TABLEAUX AND NONCOMMUTATIVE SCHUBERT POLYNOMIALS

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Schur functions constitute the natural basis of the space of symmetric polynomials. They are the irreducible characters of the symmetric or linear groups. Geometrically, they form a basis of the cohomology ring and correspond to Grassmannians.

Schubert polynomials $\underline{X}_\mu$ [1, 2] are the basis of the ring of polynomials generalizing the above interpretation of Schur functions: they are characters of some cyclic representations of the Borel group of triangular matrices [3] and their classes are the natural basis of the cohomology of the flag manifolds.

Both functions can be lifted to the algebra of the free monoid, as sums (with coefficient 0 or 1) of certain words called tableaux. Combining different interpretations of tableaux, we give in Theorem 5 explicit links between Schubert polynomials, Demazure characters, standard bases, and reduced decompositions of permutations.

Let $A = \{a_1 < a_2 < \dots\}$ be an alphabet, $\mathbb{Z}\langle A\rangle$ the associated free algebra. A word $w = x_1 \dots x_r$ is a <u>column of degree</u> r (noted $|w|$) iff $x_1 > \dots > x_r$, $x_1 \in A$. A column w <u>dominates</u> another one v: $w \gg v$ iff there exists a weakly decreasing injection of the set of letters of v into that of w. A product $t = w_1 w_2 \dots$ of columns is a <u>tableau</u> iff $w_1 \gg w_2 \gg w_3 \dots$ and its <u>shape</u> is the partition $(|w_1|, |w_2| \dots)$.

A tableau t is <u>standard</u> iff it is a permutation of $1, 2, \dots, |t|$. A <u>key</u> is a tableau such that for each k, the column w_{k+1} is a subword of w_k.

We need two congruences on free algebras: the <u>plactic congruence</u> $\equiv$, generated by the elementary relations

$$\left.\begin{array}{lll} \text{(PL1)} & a_k a_i a_j \equiv a_i a_k a_j \text{ and } a_j a_i a_k \equiv a_j a_k a_i \\ \text{(PL2)} & a_j a_i a_j \equiv a_j a_j a_i \text{ and } a_j a_i a_i \equiv a_i a_j a_i \end{array}\right\} \quad i<j<k$$

and the <u>nilplactic congruence</u> $\cong$, which is also given by (PL1) and (PL2), except when i and j are consecutive numbers, in which case we replace (PL2) by (NilPL):

$$\text{(NilPL)} \qquad a_i a_{i+1} a_i \cong a_{i+1} a_i a_{i+1} \text{ and } a_i a_i \simeq 0$$

Schensted's construction for the plactic congruence extends to the nilplastic case [4]:

<u>THEOREM 1.</u> 1) Each plactic class (resp. nilplactic not containing 0) contains one and only one tableau t.

2) The elements of the plactic (resp. nilplactic not containing 0) class of t are in bijection with standard Young tableaux (called insertion tableaux) of the same shape as t.

In fact, the insertion tableau of a word $w = x_1 x_2 \dots$ describes the sequence of shapes of the tableaux congruent to the successive words x_1, $x_1 x_2$, $x_1 x_2 x_3$, ... It is noted $w@$.

Since the plactic congruences are commutation relations, two congruent words $v \equiv w$ have the same image $\underline{v} = \underline{w}$ in $\mathbb{Z}[A]$.

The nilplactic relations being stricter than Coxeter relations (for the simple transpositions), the set of reduced decompositions of any permutation μ is a disjoint union of nilplactic classes [5].

Theorem 1 implies that, given a tableau t of shape I and an integer k there exists one and only one word $u_k t_k$ (resp. $t_k' v_k$) congruent to t such that u_k (resp. v_k) is a column of degree I_k and t_k (resp. t_k') is a tableau of shape $I_1 \dots I_{k-1} I_{k+1} \dots$. Moreover, if $k \geq h$,

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Laboratoire Informatique Théorique et Programmation, Université Paris VII, Paris, France. Published in Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 23, No. 3, pp. 63-64, July-September, 1989. Original article submitted January 10, 1989.

0016-2663/89/2303-0223$12.50

then u_k is a subword of u_h and v_k is a subword of v_h. Therefore the word $u_1u_2u_3\ldots$ is a key (called left key of t) as well as $v_1v_2\ldots$ (right key of t).

Ehresmann [6] has associated to each permutation μ and shape I, a key noted K(μ, I), by taking the sequence of left reordered factors of μ (considered as a word) of successive degrees I_1, I_2, ... For example, μ = 316452 and I = 532 give the key 65431.631.31. The Ehresmann–Bruhat order on the symmetric group $\mathfrak{S}(n)$ is defined by: $\mu \le \zeta$, iff $K(\mu, n\ldots21) \le K(\zeta, n\ldots21)$ componentwise.

Given a key K = K(μ, I), let $\mathfrak{U}$ (K) be the sum of tableaux having K as their right (plactic) key. One can check that the tableaux t appearing in $\mathfrak{U}$ (K) are exactly the tableaux of shape I which are "standard with respect to μ" in the terminology of [7]. Therefore, from [7, Theorem 9.6] we conclude that $\mathfrak{U}$ (K) is the class in $\mathbf{Z}\langle A\rangle$ of the sum of the elements of the "standard basis" of sections of the line bundle associated to I over the Schubert variety of index μ.

Two operators $\bar{\pi}_\mu$ and π_μ on $\mathbf{Z}[A]$ are associated to each permutation [8]. For a simple transposition σ_1 (exchanging a_i and a_{i+1}), $\bar{\pi}_{\sigma_1}$ (abbreviated $\bar{\pi}_1$, and acting on its left) is $f \to (f^{\sigma_1} - f)/1 - a_i/a_{i+1})$ and $\pi_1 = \bar{\pi}_1 +$ identity.

One lifts the operator $\bar{\pi}_1$ to an operator θ_1 on $\mathbf{Z}\langle A\rangle$ in the following manner: if w is a word in a_i, a_{i+1}, then $w\theta_1 = \varepsilon\Sigma v$, iff $\underline{w}\bar{\pi}_1 = \varepsilon\Sigma\underline{v}$ and $v\mathbf{@} = w\mathbf{@}$, putting ε = 1 or −1 whether w has more a_i than a_{i+1} or not [$w\theta_1 = 0$ if $\underline{w} = (a_ia_{i+1})^k$]. This works because words on two letters are determined by their commutative image and their insertion tableau. We make θ_1 act on any word w by applying it in the manner described above, to the subword of w consisting of letters a_i and a_{i+1} and letting invariant the other letters of w.

The operators π_1 and $\bar{\pi}_1$ satisfy Coxeter's relations: this allows one to define π_μ and $\bar{\pi}_\mu$ as products of π_1 or $\bar{\pi}_1$ by taking any reduced decomposition of μ. On the other hand, $\theta_1\theta_2\theta_1 \neq \theta_2\theta_1\theta_2$ and thus, one cannot define operators θ_μ; nevertheless, for dominant monomials $a^I = (a_{I_1}\ldots a_2a_1)(a_{I_2}\ldots a_2a_1)(a_{I_3}\ldots a_2a_1)\ldots$, where I is a partition (i.e., $I_1 \ge I_2 \ge\ldots$), one proves:

Proposition 2. Let $\sigma_1\sigma_j\ldots\sigma_k$ be any reduced decomposition of a permutation μ, and I be a partition. Then $\mathfrak{U}(K(\mu, ()) = a^I\theta_i\theta_j\ldots\theta_k$.

Given a key K, let $\mathfrak{D}$ (K) be the sum $\sum_{H\le K}\mathfrak{U}(H)$ on all keys H having the same shape as K and which are smaller componentwise, and let $\underline{\mathfrak{D}}$ (K) be its commutative image. Using that $\pi_\mu = \sum_{\nu\le\mu}\bar{\pi}_\nu$ one gets:

COROLLARY 3. Let H and K be two keys of the same shape, H ≤ K. Then $\underline{\mathfrak{D}}(K) - \underline{\mathfrak{D}}(H)$ is a polynomial with positive coefficients.

This corollary is a rewriting, in terms of polynomials instead of modules, of propositions 3 and 4 of [9], which are the strongest version of "Demazure character formula" (for GL_n).

Let n be the cardinal (possibly infinite) of A; then $\mathbf{Z}[A]$ is a free $\mathbf{Z}[A]^{\mathfrak{S}(A)}$-module having three natural bases: the monomials a^I, $I \le n-1\ldots10$ (i.e., $I_1 \le n-1$, ..., $I_{n-1} \le 1$, $I_n = 0$), the images of the dominant monomials a^I, $I \le n-1\ldots0$, by the π_μ, $\mu \in \mathfrak{S}(A)$, and the Schubert polynomials $\underline{X}_\mu$ [1]. Accordingly:

$$\underline{X}_\mu = \sum m((\mu, K)\ \underline{\mathfrak{D}}(K) \tag{4}$$

sum on a certain set of keys, with certain multiplicities $m(\mu, K) \in \mathbf{Z}$.

We now define the noncommutative Schubert polynomials X_μ by replacing in (4) $\underline{\mathfrak{D}}$ (K) by $\mathfrak{D}$ (K). Their main characteristic property is the following theorem, denoting the left nilplactic key of a tableau t by K(t).

THEOREM 5. Let μ be a permutation, $\mathfrak{T}$ (μ) the set of tableaux which are reduced decompositions of μ. Then

$$X_\mu = \sum_{t\in\mathfrak{T}(\mu)} \mathfrak{D}(K(t)) \tag{6}$$

The proof is by induction, using the action of the θ_1 on both members of (6). Relation (6) connects our present definition of the X_μ's with the one proposed in [4].

Example. Let $\mu = 21534$; there are two tableaux in $\mathcal{T}(\mu)$: 314 and 134. Since 314 ≅ 341, the factors extracted are 31 and 3 and thus K(314) = 313; 134 is alone in its class and K(134) = 111. Thus $X_{21534} = \mathcal{D}(111) + \mathcal{D}(313) = (a_1a_1a_1) + (a_2a_1a_1 + a_3a_1a_1 + a_2a_1a_2 + a_2a_1a_3 + a_3a_1a_3)$, the sets being the sets of all tableaux with right plastic key $\le a_1a_1a_1$ or $a_3a_1a_3$ respectively.

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