY AND FLAG MANIFOLDS

by

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In spite of its links with the symmetric group, the study of flag varieties has not yet fully used the customary technics (permutoèdre, Ehresman's order, Lehmer's code) of the theory of symmetric functions.

To the so-called Schubert cycles are associated polynomials, which are no other than Schur functions in the case of Grassmann varieties, and which can be studied through the help of symmetrizing operators, acting both on the cohomology ring, and the Grothendieck ring as special cases. Conversely, the study of representations of the symmetric group benefits from the geometrical intuition coming from the action of the symmetric group on the flag variety.

As an example, we indicate how to effectively compute the projective degrees of Schubert cycles. A note submitted to the Académie des Sciences apply these methods to the calculation of the Chern classes of the flag variety - as for its harmonic functions, the theory of which requires some properties of the plactic monoïd, they will be the subject of a separate article.

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Caution : the operators are placed on the right.

§ 1 Symmetrizing operators.

It is always delicate to distinguish between a permutation and its inverse, or between right and left multiplication for the symmetric group, if one does not take a set of "values" and a set of "places". To avoid misunderstandings, we shall consider permutations of $n+1$ elements as operators on the ring of polynomials $\mathbb{Z}[a,b,\dots]$, $\{a,b,\dots\}$ being a totally ordered alphabet of cardinal $n+1$.

Starting from the special element $a^E = a^n\, b^{n-1}\, c^{n-2} \dots$, one uses the transpositions $\sigma_{ab}, \sigma_{bc} \dots$ of consecutive letters to generate all monomials (written in the lexicographic order) whose multidegrees are a reordering of $\{0,1,\dots,n\}$.

This process gives us a ranked poset, as shown in the following figure, the permutations being considered as paths (directed downwards) in the graph of the poset (the graph is called the "permutoëdre"). The permutoëdre gives us all the reduced decompositions of the elements of the symmetric group, the length of a permutation being the length of any corresponding path; ω denotes the permutation of greatest length.

For example, the symmetric group on three letters gives

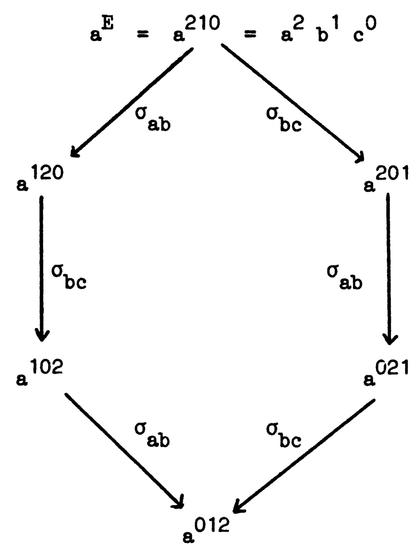


FIGURE 1

Moore's relations say that two paths having the same end points can be obtained by a sequence of elementary transformations

1.1 $\sigma_{ab} \cdot \sigma_{bc} \cdot \sigma_{ab} = \sigma_{bc}\, \sigma_{ab}\, \sigma_{bc}$

a,b,c being any triple of consecutive letters

1.2 $\sigma_{ab} \cdot \sigma_{de} = \sigma_{de}\, \sigma_{ab}$

if $\{a,b\} \cap \{d,e\} = \emptyset$

The conventions are such that if $\sigma . \sigma' . \sigma'' \ldots$ is a path from a^E to $a^I = a^{i_0} b^{i_1} \ldots$, then with $\omega w = i_0+1, i_1+1, \ldots, i_n+1$, one has $a^E\, \omega w = a^I$, $a^I\, w^{-1} = a^0 b^1 c^2 \ldots$; less trivial operators on $\mathbb{Z}[a,b,\ldots]$ appear when applying Jacobi's procedure to generate symmetric polynomials through alternating polynomials.

Define ∂_{ab} to be the operator

1.3 $\partial_{ab} : f(a,b,\ldots) \rightsquigarrow [f(a,b,c\ldots) - f(b,a,c\ldots)]/(b-a)$

and similarly for all pairs of consecutive letters, where f is an arbitrary polynomial (or rational) function.

We can interpret any path of the permutoèdre as a product of operators ∂_{ab} . Checking the relation similar to 1.1 (1.2 is trivial in this case), one gets

1.4 Lemma. *The product of operators corresponding to a path from* ω *to* w *is independant of the choice of the path and depends only upon* w . *It is denoted* $\partial_{\omega w}$.

The operators ∂_w are not always adequate because they systematically decrease the degrees. To preserve the degree, one defines

1.5 $\pi_{ab} \; : \; f \rightsquigarrow (af)\, \partial_{ab}$

$\pi_{bc} \; : \; f \rightsquigarrow (bf)\, \partial_{bc}$

. . . .

and one checks that these new operators still verify relations 1.1 and 1.2, so that a product of operators corresponding to a path depends only upon the end points : $\pi_{\omega w}$ is given by a path from ω to w , π_w by a path from w^{-1} to 123 ... .

Having at hand three operators verifying the same relations 1.1 and 1.2, one cannot resist in putting them in a single family.

Let p,q,r be fixed integers.

Define

1.6 $\quad D_{ab}(p,q,r) : f \rightsquigarrow (f)(p\partial_{ab} + q\pi_{ab} + r\sigma_{ab})$

and similarly for all pairs of consecutive letters.

It is a simple, but not totally trivial verification, that conditions 1.1 and 1.2 are still fulfilled, so that one can write $D_w(p,q,r)$.

To accelerate computations one may remark that symmetric functions are scalars with respect to the D_w :

1.7 $\quad f,g \in \mathbb{Z}[a,b,..]$, $f\sigma_{ab} = f \Rightarrow (fg)D_{ab} = gD_{ab}f$.

In fact $D_\omega(p,q,o)$ is a symmetrizer in the whole alphabet, i.e. $\forall f \in \mathbb{Z}[a,b,...]$, $\forall w$, $[f\,D_\omega(p,q,o)]w = f\,D_\omega(p,q,o)$.

One can show that, up to a change of variables, the operators $D_w(p,q,o)$ are the most general symmetrization operators verifying certain natural conditions, and thus we cannot find a family with more parameters containing them.

More precisely concerning $D_\omega(p,q,o)$, one has:

1.8 $\quad f\,D_\omega(p,q,o) = \sum\ (f\ \Delta_{pq})w$

sum on all permutations, Δ_{pq} being the generalization of Vandermonde's determinant:

$$\Delta_{pq} = \Pi_{x<y}\ (p + qx)/(x - y) .$$

1.9 Remark. $f\,\pi_\omega = (fa^E)\partial_\omega = \sum (-1)^{\ell(w)}(fa^E)w \,/\, \sum (-1)^{\ell(w)}\, a^E w$;

$a^I\,\pi_\omega$ is the classical Schur function of index $i_n, i_{n-1}, \ldots, i_0$ (cf Macdonald).

Thus the operators $D_\omega(1,0,0) = \partial_\omega$ and $D_\omega(0,1,0)$ are essentially the same, and formula 1.8 becomes in this case Jacobi's expression of Schur functions, Weyl's character formula for the linear group and Bott's theorem for the cohomology of line bundles on flag manifolds.

We did not use the square D^2_{ab} of an operator; in fact, one has

1.10 $$D^2_{ab} = q\, D_{ab} + r(q+r)$$

so that one is really working with a representation of the Hecke algebra of the symmetric group.

§ 2 Schubert polynomials.

As the action of ∂_ω , or π_ω as well, transforms a monomial into a Schur function, the operators D_w will give generalizations of Schur functions.

Following Demazure, and independently, Bernstein-Gelfand and Gelfand, we shall define, for every permutation w , the polynomial X_w by

2.1 $$X_w = a^E\, \partial_{\omega w} \quad ,$$

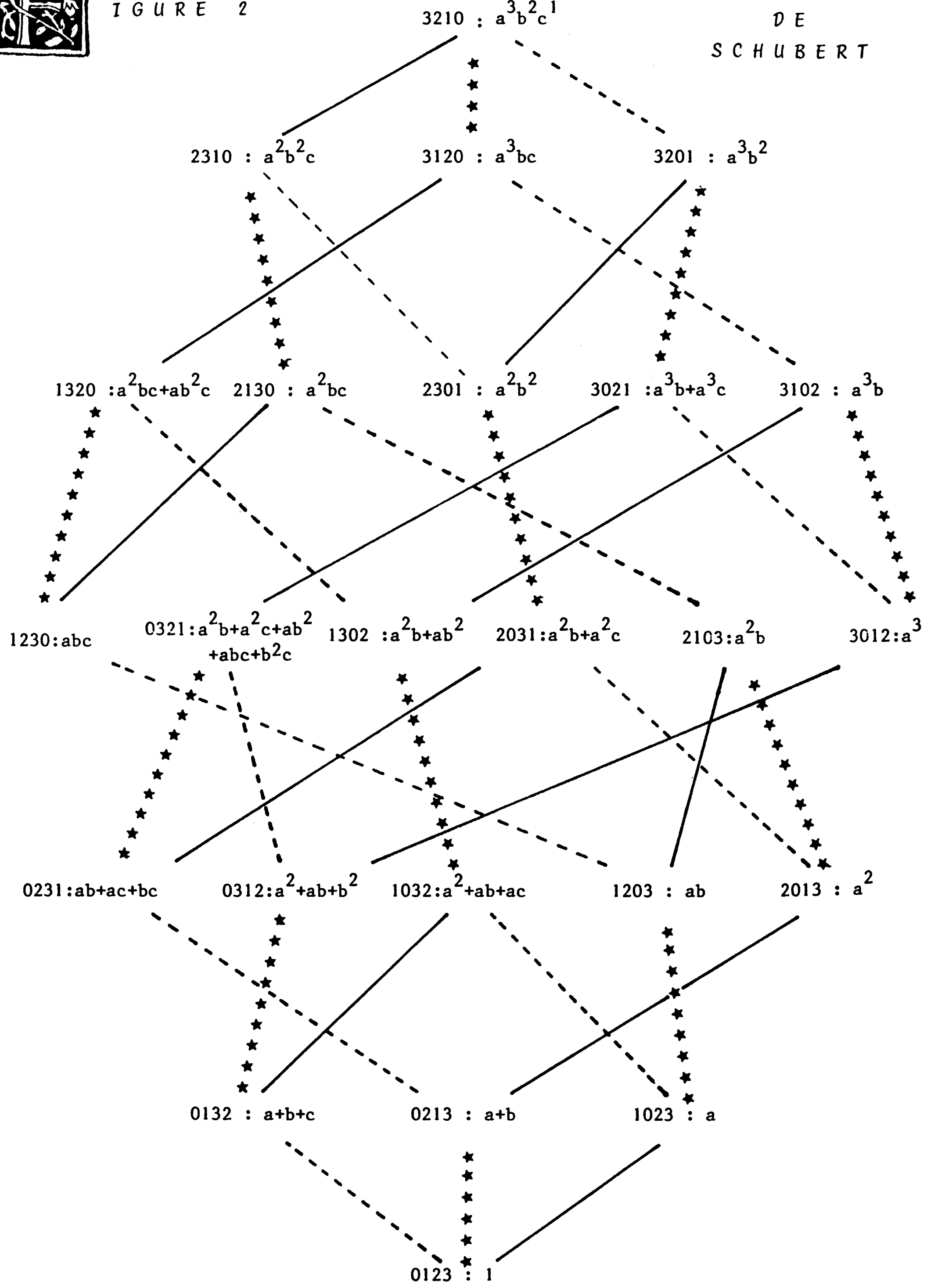
a^E being the monomial $a^n\, b^{n-1}\, c^{n-2} \ldots$. Thus, the X_w are obtained just by pushing down $X_\omega = a^E$ along the edges of the permutoëdre.

Figure 2 gives the result in the case of four letters.

One could generate the X_w through the π_w ; this more complicated process leads to a combinatorial representation of the X_w .

FIGURE 2

POLYNÔMES DE SCHUBERT



One notices on the example that X_w is a polynomial of degree $\ell(w)$ with positive coefficients, and that X_w is symmetrical in the i-th and i+1-th letters of A if and only if $w_i < w_{i+1}$; in other words, the shape of w (i.e. the sequence of its up's and down's) gives the symmetries of the polynomial X_w . For those permutations which are called grassmannian permutations (i.e. the ones which have only one descent), then X_w is a Schur function in the first letters of A, e.g. X_{2413} , which has to be symmetrical in a,b and also in c,d is indeed the Schur function

$$S_{12}(a+b) = a^2b + ab^2$$

(we identify an alphabet $A = \{a,b,\ldots\}$ and $S_1(A) = a+b+\ldots$) .

As for Schur functions, the first problem will be to multiply two polynomials. The simplest case is due to Monk, but first we need to enlarge the permutoëdre. For each pair of permutations (v,w) , such that v and w differ only by a transposition: $v = \ldots v_i \ldots v_j \ldots \curvearrowright w = \ldots v_j \ldots v_i \ldots$ and that $\ell(w) = \ell(v) + 1$, one draws between v and w j-i edges, of respective "colors" (i+1, i), ..., (j, j-1) (remember that on the permutoëdre, an edge of color (i+1, i) meant the transposition of the letters at place i and i+1).

The graph so obtained, when one forgets about the colours and the multiplicities of the edges, is due to Ehresman, and more generally, for Coxeter groups, to Bruhat. Let us call it the coloured Ehresmanoëdre.

Now, choose one colour (i+1, i) , and consider the monocolour subgraph $\Gamma_{i+1\ i}$ obtained by erasing the edges of colour different from the choosen one.

Then, writing i+1 i for the permutation 1 ... i-1 i+1 i i+2 ... , one has Monk's formula

2.2 $$X_{i+1\ i} \cdot\cdot X_v = \sum X_w$$

sum on all $w : \ell(w) = \ell(v) + 1$, vw is an edge of $\Gamma_{i+1\ i}$, i.e. there is an edge of colour (i+1 i) between v and w .

It is not too difficult to verify by induction this formula. The remarkable fact is that there is no multiplicity in this multiplication. Pieri's formula asserts that the multiplication of a Schur function by a special Schur function of any degree (i.e. elementary or complete symmetric functions, cf. Macdonald) produces no multiplicities. The same thing happens more generally for Schubert polynomials, i.e. the multiplication of a Schubert polynomial by a special one gives rise to no multiplicities (cf. L & S for the description of the w coming in the summation). Thus Monk's formula is the initial degree one-case of the general Pieri's formula.

As a by-product, one obtains a commutation property which is valid for all finite Coxeter groups.

Let $C_{21}, C_{32} \dots$ be the matrices of the directed graphs $\Gamma_{21}, \Gamma_{32}, \dots$, i.e. we put 1 at the place (v,w) of the matrix $C_{i+1\ i}$ if $\ell(v) < \ell(w)$, and vw is an edge of $\Gamma_{i+1\ i}$, and 0 otherwise. Then one has

2.3 Lemma: *The matrices* $C_{i+1\ i}$ *commute*.

Proof.: As the multiplication of Schubert polynomials by $X_{i+1\ i}$ is described by the matrix $C_{i+1\ i}$. The lemma is equivalent to the fact that the product $X_{i+1\ i} \cdot X_{j+1\ j} \cdot X_v$ is equal to $X_{j+1\ j} \cdot X_{i+1\ i} \cdot X_v$ for every v.

This specific property of Bruhat order on Coxeter groups has to be proved in itself without reference to multiplication of Schubert polynomials.

§ 3 Cohomology of the flag manifold.

The reader who wants to use Figure 2 to multiply $a^3\left(=X_{4123}\right)$ by $a\left(=X_{2134}\right)$ finds no edge with colour 21 from 4123 upwards. So he must disagree with Monk's formula as stated above. But if he enlarges the alphabet by just one letter, he certainly obtains that

$$a^3 \cdot a = X_{41235} \cdot X_{21345} = X_{51234} = a^4 .$$

More generally, to use Monk's formula for the symmetric group W_n , one must for safety reasons imbed it into W_{n+1} .

Alternatively one can also notice that

$$a^4 = S_4(a+b+c+d) - (b+c+d)\,S_3(a+b+c+d) + (bc+bd+cd)\,S_2(a+b+c+d) - bcd(a+b+c+d)$$

which has the consequence that a^4 belongs to the ideal generated by the polynomials summetrical in a,b,c,d .

Definition: The cohomology ring of the flag manifold associated to the symmetric group W_{n+1} is

<u>3.1</u> $\quad H = \mathbb{Z}[a,b,\ldots] \,/\, I$

where I is the ideal generated by the symmetric polynomials (with no constant term!) in all the variables (in other words, the ideal generated by the invariants of W_{n+1}).

It is easy to show by induction on n that H has two natural $\mathbb{Z}$-bases:

i) the monomials $a^I = a^{i_1}\, b^{i_2} \ldots$ with $0 \leq i \leq E$

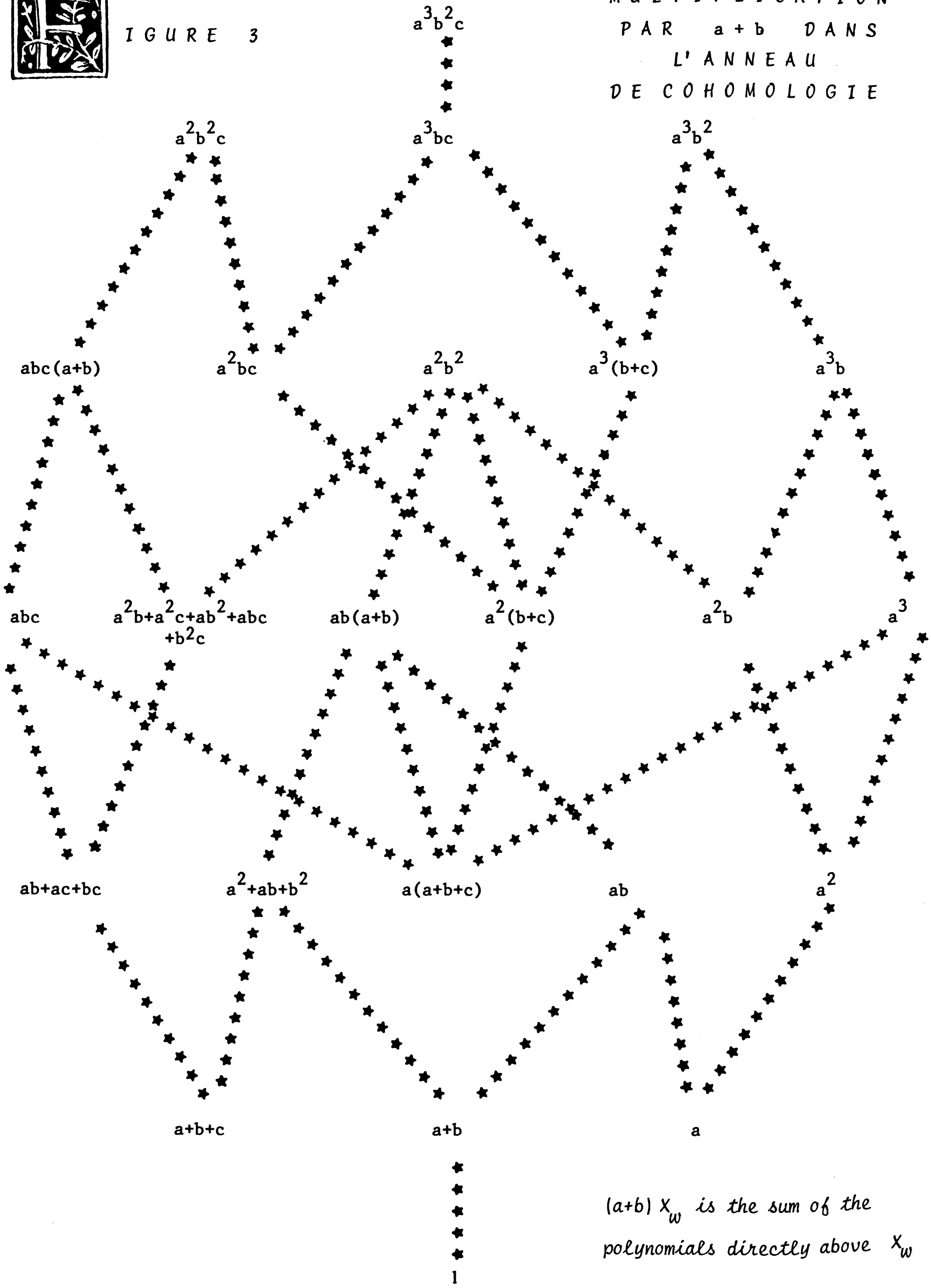
ii) the Schubert polynomials X_w (the class of a Schubert polynomial in H is called a <u>Schubert cycle</u>).

Notice that H is of rank 1 in the maximal degree $\ell(\omega) = n(n+1)/2$ and that $X_\omega = a^E$.

Now, Monk's formula is perfectly valid: passing from an alphabet of $n+1$ letters to n , one annihilates exactly the Schubert polynomials X_w

FIGURE 3

MULTIPLICATION PAR a + b DANS L'ANNEAU DE COHOMOLOGIE



(a+b) X_w is the sum of the polynomials directly above X_w

for those w such that $w_{n+1} \neq n+1$.

But another difficulty comes: how can we see that two polynomials are equivalent modulo the invariants of W_{n+1} ?

Ehresman, generalizing classical results on Grassmann varieties, has shown that the multiplication in H does induce a pairing on the basis of Schubert cycles: for w,w' such that $\ell(w) + \ell(w') = \ell(\omega)$, then

<u>3.2</u> $$X_w \cdot X_{w'} \equiv a^E \text{ or } 0$$

according to $w' = \omega w$ or not.
This result is due to Chevalley for arbitrary Coxeter groups.

Since the operators D_w preserve the ideal I ($(fg)D_{ab} = g\, D_{ab} \cdot f$ if $f\, \sigma_{ab} = f$), they are indeed operators on H .

Using $a^E \partial_\omega = 1$, one gets:

<u>3.3</u> <u>Let</u> P <u>be a homogeneous polynomial of degree</u> $\ell(\omega)$. <u>Then</u>
$P \equiv (P\, \partial_\omega)\, a^E \mod I$.

Thus, combining with 3.2, the decomposition of a polynomial in the basis X_w is given by

<u>3.4</u> $$P \equiv \sum (P \cdot X_{\omega w})\partial_\omega\, \varepsilon \cdot X_w$$

sum on all permutations w , the <u>augmentation morphism</u> $\varepsilon : \mathbb{Z}[a,b,\dots] \to \mathbb{Z}$, $a\varepsilon = b\varepsilon = \dots = 0$ taking care of the decomposition of P into its homogeneous components.

Now, if one does not have at hand the explicit expression of the Schubert polynomials, one must improve the method to be able to determine when two polynomials are equivalent modulo the ideal I . This will be done in § 6.

§ 4 Projective degree of Schubert cycles.

Consider a graded ring H , call the graduation <u>codimension</u>, and assume that H is of rank 1 in maximal codimension (assumed different from infinity!) : $H^{max} \simeq \mathbb{Z}$.

Let Y in H be an element of codimension 1, and X of codimension d . Then the <u>degree of X relative to Y</u> is the image in $\mathbb{Z}$ of $X \cdot Y^{max-d}$.

When H is the cohomology ring of a projective variety, one chooses an imbedding in a projective space and Y is the class of the intersection with an hyperplane.

In our case, for the natural embedding of the flag variety, which is due to Plücker, Y is equal to the sum of all Schubert polynomials of codimension 1 :

<u>4.1</u> $$Y = X_{2134\ldots} + X_{1324\ldots} + X_{1243\ldots} + \ldots = na + (n-1)b + (n-2)c + \ldots$$

(To distinguish between the degree of X relative to Y and the degree of X as a polynomial, we call the first <u>projective degree</u> and the second <u>codimension</u>.)

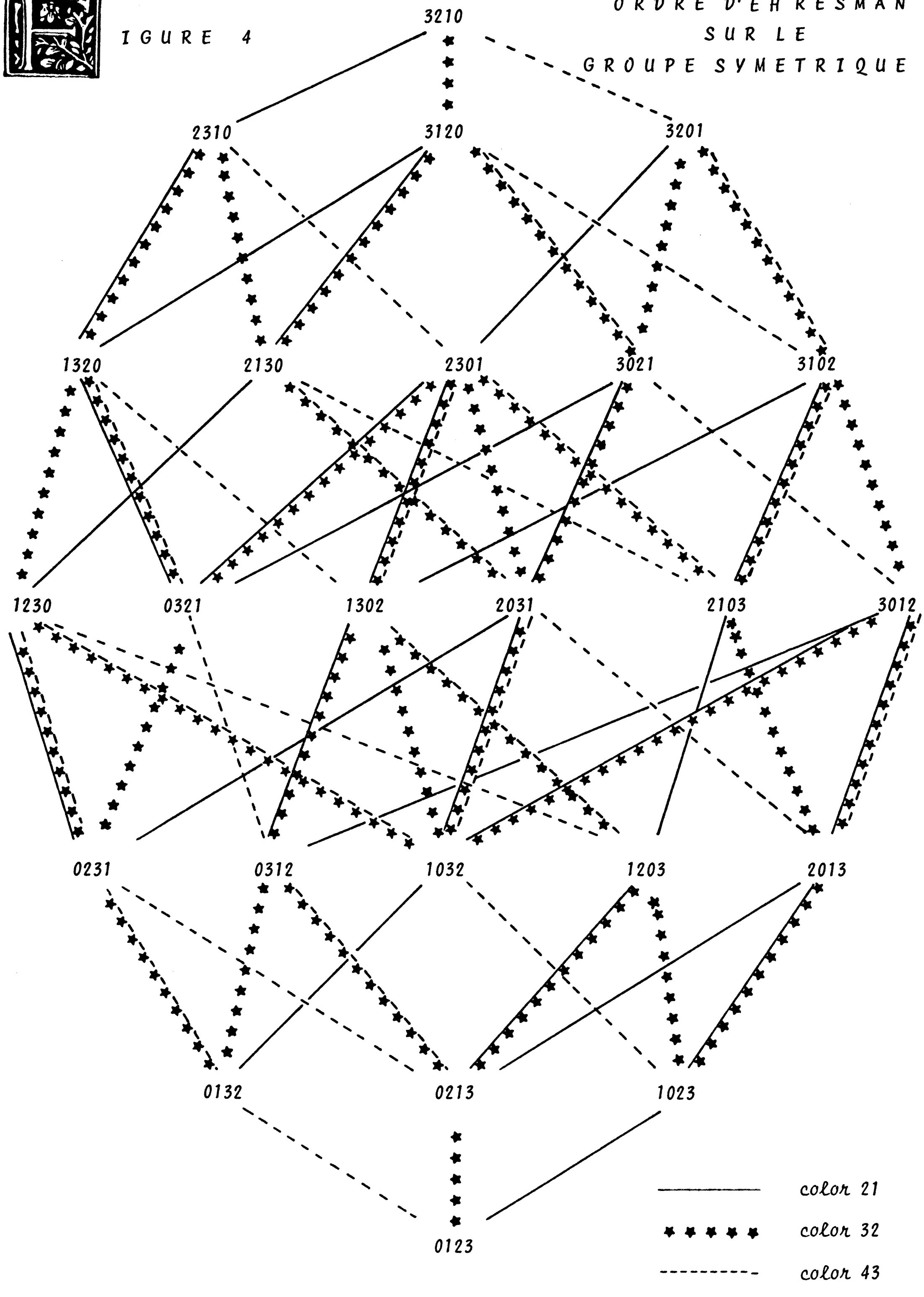
To compute the projective degrees, it is sufficient to know them for the Schubert cycles. In the case of grassmannians (a certain quotient of H) one obtains the degrees of the irreducible representations of the symmetric group, so these projective degrees should be interesting by themselves, regardless of their geometrical interpretations.

We have already done most of the work: as the multiplication by $X_{i+1\ i}$ corresponds to the edges of colour $(i+1\ i)$ in the coloured Ehresmanoëdre, one gets:

<u>4.2</u> <u>Proposition</u>. <u>The projective degree of</u> X_w <u>is the number of paths from</u> w <u>to</u> ω <u>in the coloured Ehresmanoëdre</u>.

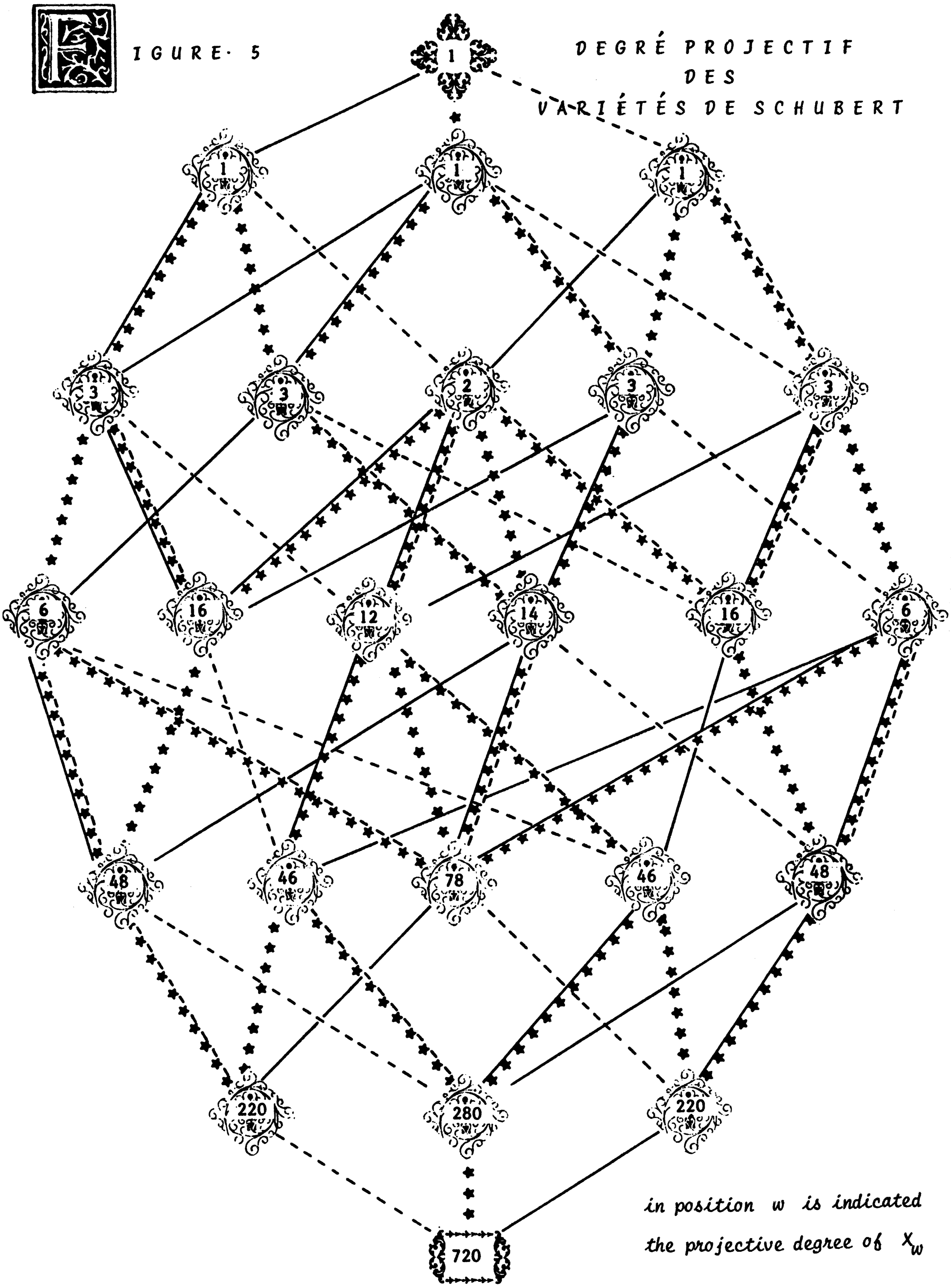
FIGURE 4

ORDRE D'EHRESMAN SUR LE GROUPE SYMETRIQUE



FIGURE· 5

DEGRÉ PROJECTIF DES VARIÉTÉS DE SCHUBERT



in position w is indicated the projective degree of X_w

This proposition is equivalent to the following induction:

<u>4.3</u> $\quad$ $\text{proj.deg}\ X_w = \sum_v m(w,v)\ \text{proj.deg}\ X_v$

sum on all permutations v: $\ell(v) = \ell(w) + 1$, with $m(w,v)$ = number of edges from w to v .

Another formulation is:

<u>4.4</u> $\quad$ $(1-Y)^{-1} = (1 - (na + (n-1)b + \ldots))^{-1} \equiv \sum \text{proj.deg}\ (X_w)\ X_{\omega w}$ in H .

For example, in the case $n = 2$
$(1 - (2a+b)^{-1} \equiv 1 + a + (a+b) + 3a^2 + 3ab + 6a^2b$ (and so the proj.deg are 1,1,1,3,3,6) taking into account that in H , $a^2 + ab + b^2 \equiv 0$, $a^3 \equiv b^3 \equiv a^2b + ab^2 \equiv 0$ modulo the symmetric functions in a,b,c.

One can show this way that proj.deg $X_{123\ldots} = \left(n(n+1)/2\right)!$

Since more information is contained in the Ehresmanoëdre, one can do better than only counting the paths, by reading the paths as words of colours. So denote colours (21), (32), ... by α, β, ..., and read a path as a sequence of colours, i.e. a word in the Greek alphabet.

Define the <u>non-commutative degree</u> of X_w as the sum of the words given by all the paths from w to ω . Then the commutation property 2.3 insures that this non commutative degree is a "Partie reconnaisable" (terminology from the theory of monoids) i.e. is <u>invariant by permutation</u>: whenever you meet the word $\alpha\alpha\beta\gamma$, you have also the word $\gamma\alpha\beta\alpha$ with the same frequency.

<u>4.5</u> <u>Example</u>. For X_{3214} , one gets the noncommutative degree

$(\alpha+\beta+\gamma)(\beta\gamma+\gamma\beta+\gamma\gamma) + (\beta+\gamma)(\alpha\gamma+\gamma\alpha) + \gamma(\alpha\beta+\beta\alpha+\beta\beta)$.

Thus, the non commutative degree is given by restricting to the <u>increasing words</u>; in the above examples, the degree is obtained by permutation of (6) $\alpha\beta\gamma$ + (3) $\alpha\gamma\gamma$ + (3) $\beta\beta\gamma$ + (3) $\beta\gamma\gamma$ + (1) $\gamma\gamma\gamma$
(inside the parenthesis, we have indicated how many words are associated in the non-commutative degree).

In other words, if $\varphi : Z\Big[[\alpha,\beta,\ldots]\Big] \to Z[\alpha,\beta,\ldots]$ is the natural morphism (the <u>evaluation</u>) from the ring of non-commuting variables to the ring of polynomials, then the non-commutative degree is the inverse image of a polynomial $Z_{\omega,w}$ that we call the <u>colour-degree</u> of X_w .

More generally, one defines the polynomials $Z_{v,w}$, when $\ell(v) \geq \ell(w)$, to be the sum of all increasing paths from v to w (this will correspond to the degree of the intersection of two Schubert cycles); put $Z_{v,w} = 0$ if $\ell(v) < \ell(w)$.

If moreover, one defines M_α to be the matrix: the entry (v,w) is α or 0 , according as $\ell(v) = \ell(w) + 1$ and there is an edge of colour α between v and w , or not, and similarly for M_β, M_γ ..., one obtains from 2.3 the commutation of the matrices M_α, M_β, ...

<u>Exercise</u>. Prove that $Z_{n+1\ 12..n,\ 12..n+1} = \sum \alpha^i \beta^j \gamma^k \ldots$, sum on all different monomials of total degree n , with $i \leq n$, $j \leq n-1$, $k \leq n-2$, ...
e.g. $Z_{4123,\ 1234} = \alpha(\alpha\alpha + \alpha\beta + \beta\beta + \alpha\gamma + \beta\gamma)$.

§ 5 The G-polynomials.

Instead of taking the cohomology ring of the flag manifold as we did in § 3, it is more fruitful to take another quotent of the ring of polynomials, which is called the <u>Grothendieck ring of the flag manifold</u>; denote the variables by L_a, L_b, ... to distinguish from the preceeding case, and keep the same notations for the operators ∂_{ab}, π_{ab}, ..., as no ambiguity is to be feared.

Call θ the <u>specialization</u> ring-morphism: $L_a\theta = L_b\theta = \ldots = 1$ and let J be the ideal generated by the relations

<u>5.1</u> $\forall\ f \in \mathbb{Z}[L_a,L_b,\ldots]$, $f\,\pi_\omega \equiv f\,\pi_\omega\theta$

i.e. the totally symmetric polynomials are equalled to their value for $L_a = L_b = \dots = 1$.

5.2 Definition. The Grothendieck ring of the flag manifold is

$$K = \mathbb{Z}[L_a, L_b, \dots] / J .$$

The properties of this ring are strongly linked with those of symmetric functions, whose theory has been formalized in the theory of λ-rings.

As J is invariant under the action of the D_w , these operators still act on K . Of course, taking relations 5.1 instead of 3.1 do not change the $\mathbb{Z}$-bases of the quotient ring, so that one has

5.2 The set of monomials $L^I = L_a^{i_1} L_b^{i_2} \dots$, for $0 \le I \le E$. is a $\mathbb{Z}$-basis of K .

5.3 The Schubert polynomials (in the alphabet $L_a, L_b, \dots$) are a $\mathbb{Z}$-basis of K .

As $L_a L_b L_c \dots = 1$, we see that the ring K contains the inverse of the variables $L_a, L_b, \dots$. The inversion of $L_a, L_b, \dots$ extends to an involution morphism of K which is called duality by reference to vector bundles ($L_a, L_b, \dots$ are the tautological line bundles of the flag manifold).

It is convenient to introduce new variables $x = 1 - L_a^{-1}$, $y = 1 - L_b^{-1}$, $z = 1 - L_c^{-1}$ ... . The symmetrizers associated to $x, y, \dots$ are related to those of $L_a, L_b, \dots$; One checks

5.4 $$\forall f \in K , \quad f\,\pi_{ab} = (f - fy)\partial_{xy}$$

and similarly for all pairs of consecutive letters,which, incidentally, shows that a change of variables induces a non trivial transformation of the symmetrizers. As in § 2, we choose a maximal element

$$G_\omega = x^E = x^n\, y^{n-1}\, z^{n-2} \dots$$

and we define the G-polynomial indexed by the permutation w by

5.5 $$G_w = G_\omega \, \pi_{\omega w} \quad .$$

Figure 6 gives the case n = 3.

It is clear from lemma 5.4 that the homogeneous part of smallest degree (= $\ell(w)$) of G_w is the Schubert polynomial X_w (in the variables x,y,... instead of a,b,...). Thus the Schubert polynomials are nothing but the leading term of the G_w .

Schubert polynomials (in x,y,...) being a basis of K , one can express the G_w in term of the X_w , or conversely, the X_w in term of the G_w , the matrix being triangular.

e.g. for (w) = 4 , one has

$$X_{2413} = G_{2413} + G_{3412} \quad ; \qquad X_{2341} = G_{2341} \quad ;$$

$$X_{3142} = G_{3142} + G_{3241} \quad ; \qquad X_{3214} = G_{3214} \quad ;$$

$$X_{1432} = G_{1432} + 2G_{2431} + G_{3421} + G_{3412} \; ; \qquad X_{4123} = G_{4123} \quad .$$

To express the multiplication in the basis of G_w is more complicated than in the basis X_w . We shall give elsewhere the corresponding "Pieri-formula". For example, one reads on figure 5 that

$$G_{1324} \, G_{1324} = (x+y - xy)^2 = G_{2314} + G_{1423} - G_{2413} \; .$$

(We previously had $X_{1324} \cdot X_{1324} = X_{2314} + X_{1423}$; here we had to substract the supremum of 2314 and 1423 which is 2413 ; bigger intervals are involved in general.)

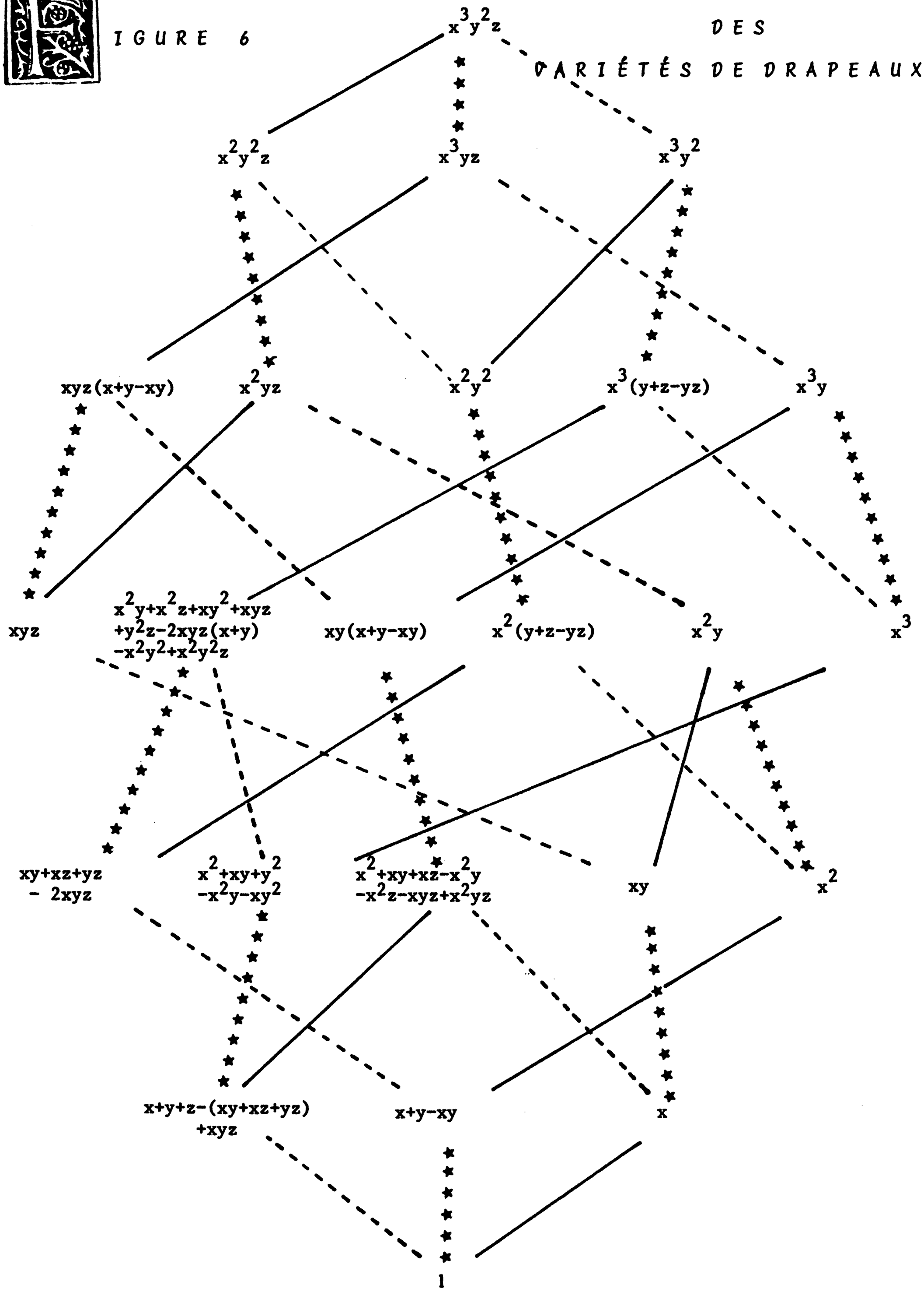
To understand the link between the two rings H and K , one must recall the existence and properties of Chern classes:

Denote by $1 + H^+$ the multiplicative monoid of polynomials with constant term 1 (and coefficients in $\mathbb{Q}$). There exists a homomorphism:

$$c \; : \; K \to 1 + H^+$$

FIGURE 6

ANNEAU DE GROTHENDIECK DES VARIÉTÉS DE DRAPEAUX



such that $c(1) = 1$, $c(E+F) = c(E) \cdot c(F)$. On the basis L^I , it takes the values

$$c(L_a{}^i \, L_b{}^j \, L_c{}^k \ldots) = 1 + ia + jb + kc + \ldots .$$

(Of course the multiplication in K induces a "product" in $1 + H^+$, for whose explicit description one needs Schur functions - see Macdonald.)
Now one can check

$$c(G_w) = 1 - (-1)^{\ell(w)} \; (\ell(w)-1)! \;\; X_w + X'$$

where X' is a polynomial of degree $> \ell(w)$. (cf. SGA6, exp. 0 formule 1.18).

If G is a sum $\Sigma\, n_w\, G_w$ with w of constant length $\ell(w) = d$ (one says $G \in K^d$)

$$c(G) = 1 - (-1)^d \;\; (d-1)! \;\; \Sigma\, n_w\, X_w + X'$$

and

$$c(G\,\pi_{ab}) = 1 - (-1)^{d-1} \, (d-2)! \;\; \Sigma\, n_w\, X_w\, \partial_{ab} \;\; + \;\; X'\, \partial_{ab}$$

so that one sees that:

5.6 Proposition: $-(d-1/^{-1}\, \partial_{ab}$ is the image by the Chern homomorphism of π_{ab} acting on K^d .

§ 6 Quadratic form on the cohomology ring.

Most of the preceeding description relies heavily upon the natural bases of the cohomology or Grothendieck ring of the flag variety. To be able to compute without restriction in these rings, one must be able to express a general element (in a finite time) in the bases already defined.

The operators corresponding to the permutation of greatest length are the most effective rool for this purpose. It amounts to define on each of the spaces H and K a quadratic form.

We consider sequences as vectors in $\mathbb{Z}^{n+1}$ and thus can write $I \pm J$; through the identification $I \leftrightarrow a^I$, the symmetric group acts on sequences: $I \rightsquigarrow Iw$;

recall that E is the sequence $n, n-1, \ldots, 0$.

Now, when $-E \leq I \leq E\omega$, one checks from formula 1.9 that

6.1 $$a^I \pi_\omega = a^{I+E} \partial_\omega \begin{cases} = (-1)^{\ell(w)} & \text{if there exists } w \text{ such that } I + E = Ew \\ = 0 & \text{otherwise.} \end{cases}$$

Moreover, $a^I \pi_\omega = (a^{-I\omega})\pi_\omega$, and $a^I \pi_\omega = 0$ if $i_1 + \ldots + i_{n+1} (=|I|) \neq 0$.

E.g. $I = -3102 \Rightarrow I+E = 0312$ is a permutation of E and so is $-I\omega + E$ $(= -2\ 0\ -1\ 3\ +\ 3\ 2\ 1\ 0\ =\ 1\ 2\ 0\ 3)$.

Owing to this symmetry between I and $-I\omega$, one defines a scalar product on $\mathcal{H}$ by its values on the basis a^I (for $0 \leq I \leq E$) :

6.2 $$(a^I, a^J) = a^{I\omega - J}\, \pi_\omega .$$

For example, for four letters and degree 3

2100	1110	2010	1200	3000	0210	I / J
0	0	0	0	0	-1	2100
0	0	0	0	-1	0	1110
0	0	0	-1	1	1	1200
0	0	-1	0	1	1	2010
0	-1	1	1	-1	0	3000
-1	0	1	1	0	-1	0210

On the example one sees that the quadratic form, for a given weight, is positive or negative definite, and triangular for an appropriate ordering of the monomials.

Instead of describing the ordering, which directly comes from the interpretation of sequences I such that $0 \leq I \leq E$ as coding permutations, one can do better and give the adjoint basis of a^I .

6.3 Let $P_I = \Pi_{0 \leq p \leq n} \Lambda_{i_p} (A_{n-p})$.

A_p being the alphabet of the first p letters, and Λ_i the elementary symmetric function of degree i ; then one has

6.4 Proposition: The family $(-1)^{|I|}\{P_I\}$ with $0 \leq I \leq E$, is the adjoint basis of $\{a^I\}$.

For example, for $n=3$, $I = 2010$, $P_I = \Lambda_2 (a+b+c) \Lambda_1(a) = a^2b + a^2c + abc$, and one checks from the preceeding table that

$$(a^I, P_I) = -1 \quad , \quad (a^J, P_I) = 0 \quad \text{if} \quad J \neq I .$$

6.5 Corollary (Bott-Rota's straightening).

If Y is an homogeneous polynomial of degree d , then in H , $(-1)^d Y \equiv \equiv \Sigma (Y, P_I) a^I \equiv \Sigma (Y, a^I) P_I$ sum on all sequences I such that $0 \leq I \leq E$.

Thanks to 6.1, this straightening is an efficient way of decomposing in H the class of a polynomial. For the decomposition in the basis X_w , we already have 3.4.

We note that $X_w \partial_v = 0$ if $\ell(v) = \ell(w)$ and $v \neq w^{-1}$, because $X_w \partial_v = X_\omega \partial_{\omega w} \partial_v$, so that either $\omega w v = \omega$, or $\ell(\omega w v) < \ell(\omega) \Leftrightarrow \partial_{\omega w} \partial_v$ can be written $\partial_{w'} \partial_u \partial_u \partial_{w''}$ $(= 0)$. Thus we have the other way of decomposing a polynomial Y .

6.6 $$Y \equiv \Sigma \ (Y \cdot \partial_{w^{-1}}) \varepsilon \ X_w$$

sum on all permutations w , which $P\varepsilon$ = term of degree o of P , as in 3.4.

§ 7 Quadratic form on the Grothendieck ring.

As Schubert polynomials are still a basis of the Grothendieck ring K, one could still keep the scalar product for which the Schubert polynomials are an orthonormal basis. This would not fit well with the action of the operators on K.

Remembering that π_ω sends K to its subring $\mathbb{Z}$, one can define

7.1 $\quad \forall\ P, Q \in K\ ,\ \langle P, Q \rangle = (PQ)\,\pi_\omega\ ;$

on the basis G_w, the quadratic form takes only the values 0 or 1.

For example, for S_3, the multiplication table is

	G_{123}	G_{213}	G_{132}	G_{231}	G_{312}	G_{321}
G_{123}	G_{123}	G_{213}	G_{132}	G_{231}	G_{312}	G_{321}
G_{213}	G_{213}	G_{312}	$G_{312}+G_{231}-G_{321}$	G_{321}	0	0
G_{132}	G_{132}	$G_{312}+G_{231}-G_{321}$	G_{231}	0	G_{321}	0
G_{231}	G_{231}	G_{321}	0	0	0	0
G_{312}	G_{312}	0	G_{321}	0	0	0
G_{321}	G_{321}	0	0	0	0	0

and the quadratic form is the image of this table by π_ω (noting that $\forall w\ ,\ G_w\,\pi_\omega = 1$):

1	1	1	1	1	1
1	1	1	1	0	0
1	1	1	0	1	0
1	1	0	0	0	0
1	0	1	0	0	0
1	0	0	0	0	0 .

We shall not prove here the two following propositions which generalize 3.4 and 6.6.

7.2 Proposition. For any w, let H_w be the sum of G-polynomials $\Sigma_{v \geq w} \; (-1)^{\ell(v)-\ell(w)} \; G_v$.

Then $\{H_{\omega w}\}$ is the adjoint basis of $\{G_w\}$ (with respect to $\langle \, , \, \rangle$).

For example,

$\langle G_{132} - G_{231} - G_{312} + G_{321} \, , G_w \rangle = 0$ except for $w = 312 = \omega \cdot 132$.

7.3 Proposition. Let θ be the specialization morphism $L_a \, \theta = L_b \, \theta = \ldots = 1$. Then $\forall \, P \in K$, $\forall w$, $\langle Y, G_w \rangle = Y \, \pi_{w^{-1}} \, \theta$.

As for every w , $\langle G_w, 1 \rangle = (G_w \cdot 1) \, \pi_\omega = 1$ the fact that $\langle H_w, 1 \rangle = 0$ generalizes the property of the Moebius function (for the Bruhat order) to be ± 1 .

§ 8 Applications.

We have mainly described the tools to study the cohomology or Grothendieck ring of the flag manifold. Many questions arising from the theory of groups or algebraic geometry can be then easily studied.

8.1 The representation of the symmetric group W_{n+1} on H or K . One must note that as $\mathbb{Z}$-modules, H and K are isomorphic to the regular representation of W_{n+1} but that the degree gives us an extra information; in fact, the multiplicity of an irreducible representation can be considered as a polynomial (which happens to be a Kostka-Foulkes polynomial coming in the theory of representation of the finite linear groups). More generally, De Concini et Procesi have studied the quotients of H associated to the variety of flags fixed by a given unipotent matrix.

8.2 Enumerative geometry on the flag manifold.

We have only given the projective degree of a Schubert cycle in § 4.

One needs also the postulation of the cycle X_w with respect to a line bundle L^I : by definition, it is $\Sigma (-1)^i \dim H^i (\mathcal{O}_w, L^I)$; once given the rules of the translation, it simply becomes $(L^I G_w)\, \pi_\omega$, which is also equal to $L^I\, \pi_{w-1}\, \theta$, as asserted in 7.3.

The Chern classes of a variety are the first invariants of it that one tries to get. In the case of the flag manifold, the tangent bundle T has class

$$L_a \cdot L_b^{-1} + L_a \cdot L_c^{-1} + L_b L_c^{-1} + \dots \quad \text{in } K$$

so that its Chern class is

$$c(T) = (1+a-b)\,(1+a-c)(1+b-c) \dots$$

and it remains to compute $c(T)$ in the basis of Schubert cycles. This will be done elsewhere.

8.3 Representations of the linear group $G\ell(\mathbb{C}^{n+1})$.

One can consider the ring of invariants of W_{n+1}: $\mathbb{Z}[a,1/a,b,1/b,\dots]^W$ to be the ring of formal sums of representations of $G\ell(\mathbb{C}^{n+1})$.

Bott's theorem evaluates in this ring, for any line bundle L^I, $I \in \mathbb{Z}^{n+1}$, and any i , the representation $H^i(X, L^I)$, X being the flag manifold.

We have obtained here a little less:

$$\Sigma\, (-1)^i\, H^i(X, L^I) = (L^I)\, \pi_\omega ,$$

(in fact, all the $H^i(X, L^I)$ are {0} except at most one, so that the two computations are not very different).

One can also look for syzygies of the Schubert variety corresponding to w , i.e. try to get a complex of locally free bundles which "solves" the ring of the Schubert variety. The class of the complex in $\mathbb{Z}[a,1/a,b,1/b,..]$ is given by

$$[\,(1-L_a^{-1})^n\,(1-L_b)^{n-1} \dots]\,\pi_{\omega w}$$

but, of course there remains to describe the morphisms inside the complex. This, we shall not do.

8.4 Root systems and Coxeter groups.

Most of the properties of the operators D_w can be extended to other finite Coxeter groups, as shown first by Demazure and independently Bernstein, Gelfand-Gelfand.

In the case of the symmetric group, if α, β, ..., are the simple roots, and ρ half the sum of positive roots, then e^α, e^β, ... are respectively $L_a L_b{}^{-1}$, $L_b L_c{}^{-1}$, ..., and e^ρ is equal to L^E up to a power of $L_a L_b L_c$ ... . If I is weakly decreasing (L^I is dominant), then Weyl's character formula for the corresponding irreducible representation E_I is

$$ch(E_I) = \Sigma\,(-1)^w\,(L^I L^E)w \;/\; \Sigma\,(-1)^w\,L^E w$$

and as we have remarqued in 1.9, it can be written

$$ch(E_I) = (L^I L^E)\,\partial_\omega \quad ;$$

an equivalent result, using instead the operator π_ω, is

$$ch(E_I) = L^I\,\pi_\omega$$

which is in fact Bott's formula.

8.5 Determinants.

For $m : 0 \leq m \leq n+1$, one can consider The Grassmann variety of subvector spaces of dim $m+1$ of $\mathbb{C}^{n+1}$. The associated cohomology ring is the subring $H^{W' \times W''}$ of H invariant under the product of symmetric groups $W' \times W''$ (W' being the group on the first $m+1$ letters, W'', on the remaining letters).

A $\mathbb{Z}$-basis of $H^{W' \times W''}$ is the set of Schubert polynomials X_w for the w of minimum length in their class modulo $W' \times W''$. In this case X_w is a Schur function on the alphabet of the first $m+1$ letters, and all the properties of the cohomology ring of the Grassmann variety (or of the Grothendieck ring) can be translated in term of Schur functions. In particular, the determinantal expression of Schur function gives rise to a determinantal expression for X_w (due to Giambelli), for G_w , for the postulation (due to Hodge), etc...

Unfortunately, not all permutations in general give determinants. We have given in [L & S] several characterizations of those permutations for which X_w, G_w, ... are determinants (permutations vexillaires); for them, the computations are very similar to the ones in the more special case of Grassmannvariety.

8.6 Combinatorics.

A combinatorial and powerful description of Schur functions is given by Ferrers diagrams and Young tableaux. One can similarly associate to any permutation a diagram (due to Riguet), and fill it according to rules deduced from Pieri-Monk's formula 2.2. This seems to be an interesting generalization of Young tableaux and the plactic monoid.

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