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Schubert Polynomials and the Littlewood–Richardson Rule★

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Abstract. The decomposition of a product of two irreducible representations of a linear group Gl(N, $\mathbb{C}$) is explicitly given by the Littlewood–Richardson rule, which amounts to finding how many *Young tableaux* satisfy certain conditions. We obtain more general multiplicities by generating 'vexillary' permutations and by using partially symmetrical polynomials (*Schubert polynomials*).

0. Introduction

Character expansion techniques are widely used in both particle and nuclear physics (cf. [1, 6]). For what concerns SU(N) or Gl(N), the fundamental functions are the *Schur functions*, which also describe the irreducible representations of the symmetric group. As a matter of fact, the ring generated by Schur functions is the ring of symmetric polynomials (in an infinite number of variables), which can be given many interpretations, among which are the representation ring of a symmetric group, of a linear group, and the cohomology ring of a Grassmannian, i.e., of the classifying space BU_n, etc.

The images of Schur functions in all these isomorphic rings are each time the fundamental basis, but the product takes different interpretations (Table I).

Table I.

Interpretation	Symmetric polynomials	Representation of the sym. group	Representation of the linear group	Cohomology of the Grassmannian
Fundamental basis	Schur functions	Irreducible rep. over $\mathbb{C}$	Irreducible rep. over $\mathbb{C}$	Schubert cycles
Product	Product of polynomials	Branching $\mathfrak{S}_M \times \mathfrak{S}_N \hookrightarrow \mathfrak{S}_{M+N}$	Tensor product of vector spaces	Intersection

The product was determined by Pieri, an Italian geometer, a century ago. He gave the intersection of all Schubert cycles by special ones. The general intersection, or the decomposition of the tensor product of two irreducible representations of the linear

★ A la mémoire de S. Ulam, exemple et ami.

group was given 50 years later by Littlewood and Richardson [4], and satisfactory proof of the so-called Littlewood–Richardson rule was given only recently (cf. [5]). Algebraically, it amounts to considering Schur functions as sums of Young tableaux of a fixed shape, i.e., to give a noncommutative version of symmetric polynomials.

A different approach consists, not of studying symmetric polynomials, but rather the action of a symmetric group on a ring of polynomials. The necessary objects for this action are now the *Schubert polynomials* Y_I (among which are the Schur functions) indexed by sequences of positive integers. The Schubert polynomials have only partial symmetries. Nevertheless, by restriction to a subset of variables, one recovers symmetric functions which, of course, can be decomposed into a sum of Schur functions S_J. In other words, one has a morphism $Y_I \to \Sigma\, m(I, J) S_J$, the $m(I, J)$ being positive integers among which one can find the Littlewood–Richardson multiplicities, as will be explained below.

Indeed, we can do better than compute multiplicities: to each sequence, we attach a set of 'vexillary' sequences. Now, a Schubert polynomial indexed by a vexillary sequence gives exactly one Schur function by restriction, and the above multiplicity $m(I, J)$ is just the cardinal of vexillary sequences giving the Schur function S_J. Vexillary sequences play a role in geometry and correspond to flags of modules.

The algorithm for producing these vexillary sequences is much faster than all those derived from the Littlewood–Richardson rule, because it involves no 'trial and error' steps. In fact, we could totally ignore the Schur functions or Schubert polynomials. What we are doing is describing certain properties (consequences of formula (3.3)) of the so-called Ehresman–Bruhat order on the symmetric group, exactly as branching rules for the symmetric or linear group can be described by paths in the lattice of partitions, i.e., Young tableaux, Yamanouchi symbols, or equivalently Gelfand patterns. Nevertheless, manipulating polynomials is more elementary!

1. Schubert Polynomials

Let $A = \{a_1, \ldots, a_n\} = A_n$ be a set of variables and $\mathbb{Z}(A)$ the corresponding ring of polynomials. The Schubert polynomials are a $\mathbb{Z}$-basis of $\mathbb{Z}(A)$ (cf. [2]); its elements are indexed (bijectively) by sequences (or vectors) $I = I_1, \ldots, I_n) \in \mathbb{N}^n$ and each Y_I is homogeneous of degree $|I| = I_1 + \cdots + I_n$ = the weight of I. The indexing is such that if $I_{m+1} = I_{m+2} = \cdots = I_n = 0$, the polynomial Y_I does not depend upon $a_{m+1}, \ldots, a_n$ and, in fact, $Y_I = Y_{I'}$, where $Y_{I'}$ is the Schubert polynomial in the variables of $A_m = \{a_1, \ldots, a_m\}$ that is indexed by the restriction $I' = (I_1, \ldots, I_m) \in \mathbb{N}^m$ of I to its m first components. Accordingly, one usually omits the zeroes on the right of a sequence indexing a Schubert polynomial.

We shall use, without proof, two basic facts (we refer to [2, 3]):

PROPOSITION (1.1). *If I is increasing (i.e., if $I_1 \leqslant I_2 \leqslant \cdots \leqslant I_n$), then Y_I is equal to the Schur function $S_{I'}(A_n)$, where I' is the partition of $|I|$ obtained by omitting the initial zeroes in I.*

Thus, for instance, $Y_{00124} = S_{124}(A_5)$.

PROPOSITION (1.2). *Assume that $I, J \in \mathbb{N}^n$ satisfy the following conditions for some m: $I_{m'} = 0$ for every $m' > m$; $J_p = 0$ for any $p \leqslant \mathrm{Max}\{j + I_j : 1 \leqslant j \leqslant m\}$. Then $Y_K = Y_I \cdot Y_J$ where $K = I + J$.*

For instance, $I = 0141$ and $J = 0^7 21 = 000000021$ satisfy these conditions with $m = 4$ since 8 is larger than $1 + 0$, $2 + 1$, $3 + 4$, and $4 + 1$. According to Proposition (1.2), $Y_{014100021}$ is equal to the product of Y_{0141} by $Y_{000000021}$.

In fact, one could rather easily derive the two preceding properties from our transition equations (Section 3) and the determinantal expression of vexillary polynomials (1.4). However, we abstain from doing so because these properties are special cases of a more general type using concepts outside the scope of this Letter.

The only other result that we shall use without proof is Monk's rule giving the product of a Schubert polynomial by a variable. It is given in Section 3, but we already have all the tools needed to describe our argument.

Let I' and J' be partitions. Take $m \geqslant |I'| + |J'|$, $p \geqslant m + 1 + |I'|$, $n \geqslant p^+ |J'|$. By introducing sufficiently many zeroes, we can find $I, J \in \mathbb{N}^n$ such that $Y_I = S_{I'}(A_m)$, $Y_J = S_{J'}(A_n)$ and that the conditions of Proposition (1.2) are satisfied. Therefore, $Y_K = Y_I \cdot Y_J$, where $K = I + J$. Denote by $Y \cap A_m$ the restriction to the variables in $A_m = \{a_1, \ldots, a_m\}$ of any polynomial Y. Equivalently, $Y \cap A_m$ is the value of Y for $a_{m+1} = \cdots = a_n = 0$. We have

$$Y_K \cap A_m = (Y_I \cap A_m) \cdot (Y_J \cap A_m), \tag{1.3}$$

where $Y_I \cap A_m = S_{I'}(A_m) = Y_I$ by definition, and where $Y_J \cap A_m = S_{J'}(A_n) \cap A_m = S_{J'}(A_m)$ since the Schur functions are symmetric in their arguments. The right member of (1.3) can be expressed in a unique manner as the sum of Schubert polynomials $Y_{K'}$, because these polynomials constitute a basis of $\mathbb{Z}(A_m)$. Since the Schur functions themselves are a basis of the ring of symmetric polynomials, the $Y_{K'}$ appearing in this sum are also Schur functions. Therefore, the desired expression of the product $S_{I'} \cdot S_{J'}$ as the sum of Schur function $S_{K'}$ is obtained by computing $Y_K \cap A_m$, i.e., by developing the Schubert polynomial Y_K in $a_n, a_{n-1}, \ldots, a_{m+1}$ and keeping only the terms in which none of these letters appear. An algorithm to do so is given in Section 4. The reader will see that it would suffice for this goal to consider only the maximal ones among what we call *transition* equations. However, we have found it more interesting to treat these equations in full generality, at the cost of proving some technical results in Section 2 which would not otherwise be needed.

Vexillary polynomials are special Schubert polynomials which constitute the simplest generalization of Schur functions. Given a set of variables B, and a partition $K = K_1 \leqslant K_2 \leqslant \cdots \leqslant K_m$, the Schur function $S_K(B)$ has the determinantal expression (cf. [5])

$$S_K(B) = |S_{K_j + j - i}(B)|_{1 \leqslant i, j \leqslant m}. \tag{1.4}$$

We can generalize it to the case of m sets $B^{(1)}, \ldots, B^{(m)}$ of variables by defining

$S_K(B^{(1)}, \ldots, B^{(m)})$ to be the determinant $|S_{K_j+j-i}(B^{(j)})|$. Now, the only Schubert polynomials which are of this last type are the vexillary polynomials. In this case, $B^{(1)}, \ldots, B^{(m)}$ is a *flag*, i.e., $B^{(1)} \supset \cdots \supset B^{(m)}$. As the restriction of $S_K(B^{(1)}, \ldots, B^{(m)})$ to a set A is the Schur function $S_K(A)$ if A is contained in $B^{(1)}, \ldots, B^{(m)}$, *a fortiori*, the restriction of a vexillary polynomial is a Schur function.

There is another notation for Schubert polynomials which simplifies the combinatorial constructions. Let $\mu = \mu_1 \ldots \mu_m$ be a permutation of $\{1, 2, \ldots, m\}$. Define the *code* $\mu\mathbf{L}$ of μ as the vector $\mathbf{L} \in \mathbb{N}^m$ such that, for any i: $1 \leqslant i \leqslant m$,

$$\mathbf{L}_i = \mathrm{Card}\{j > i : \mu_j < \mu_i\}\,. \tag{1.5}$$

Thus, $\mathbf{L}_i \leqslant m - i$ identically and $|\mu\mathbf{L}|$ is the number $ht(\mu)$ of inversions of μ (which some authors call the *length* $l(\mu)$ of μ).

For instance, (41532)**L** = (30210); (14532)**L** = (02210); (41325)**L** = (30100).

Note that if we consider μ as a permutation on $\{1, \ldots, m'\}$ where $m' > m$ by adding fixed points to its right, it has no other effect on the code than to add zeroes on its right. For instance, (4132)**L** = (3010), (41325)**L** = (30100), (413256)**L** = (301000) etc. Note also that with this convention, the correspondence **L** between sequences and permutations is bijective up to adjunction on the right of the zeroes to the sequences, or of fixed points to the permutations. Indeed, given a sequence J, we retrieve the permutation μ such that $\mu\mathbf{L} = J$ by computing successively $\mu_1 = J_1 + 1$, $\mu_2 = J_2 + 1$ or $J_2 + 2$ depending upon $J_2 < J_1$ or not, $\mu_3 = J_3 + 1$, $J_3 + 2$ or $J_3 + 3$ depending upon the inequalities satisfied by J_1, J_2 and J_3, etc. This nice trick to encode permutations is due to A. Lehmer and it is classical among computer scientists.

It is customary to use the notation X_μ to denote the Schubert polynomial $Y_{\mu\mathbf{L}}$. Therefore, $X_{4132}, X_{41325}, X_{413256} \ldots$ and $Y_{201}, Y_{2010}, Y_{20100} \ldots$ are just different names for the same polynomial (which is, in fact, $a_1^2 a_2 + a_1^2 a_3$). With this notation, the set of polynomials X_μ, where μ is a permutation of $\{1, \ldots, n\}$ is for any given n a basis of the module spanned by the Y_I where $I \in \mathbb{N}^n$ identically satisfies $I_k \leqslant n - k$, $1 \leqslant k \leqslant n$.

2. Vexillary Permutations

To each $I = (I_1, \ldots, I_n) \in \mathbb{N}^n$ we associate the sequence $I\mathbf{R}$ obtained by rearranging its coordinates I_k in an increasing order. Therefore, $I\mathbf{R}$ can be identified with a partition of $|I|$.

We recall that the partitions of an integer p are a lattice when they are ordered in the following manner:

$$\begin{aligned} &J = (J_1, \ldots, J_n) > J' = (J'_1, \ldots, J'_n) \quad \text{iff} \\ &J_k + J_{k+1} + \cdots + J_n \leqslant J'_k + J'_{k+1} + \cdots + J'_n \quad \text{for all } k \leqslant n\,. \end{aligned} \tag{2.1}$$

We denote by $\tilde{}$ the *transposition* of the partitions (which is the exchange of rows and columns in the Ferrers diagram representing them). This involution reverses the order, i.e.,

$$J > J' \Leftrightarrow J'^{\sim} > J^{\sim} \tag{2.2}$$

For example, $1125 > 126$ and $111123 > 11124$.

We now turn to the definition of vexillary permutations. Note that the simplest sequence K to admit a (nontrivial) decomposition $K = I + J$, as in Proposition (1.2), is $K = 101$. It is the code of the permutation 2143. We call *vexillary* every permutation μ which does not contain any subpermutation isomorphic to 2143, i.e., such that there does not exist four integers $i < j < h < k$ such that $\mu_j < \mu_i < \mu_k < \mu_h$. It is clear that μ is vexillary iff its inverse μ^{-1} is such.

LEMMA (2.3). *Let μ be permutation, $\mu\mathbf{L}$ its code, $\mu\mathbf{LR}$ the associated partition. The $\mu^{-1}\mathbf{LR} \geqslant \mu\mathbf{LR}^{\sim}$ and there is equality if and only if μ is vexillary.*

Proof. We suppose that the lemma is true for any $\zeta \in \mathfrak{S}_{n-1}$. Take $\mu = \zeta_1 \ldots \zeta_{n-k-1} n \zeta_{n-k} \ldots \zeta_{n-1}$. One checks that μLR is deduced from $\zeta\mathbf{LR}$ by adding a part equal to k, and thus $\mu\mathbf{LR}^{\sim}$ is deduced from $\zeta\mathbf{LR}^{\sim}$ by increasing by one the k largest parts of this last partition.

On the other hand, $\mu^{-1}\mathbf{L}$ is obtained from $\zeta^{-1}\mathbf{L}$ by increasing the k parts by one. If $\zeta^{-1}\mathbf{LR}$ is strictly larger than $\zeta\mathbf{LR}^{\sim}$, then the strict inequality remains valid for μ. Conversely, if ζ is vexillary, and $k \neq 0$, then μ^{-1} contains at least a subpermutation isomorphic to 2143 iff there exist $i, j, h : i < j < h < n$ such that $\mu_i^{-1} > \mu_j^{-1}$, $\mu_i^{-1} < n + 1 - k < \mu_h^{-1}$. Take a subpermutation isomorphic to 2143 for which h is maximal. The inequality $\mu_i^{-1} < n + 1 - k < \mu_h^{-1}$ implies that $x = \zeta^{-1}L_i > \zeta^{-1}L_{h-1}$. Thus, one gets $\mu^{-1}\mathbf{LR}$ from $\zeta^{-1}\mathbf{LR}$ by increasing by one the value $y < x$ without increasing x. This forces the inequality $\mu LR^{\sim} > \mu^{-1}LR$ to be strict.

Finally, if μ is vexillary, one must have $n + k - 1 < \mu_i$ for every triple $i < j < h < n$ such that $\mu_i^{-1} > \mu_j^{-1}$ and $\mu_h^{-1} > \operatorname{Max}\{n + k - 1, \mu_n^{-1}\}$. One checks that in this case, $\zeta^{-1}\mathbf{LR} \to \mu^{-1}\mathbf{LR}$ consists of increasing by one the k largest parts of $\zeta^{-1}\mathbf{LR}$, and thus $\mu\mathbf{LR}^{\sim} = \mu^{-1}\mathbf{LR}$. □

EXAMPLE. Let $\zeta = 361245$; its code is 240000; the inverse ζ^{-1} 341562 has code 220110. The partition 1122 is the transposition of 24; therefore, ζ is vexillary. We indicate μ, μ^{-1}, $\mu\mathbf{L}$, $\mu^{-1}\mathbf{L}$, $\mu\mathbf{LR}$, $\mu^{-1}\mathbf{LR}$ and $\mu\mathbf{LR}^{\sim}$ for the insertion of 7 at the places 2, 3, 4, 5 in ζ:

3761245	4516732	254	330221	245	12233	= 12233
3671245	4516723	244	33022	244	2233	= 2233
3617245	3516724	2403	23022	234	2223	≠ 1233
3612745	3416725	24002	22022	224	2222	≠ 1133 .

The first two permutations are vexillary, contrary to the last two (which contain the subpermutation 3174).

We can see that a permutation is vexillary on its code. But first, we geometrically represent the code of a permutation.

To a permutation μ, we associate the set of points ★, $(1, \mu_1)$, $(2, \mu_2)$, $(3, \mu_3)$, … in the integral plane $\mathbb{N} \times \mathbb{N}$. Each point creates a shadow on its right on the same horizontal, and above on the same vertical. The set of integral points $\diamondsuit$: (i, j), $i, j > 0$, which are left in the light is the *diagram* of the permutation (it generalizes the Ferrers' diagram of a partition). We can recover the code of the permutation by reading the number of $\diamondsuit$ in each column: $\mu\mathbf{L}_i$ is the number of $\diamondsuit$ which have the absciss i. For example, $\mu = 34165278\ldots$ has the diagram given in Figure 1 and the code $220210000\ldots$

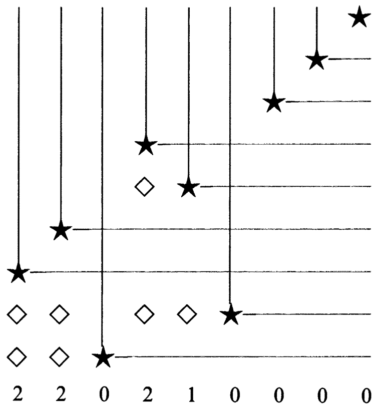

Fig. 1.

Now, it is easy to check that μ does not admit a subpermutation of the type 2143 iff any two columns of the diagram of μ are *comparable for the inclusion*, i.e., given i, j such that $\mu L_i \leqslant \mu L_j$, then for each $\diamondsuit$ of the absciss i, there exists a $\diamondsuit$ of the absciss j on the same horizontal. This condition is, in fact, symmetrical with respect to the symmetry exchanging the axes, i.e., with $\mu \to \mu^{-1}$. Translating the condition in terms of codes, we obtain the following characterization of vexillary permutations.

PROPOSITION (2.4). *The sequence* $K = (K_1, \ldots, K_n) \in \mathbb{N}^n$ *is vexillary, i.e., the code of a vexillary permutation iff*

(1) *if* i *is such that* $K_i > K_{i+1}$, *then* $K_i > K_j$ *for any* $j > i$,

(2) *if* i, h *are such that* $K_i \geqslant K_h$, *then card* $\{j : i < j < h, K_j < K_i\} \leqslant K_i - K_h$.

3. Transitions

We first state without proof Monk's rule giving the multiplicative structure of $\mathbb{Z}(A)$ in terms of Schubert's basis. If μ is a permutation of $\{1, \ldots, n\}$ and if $\tau = \tau_{j,k}$ is the transposition exchanging the values located at the places j and k, one says that τ is *p.p.* on μ iff $ht(\mu\tau) = 1 + ht(\mu)$. It is known that this is equivalent to the two following conditions:

$$\operatorname{sign}(j - k) = \operatorname{sign}(\mu_j - \mu_k) \tag{3.1}$$

$$\text{for each } i \neq j, k \text{ in the interval } [j, k],\ \mu_i \text{ is } \textit{outside} \text{ the interval } [\mu_j, \mu_k]\,. \tag{3.2}$$

MONK'S RULE. *Assume $k < n$ and $\mu_n = n$. Then*

$$X_\mu \cdot a_k = \sum \operatorname{sign}(j-k) \cdot X_{\mu\tau_{jk}} \tag{3.3}$$

where the summation is over all the places j such that $\tau_{j,k}$ is p.p. on μ.

For instance, if $\mu = 41325$, one finds:

$$X_\mu a_1 = X_{51324}\,; \qquad X_\mu a_2 = X_{43125} + X_{42315}\,;$$

$$X_\mu a_3 = -X_{43125} + X_{41523}\,; \qquad X_\mu a_4 = -X_{42315} + X_{41352}\,.$$

The condition $k < n$ and $\mu_n = n$ is sufficient to insure that no new terms would arise in the multiplication if further fixed points were added to the right of μ. For instance, $X_{413256} a_3 = -X_{43126} + X_{415236}$.

We shall call *transition* any relation of the type

$$X_\mu = X_\zeta \cdot a_j + \sum_{\psi \in \Psi} X_\psi \tag{3.4}$$

the summation being on a set Ψ of permutations depending upon j, ζ and μ.

From (3.3), it follows that the product $X_\zeta \cdot a_j$ gives a transition iff

there exists a unique $k > j$ such that $\tau_{j,k}$ is p.p. on ζ (3.5)

(for this k, $\zeta\tau_{j,k} = \mu$ gives the left-hand side of Equation (3.4));

Ψ is the family of permutations $\zeta\tau_{j,i}$ such that $i < j$ and $\tau_{j,i}$ is p.p. on ζ. (3.6)

Graphically (Figure 2), if we represent the permutation μ as the set of points ★ $(1, \mu_1), (2, \mu_2), \ldots$, then condition (3.5) is equivalent to the condition that the area **0** is void. The family (3.6) includes those permutations $\zeta\tau_{j,i}$, $i < j$, for which the area **4** is void. In Ψ, we distinguish the element $\zeta\tau_{j,i}$ for which i is maximal, and call it the *leader*. It amounts to imposing that the area **1** is void. We shall always suppose that Ψ is nonvoid by eventually embedding $\mathfrak{G}_n$ in $\mathfrak{G}_1 \times \mathfrak{G}_n$.

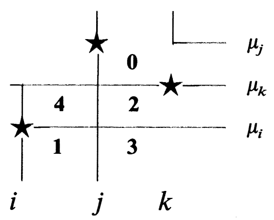

Fig. 2.

EXAMPLE. Let $\mu = 52186347$; there are three transitions:

$$X_\mu = X_{52184367} a_5 + X_{52481367} + X_{54182367}\,,$$

$$X_\mu = X_{52176348} a_4 + X_{52716348} + X_{57126348} + X_{72156348}\,,$$

$$X_\mu = X_{51286437} a_2\,.$$

In this last case, the embedding $\mu \to 163297458$ changes the transition into $X_{163297458} = X_{162397548} a_3 + X_{261397548}$.

REMARK (3.7). Transitions are compatible with the inversion of permutations:

$$X_\mu = X_\zeta a_j + \sum X_\psi \Rightarrow X_{\mu^{-1}} = X_{\zeta^{-1}} a_h + \sum X_{\psi^{-1}},$$

with $h = \mu_j$. Indeed, the geometrical conditions (3.5) and (3.6) are preserved by symmetry (with respect to the principal diagonal), and this symmetry is precisely $\mu \to \mu^{-1}$.

The following lemma shows that transitions are also compatible with the order of partitions. We call the *critical place* of a permutation μ the largest r for which $\mu\mathbf{L}_1 \leqslant \mu\mathbf{L}_2 \leqslant \cdots \leqslant \mu\mathbf{L}_r$.

LEMMA (3.8). *Let* $X_\mu = X_\zeta a_j + \Sigma X_\psi$ *be a transition. Then for every* $\psi \in \Psi$, $\mu\mathbf{LR} > \psi\mathbf{LR}$; *the inequality is strict iff* ψ *is not the leader*; *if* $j > r$ (= *the critical place of* μ), *the critical place of every* $\psi \in \Psi$ *is* $\geqslant r$.

Proof. Let $\psi \in \Psi$, and i be the corresponding integer: $\psi = \zeta\tau_{i,j}$. Let m_i, n_i, p_i be the respective number of stars ★ in the open zones **1**, **2**, **3** of the graphical representation of μ (Figure 2). One easily checks that $\psi\mathbf{L}_h = \mu\mathbf{L}_h$ if $h \neq i, j$ and that

$$\mu\mathbf{L}_i = m_i + p_i, \qquad \mu\mathbf{L}_j = n_i + p_i + 1,$$

$$\psi\mathbf{L}_i = m_i + p_i + n_i + 1, \qquad \psi\mathbf{L}_j = p_i.$$

Let us suppose that the difference $d = \mu\mathbf{L}_i - \mu\mathbf{L}_j$ is $\geqslant 0$, which implies that $m_i > 0$. Then $\psi\mathbf{L}_i - \psi\mathbf{L}_j = d + 2n_i + 1 > 0$ and, thus, $\mu\mathbf{LR} > \psi\mathbf{LR}$, the inequality being strict.

In contrast, let $d < 0$; then $\psi L_i - \psi L_j = \mu L_j - \mu L_i + 2m_i$. The same conclusion holds if $m_i \neq 0$; if $m_i = 0$, which exactly means that ψ is the leader, then $\psi\mathbf{L}$ is deduced from $\mu\mathbf{L}$ by the exchange of $\mu\mathbf{L}_i$ and $\mu\mathbf{L}_j$, an operation which leaves invariant the associated partitions $\psi\mathbf{LR} = \mu\mathbf{LR}$.

We now verify the assertion concerning the critical places. The preceding computations have shown that $\psi\mathbf{L}_s = \mu\mathbf{L}_s$ for each $s \neq i, j$ and that $\psi\mathbf{L}_i > \mu\mathbf{L}_i$. Assume that $j > r$. The fact that the critical place r' of ψ is $\geqslant r$ is clear if $i \geqslant r$, therefore we have only to consider the case when $i < r$. Because of our hypothesis $j > r$, it implies $i + 1 < j$ and $\mu\mathbf{L}_i \leqslant \mu\mathbf{L}_{i+1}$. Since zone **4** is void, this last relation is equivalent with $\mu_{i+1} > \mu_k$. It follows that $\psi_i (= \mu_k) < \psi_{i+1} (= \mu_{i+1})$ and, accordingly, $\psi\mathbf{L}_i \leqslant \psi\mathbf{L}_{i+1}$ concluding the proof. □

COROLLARY (3.9). *For any* $\psi \in \Psi$, $(\mu\mathbf{LR}, \mu^{-1}\mathbf{LR}) > (\psi\mathbf{LR}, \psi^{-1}\mathbf{LR})$; *the inequality is strict iff the cardinal of* Ψ *is different from* 1.

The assertion follows from the fact that ψ and ψ^{-1} are simultaneously leaders only in the case that the cardinal of Ψ is 1.

Transitions are especially simple in the case of vexillary permutations as the following property shows.

LEMMA (3.10). *For a transition such that* μ *is vexillary, then* Ψ *is reduced to one element* ψ *which is vexillary, and* $\mu\mathbf{LR} = \psi\mathbf{LR}$.

Proof. According to Lemma (2.3), $\psi\mathbf{L\breve{R}} > \psi^{-1}\mathbf{LR}$, or equivalently, $\psi^{-1}\mathbf{L\breve{R}} > \psi\mathbf{LR}$ for any $\psi \in \Psi$. On the other hand, Remark (3.7) implies $\psi\mathbf{LR} > \mu\mathbf{LR}$ and $\psi^{-1}\mathbf{LR} > \mu^{-1}\mathbf{LR}$, i.e., $\mu^{-1}\mathbf{L\breve{R}} > \psi^{-1}\mathbf{L\breve{R}}$. It follows that $\mu^{-1}\mathbf{L\breve{R}} > \psi^{-1}\mathbf{L\breve{R}} > \psi\mathbf{LR} > \mu\mathbf{LR}$; μ being vexillary, these four partitions are equal, ψ is also vexillary and from Corollary (3.7), the family Ψ is reduced to one element. □

A special type of vexillary permutation is associated to the Grassmann varieties. We say that μ is a *Grassmann permutation* iff there exists j: $\mu\mathbf{L}_1 \leqslant \cdots \leqslant \mu\mathbf{L}_j$ and $\mu\mathbf{L}_k = 0$ for all $k > j$. Equivalently, μ is Grassmannian iff its critical place r is the largest j for which $\mu\mathbf{L}_j > 0$. In that case (cf. [3]), X_μ is the Schur function which is indexed by the partition $\mu\mathbf{L}_1, \ldots, \mu\mathbf{L}_j$ in the variables $\{a_1, \ldots, a_j\}$. For example, $X_{126934578}$ is the Schur function $S_{35}(a_1, a_2, a_3, a_4)$, because the code of μ is 003500000, and $j = 4$.

From now on, we suppress the freedom of choice and take only the transition (the *maximal* transition) $X_\mu = X_\zeta a_j + \Sigma X_\psi$ for which j is maximal. This number j is precisely the largest one such that $\mu\mathbf{L}_j \neq 0$; call it the *maximal place* of μ.

LEMMA (3.11). *Let $\mu = \mu^{(0)} \to \mu^{(1)} \to \cdots \to \mu^{(p)}$ be a sequence of permutations, where every term $\mu^{(k+1)}$ is a member of Ψ in the maximal transition of $\mu^{(k)}$, and none of the $\mu^{(k)}$ is Grassmannian $(0 \leqslant k \leqslant p)$. Then $p < (m - r)ht(\mu)$.*

Proof. The last statement in Lemma (3.8) shows that $r^{(0)} \leqslant r^{(1)} \leqslant \cdots \leqslant r^{(p)}$, where $r^{(k)}$ is the critical place of $\mu^{(k)}$. Since none of the $\mu^{(k)}$ is Grassmannian, we have identically $m^{(k)} > r^{(k)}$ for $k < p$, where $m^{(k)}$ is the maximal place of $\mu^{(k)}$. Because $\mu^{(k+1)}$ is a term in the maximal transition of $\mu^{(k)}$, we have either $m^{(k+1)} < m^{(k)}$ or $m^{(k+1)} = m^{(k)}\ (=m)$, and then $\mu^{(k+1)}\mathbf{L}_m < \mu^{(k)}\mathbf{L}_m$. □

4. The Algorithm

We associate the following tree to any permutation μ: if μ is vexillary, the tree is reduced to one vertex $\{\mu\}$; otherwise, we take the maximal transition $X_\mu = X_\zeta a_j + \Sigma X_\psi$, draw the edges $[\mu\psi]$ for all $\psi \in \Psi$ and attach to each ψ its tree. This is well defined, as in any sufficiently long sequence of transitions, one meets a Grassmann permutation according to Lemma (3.11). Proposition (2.4) would give a way to deal exclusively with sequences, without using permutations. However, such a procedure is rather clumsy for human computations.

THEOREM. *Let μ be a permutation, r its critical place, m an integer r, $\{\theta\} = \Theta$ the set of end points of the tree of root μ. Then*

$$X_\mu \cap A_m = \sum_{\theta \in \Theta} S_{\theta LR}(A_m)\,.$$

Proof. To build the tree, we have taken the maximal transitions $X_{\mu'} = X_{\zeta'} a_{j'} + \Sigma X_{\psi'}$ for which j' is greater than the critical place of μ'. According to Lemma (3.11), it implies $j' > r$. Thus, for all these transitions

$$X_{\mu'} \cap A_m = \Sigma X_{\psi'} \cap A_m\,,$$

and finally,

$$X_\mu \cap A_m = \sum X_\theta \cap A_m = \sum S_{\theta LR}(A_m)\,. \qquad \square$$

EXAMPLE (Figure 3) (above each permutation, we have written its code):

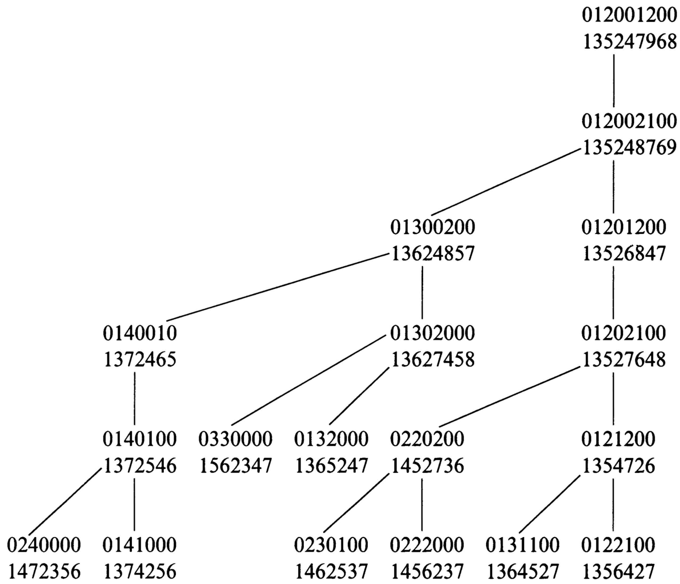

Fig. 3.

The codes of the end points are 024, 0141, 033, 0132, 02301, 0222, 01311, 01221 and, consequently, $S_{12}\cdot S_{12} = S_{24} + S_{33} + S_{114} + 2\,S_{123} + S_{222} + S_{1113} + S_{1122}$, as $Y_{012001200}$ factorizes into $Y_{012}\cdot Y_{000001200}$.

5. Vexillary Classes

From now on, P is a fixed partition and $\mathscr{W} = \mathscr{W}(P)$ is the set of all vexillary sequences K having P as their associated partition, i.e., $P \simeq K\mathbf{R}$ where $\simeq$ means equality up to adjunction or deletion of the zeroes on the left.

The purpose of this section is to show that transitions induce a natural order on $\mathscr{W}$ that turns $\mathscr{W}$ (or more accurately, a certain section $\mathscr{V}$ of $\mathscr{W}$ with respect to the equivalence $\simeq$) into a distributive lattice having the remarkable generating function described in Corollary (5.8).

This requires some definitions. First, the *inversion number* of a sequence $K \in \mathbb{N}^n$ is the number of pairs (i, k) such that $K_i > K_k$ where $1 \leqslant i < k \leqslant m$, m being the maximal place of K.

For example, $\mathrm{Inv}(0353012100) = 4 + 5 + 4 + 0 + 0 + 1 = 14$. It is clear that $\mathrm{Inv}(K)$ does not depend upon the zeroes on the right or on the left of K.

Secondly, to each sequence $K \in \mathbb{N}^n$, we associate another sequence $KD \in \mathbb{N}^n$ by putting, for $1 \leqslant i \leqslant n$:

$$\begin{cases} KD_i = 0 & \text{if } K_i = 0\,, \\ KD_i = \mathrm{Max}\{j\colon i \leqslant j \leqslant n : K_i < K_j\} & \text{otherwise}\,. \end{cases} \tag{5.1}$$

Finally, the 'drapeau' $K\mathbf{DR}$ of K is the rearrangement of $K\mathbf{D}$ (in increasing order). For instance, if $K = 0003530210\ldots0$, one finds $K\mathbf{D} = 0006560890\ldots0$, $K\mathbf{DR} = 0\ldots056689$.

We now give a rather pedestrian description of $\mathcal{W}$. Let p be the largest part of P, r its multiplicity and let P' be the partition obtained by suppressing one part 'p' in P.

REMARK (5.2). *If $r > 1$ there is a bijection between $\mathcal{W}(P)$ and $\mathcal{W}(P')$, each sequence $K \in \mathcal{W}(P)$ being obtained by inserting a new value equal to p to the right of the rightmost value equal to 'p' in any $K' \in \mathcal{W}(P)$.*

If $D' = K'\mathbf{BR}$, one obtains $D = K\mathbf{DR}$ by increasing by one every positive part of D' and then repeating the smallest one.

Proof. It is an immediate consequence of condition (1) in Proposition (2.4) that the r components of K which are equal to p, are located at adjacent places. The result follows. □

EXAMPLE. Let $K' = 000353021$. Its drapeau is $D' = 000056689$. We have $p = 5$, $r - 1 = 1$ and we deduce $K = 0003553021$, having drapeau $D = 00006677\,9\,10$.

We assume now $r = 1$ and let $p' < p$ be the largest component of P' and r' be its multiplicity.

REMARK (5.3). *If $r = 1$, every $K \in \mathcal{W}(P)$ is obtained bijectively from a $K' \in \mathcal{W}(P')$ by inserting a component equal to p either to the right of the rightmost component equal to p' in K', or to the left of this part at a distance $\leqslant p - p' + r'$. The drapeau D is obtained by adding to D' a new least part equal to the place j where the value p has been inserted.*

Proof. It again immediately follows from the two conditions in Proposition (2.4). □

EXAMPLE. Let the new part be $p = 5$ and let $K' = 00440121$. Its drapeau is 00044778. We have $p' = 4$, $r' = 2$. Adding the new part $p = 5$ gives the four following sequences:

K	D
004450121	000555889
004540121	000455889
005440121	000355889
050440121	000255889

We introduce an order on $\mathscr{W}$ by defining its consecutivity relations as follows:

DEFINITION (5.4). If $K, K' \in \mathscr{W}(P) = \mathscr{W}$, K is *immediately above* K' iff, on the one hand, $\operatorname{Inv}(K') > \operatorname{Inv}(K)$ and, on the other hand, $Y_{K'}$ is the leader in a transition equation of Y_K.

As we have seen in Proposition (2.4), this last condition implies that $K' = K\tau_{ik}$ where τ_{ik} is a transposition. More accurately, one has:

REMARK (5.5). *Assuming that K is vexillary, a necessary and sufficient condition that K be immediately above K′ is that* $K' = K\tau_{ik}$ $(i < k)$ *where the following conditions are satisfied*:

(1) *i is the largest place* $<k$ *for which* $K_i < K_k$.

(2) *if k is the maximal place of K, then* K_k *is not the least positive component of K.*

(3) $k = n$ *or* $K_k > K_{k+1}$.

(4) $\operatorname{Card}\{j : i < j < k : K_j < K_i\} \leqslant K_k - K_i$.

Assuming that it is so, K′DR is obtained from KDR by reducing by one unit all the parts of KDR that are equal to k except the last one.

Proof. Again, by induction on the largest part of P using Proposition (2.4) and the explicit description provided by the last two remarks. □

For instance $K = 0081242$ is immediately above the following three sequences K': 0801242, 0081422, 0082241. We have $KDR = 0036777$ and the drapeaux of the above sequences K' are 0026777, 0035777, 0036667.

Let $I = I(P)$ be defined by the condition that $\operatorname{Inv}(I) = \operatorname{Max}\{\operatorname{Inv}(K) : K \in \mathscr{W}(P)\}$ and that $I_1 > 0$. Using the same induction technique, one finds that $I(P)$ can be described symbolically by

$$I = p_s^{r_s} 0^{d_s} p_{s-1}^{r_{s-1}} \ldots 0^{d_2} p_1^{r_1}$$

where $0 < p_1 < p_2 < \cdots < p_s$ are the parts of P, r_i is the multiplicity of p_i and d_i the difference $p_i - p_{i-1}$. This abbreviated notation means, as usual, that the r_s first component of I are equal to p_s, the next d_s ones are equal to 0, etc. Thus $I \in \mathbb{N}^n$ where $n = p_s - p_1 + \Sigma\, r_i$ and the number of components of I which are equal to 0 is $d = p_s - p_1$.

For instance, if $P = 12455$, one finds that $n = 9$, $d = 4$, $I(P) = 550400201 \in \mathbb{N}^9$.

In the opposite direction, let $J = J(P)$ be the sequence in $\mathbb{N}^n$ obtained by prefixing P with d zero components. One has $\operatorname{Inv}(J) = 0$, by definition, and it is readily seen that there exists a chain from $J(P)$ to $I(P)$ in which every term is immediately below the preceding one. Therefore, the interval $\mathscr{V}(P)$ with top element $J(P)$ and bottom element $I(P)$ is a section of $\mathscr{W}(P)$. As an example, we have given in Figure 4 below the diagram corresponding to $\mathscr{V}(1224)$.

We now examine the drapeau of the sequences in $\mathscr{V}(P)$. It follows immediately from the definition that $J(P)\mathbf{DR}$ is the vector $0^d n^{n-d}$ made up of d components 0 and $n - d$ components equal to n. Similarly, one finds that $I(P)\mathbf{DR}$ is a vector $E = E(P) \in \mathbb{N}^{n-d}$ having positive components prefixed by d zero components.

We take as order on the drapeaux the natural one, i.e., we let $D \geqslant D'$ iff $D_i \geqslant D'_i$ for each place i and we introduce the following

DEFINITION (5.6). $\mathscr{U}(P)$ is the set of all drapeaux $D \in \mathbb{N}^{n-d}$ which satisfy the two conditions

(1) $E \leqslant D \leqslant n^{n-d}$,
(2) $D_{i+1} - D_i \leqslant E_{i+1} - E_i$ for all $i < n - d$.

THEOREM (5.7). *The operator* **DR** *establishes an isomorphism from* $\mathscr{V}(P)$ *onto* $\mathscr{U}(P)$.

Proof. That **DR** is a bijection follows from Remarks (5.2) and (5.3), and that it is an isomorphism follows from Remark (5.5). □

It is clear that $\mathscr{U}(P)$ is a distributive lattice, since its definition implies that it is an interval of $\mathbb{N}^{n-d}$ considered as an ordered set, i.e., considered as a direct product of the chains.

COROLLARY (5.8). *The Poincaré polynomial of* $\mathscr{V}(P)$ *is*

$$(1 - q^{(p_s - p_{s-1} + r_{s-1})})(1 - q^{2(p_{s-1} - p_{s-2} + r_{s-2})}) \dots (1 - q^{(s-1)(p_2 - p_1 + r_1)}) / (1 - q)(1 - q^2) \dots (1 - q^{s-1}) = H(q).$$

Proof. Condition (2) in Definition (2.6) implies that if any two adjacent parts E_i and E_{i+1} are equal, the same is true of any drapeau in $\mathscr{V}$. Therefore, $\mathscr{V}(P)$ has the same order structure as $\mathscr{V}(P')$ when P' is the partition such that the corresponding extremal sequence $J(P') = J'$ satisfies the following two conditions:

(i) $J(P')$ has the same maximal place n as $J = J(P)$,

(ii) $J'_i = J_i$ if $i = n$ or if $i < n$ and $J_{i+1} = 0$; otherwise, $J'_i = 0$.

In equivalent fashion, P' is the unique partition with *unequal parts* $0 < p'_1 < p'_2 < \cdots < p'_s$ is such that $p'_{i+1} - p'_i + 1 = p_{i+1} - p_i + r_i$ where $0 < p_1 < \cdots < p_s$ are the distinct parts of P. We assume henceforth that $P = P'$, i.e., that all parts in P are different and, for convenience, we let $m = n - d$.

Let $D = n^m$; since $\mathscr{V}$ is a distributive lattice there is a minimal $D \in \mathscr{V}$ such D is NOT $\leqslant$ the drapeau $D' = (n-1)^{m-1}n$. Call it F and let D'' be the drapeau $n'^{m-1}n$, where n' is the second largest part of E, i.e., $n' = E_{m-1}$.

The intervals $[D, F]$ and $[D'', E]$ are isomorphic because the differences $D - F$ and $D'' - E$ are equal vectors in $\mathbb{N}^m$.

By construction $\mathscr{V}$ is equal to the disjoint union of the intervals $[D, F]$ and $[D', E]$. Set $\overline{D}' = (n-1)^m$ and let $\overline{E}$ be obtained by replacing in E the largest part n by $n - 1$. By induction, we have the explicit expression of the Poincaré polynomial $H_1(q)$ of the interval $[\overline{D}', E]$. In similar manner, if $\overline{D}''$ and $\overline{E}'' \in \mathbb{N}^{m-1}$ are obtained from D'' and E'' by dropping their last part n, we find that $[\overline{D}'', \overline{E}'']$ has the Poincaré polynomial $H_2(q)$. Letting $h = |F| - |E|$ be the height of F in $\mathscr{V}(P)$, we finally obtain the desired result $H = H_1 + q^h H_2$.

EXAMPLE (Figure 4). Let $P = 0001224$. $\mathscr{V}(P)$ is the following set, with the drapeaux written between brackets. The Poincaré polynomial is $(1 - q^5)(1 - q^6)/(1 - q)(1 - q^2)$.

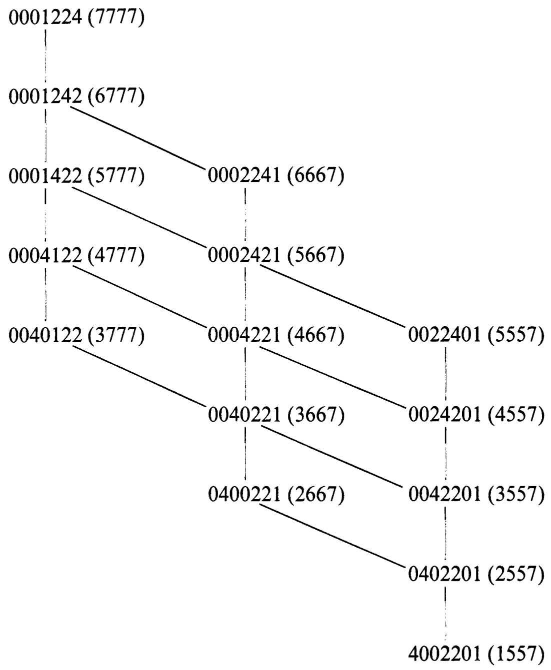

Fig. 4.

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