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Dennis Stanton
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Invariant Theory and Tableaux

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KEYS & STANDARD BASES

ALAIN LASCOUX AND MARCEL-PAUL SCHÜTZENBERGER*

1. Introduction. The irreducible characters of the linear group on C {*Schur Functions*} are combinatorially interpreted as sums of *Young tableaux*.

Demazure **[D1]** **[D2]** has given a "Formule des caractères" which interpolates between a dominant weight, corresponding to a partition I, and the Schur function of index I. For every permutation μ, he obtains a "partial" character which can be interpreted as the class of the space of section $\mathcal{V}_{I,\mu}$ of the line bundle associated to I over the Schubert variety of index μ, in an appropriate Grothendieck ring; identifying this ring with the ring of polynomials, we can view $\mathcal{V}_{I,\mu}$ as a polynomial $\mathcal{D}(\mu, I)$.

An independent study of the same spaces $\mathcal{V}_{I,\mu}$, and more precisely, of their "standard bases", is due to Lakshmibai-Musili-Seshadri **[L-M-S]**. Extending the work of Hodge, they interpret Young tableaux as products of *Plücker coordinates* of the flag variety and associate to them *chains* of permutations to describe the different bases (see also **[L-W]**).

The link between the two constructions is not immediate. Moreover, none of these two point of view furnishes the multiplicative structure of sections which is needed in geometry to describe the *postulation* of Schubert varieties. Indeed, the product $\mathcal{V}_{I,\mu}\otimes\cdots\otimes\mathcal{V}_{I,\mu}$ contains more than the sections corresponding to a multiple of the weight I and thus the products of standard bases are not standard bases.

The answer comes from working in the free algebra rather than the Grothendieck ring or the ring of coordinates. Young tableaux (2.1) are now *words* which are representatives of certain congruence classes (th.2.4). More general words (*frank words*, 2.7) obtained from tableaux by permutation allow to associate to each congruence class two special tableaux *right* and *left keys* (2.9). The set of keys (2.12) is in fact the image of the embedding of the symmetric group in the set of tableaux (embedding originally defined by Ehresmann **[E]** to describe how cells attach in a cellular decomposition of the flag variety).

Now, a standard basis is a set (or a sum) of tableaux having the same right key (th.3.6). To generate it, one uses *symmetrizing operators* (3.5) on the free algebra which lift the operators on the ring of polynomial used by Demazure and Bernstein-Gelfand-Gelfand. Thus, the polynomial $\mathcal{D}(I, \mu)$ is just the commutative image of a sum of standard bases (th.3.8).

For what concerns the multiplicative structure of sections, the answer is also given by keys: the product of two tableaux t, t' belongs to a standard basis iff the right key of t is less than the left key of t' (2.11 and 4.2) . This allows us in section 4 to give a combinatorial interpretation of the Hilbert function associated to a weight

*L.I.T.P., Université Paris 7, 2 Place Jussieu, 75251 PARIS Ced 05, France, supported by the P.R.C. Mathématiques-Informatique

as an enumeration of chains of tableaux (th.4.3 and 4.4). See **[L-S6]** for the related order on the symmetric group and its Eulerianity properties. The link between keys, reduced decompositions of permutations and the Schubert cycles (i.e. the classes of the Schubert varieties in the cohomology ring of the flag manifold) is given in **[L-S4]**.

In section 5, we explicit some different ways of describing the standard bases.

In appendix 6, we have isolated a property of actions of the symmetric group which is of independent interest.

Caution. As usual, operators operate *on their left.*

2. Frank words and keys. Let A^* be the free monoid generated by the alphabet $\mathsf{A} = \{a_1 < a_2 < ...\}$. A word $v = x_1...x_r (x_i \in \mathsf{A})$ is called a *column* iff $x_1 > ... > x_r$ and a *row* iff $x_1 \leq x_2 \leq \cdots \leq x_r$. Let V denote the set of all columns. Every word $w \in \mathsf{A}^*$ admits a unique factorisation as a product of a minimal number of columns : $w = v_1 v_2 \cdots v_k (v_i \in \mathsf{V})$. We shall call it the *column factorisation* of w and denote it occasionally by $w = v_1 \cdot v_2 \cdot \ldots \cdot v_k$, v_1 being the *left column* $w\pounds$ of w and v_k the *right column* $w\$$ of w . The *shape* of w is the sequence $\|w\| = (|v_1|, ..., |v_r|)$ of the degrees (or lengths) of the column factors of w.

To use a traditional term (see **[Mc]**), $\|w\|$ is a *composition* of the integer $|w|$ and the $|v_i|$ are the parts of $\|w\|$. On the set of compositions, one has the following preorder: $I \geq J$ iff for every k, the sum of the k biggest parts of I is bigger than the sum of the k biggest parts of J. It is clear that if $I \geq J$ and $J \geq I$, then I is a permutation of J , and that if H is any composition, $IH \geq JH \Leftrightarrow I \geq J$. One can imbed the set of compositions into the set of words : $I = (1I, 2I, ..., rI) \rightarrow (1I \cdots 1)\big((1I+2I) \cdots (1I+1)\big) \cdots \big((1I+2I+\cdots+rI) \cdots (1I+2I+\cdots+(r-1)I)\big)$. We note this word $I\mathrm{M}$ and call it a *composition word*. It can be looked as the maximal element (as a permutation) of the Young group $\mathfrak{S}_I \hookrightarrow \mathfrak{S}_{1I+\cdots+rI}$. For instance, the composition word $(2,4,1)\mathrm{M}$ is $(21)(6543)(7)$, which is the maximal element of the subgroup $\mathfrak{S}_2 \times \mathfrak{S}_4 \times \mathfrak{S}_1$ of $\mathfrak{S}_7$.

Taking the underlying set of a column defines a bijection $v \rightarrow \{v\}$ between the set V of the columns and the family 2^{A} of the subsets of A ; one extends to V the order $\leq$ on A by letting $u \leq v$ iff there is an increasing injection of $\{u\}$ into $\{v\}$. Thus $u \leq v$ is the least order on V that contains both the inclusion order $\{u\} \subset \{v\}$ and the term to term order between equipotent subsets of A.

DEFINITION 2.1. A *contretableau* is a word which is an increasing product of columns.

For instance, if $\mathsf{A} = \{1 < 2 < 3 < \cdots\}$, the word 2 3 41 421 is a contretableau because of $2 \leq 3 \leq 41 \leq 421$.

It is convenient to define another order $\triangleright$ on V by letting $u \triangleright v$ iff there is a decreasing injection of $\{v\}$ into $\{u\}$ and to call a *tableau* any product $u_1 u_2 \cdots u_k$ where the columns u_i are decreasing for $\triangleright$, i.e. $u_1 \triangleright u_2 \triangleright \cdots \triangleright u_k$. For instance, 321 31 2 4 is a tableau because $321 \triangleright 31 \triangleright 2 \triangleright 4$.

The two orders $\leq$ and $\triangleright$ are linked by the fact that $u \triangleright v$ iff $v' \leq u'$ where $v \to v'$ is the bijection on V induced by the "reversal" of the alphabet A assumed to be finite (i.e. by the morphism $a_i \to a_{n+1-i}$, n being the cardinal of A).

It follows from the definitions that the shape of any tableau t is a partition of its degree. This suggests writing the letters of t into the boxes of the Ferrers diagram of $\|t\|$. For instance, 321 31 2 4 can be represented by $\begin{matrix}3\\23\\1124\end{matrix}$. A similar remark holds for contretableaux and we can represent 2 3 41 421 by $\begin{matrix}2344\\12\\1\end{matrix}$. It is a direct consequence of the definition of the orders $\leq$ and $\triangleright$ that each row of the planar writings of tableaux or contretableaux is a weakly increasing sequence of letters.

There is essentially one natural congruence $\equiv$ on A^* that admits as a section the set of contretableaux, where *natural* means that it commutes with the order preserving injections of alphabets. It is called the *plactic congruence*. As shown by D. Knuth, it can be defined by the following identities where a, b, c are any three letters of A such that $a < b < c$:

(2.2) $$baa \equiv aba \; ; \; bba \equiv bab \; ; \; cab \equiv acb \; ; \; bca \equiv bac$$

As it has been observed by **[K-L]**, these generating congruences are exactly all the pairs of words of degree 3 that are not a column nor a line and that differ by the transposition of two adjacent letters. Thus one member of each pair is a tableau (e.g. $\begin{matrix}b\\ac\end{matrix}$ or $\begin{matrix}b\\aa\end{matrix}$) and the other one, a contretableau (e.g. $\begin{matrix}bc\\a\end{matrix}$ or $\begin{matrix}ab\\a\end{matrix}$) .

It turns out that the set of all tableaux is also a section of the plactic congruence and the defining relations (2.2) are simply the expression of this fact for the words of degree 3.

Another remarkable property of the plactic congruence is that every tableau (or contretableau) is congruent to the word obtained when reading by rows (from top to bottom) its planar representation. For instance, 321 31 2 4 $\equiv$ 3 23 1124 or 2 3 41 421 $\equiv$ 2344 12 1.

This phenomenon is closely tied with another definition of the plactic congruence $\equiv$ as one of the least congruences on A^* such that the subalgebra of $\mathbb{Z}(\mathsf{A}^*/\equiv)$ generated by the (non commutative) symmetric sums

$$\Lambda_p = \sum \{v : v \in V \;, |v| = p\} \;,\; p = 1, 2,$$

is a commutative algebra isomorphic to the usual algebra of symmetric polynomials in the letters of A .

To obtain the complete characterisation of the plactic congruence, one needs to add a further condition which follows immediately from 2.2 and which will be used later.

PROPOSITION 2.3. *The plactic congruence $\equiv$ is the least congruence on A^* for which $\Lambda_1\Lambda_2 \equiv \Lambda_2\Lambda_1$, and which moreover satisfies for any interval B of A the relation*

$$w \equiv w' \Rightarrow w \cap \mathsf{B}^* \equiv w' \cap \mathsf{B}^*$$

($w \cap \mathsf{B}^*$ *denotes the word obtained by erasing the letters not in* B).

Taking B equal to a single letter, 2.3 implies that the plactic congruence commutes with the natural morphism $w \to \underline{w}$ of A^* onto the free commutative monoid; this can also be directly checked on relations 2.2.

The plactic congruence is no other than the algebraic formalization of Schensted's construction, whose main result can be summarized in the following theorem (**[Sche]**, **[L-S1]**).

THEOREM 2.4. 1) *Each plactic class contains a unique tableau* t *and a unique contretableau.*

2) *The elements of the class of* t *are in bijection with the set of permutation tableaux (called* ***insertion tableaux****) of the same shape as* t .

By a permutation tableau, we mean, of course, a permutation (of any alphabet) which is at the same time a tableau. Given any word w, we denote $w\mathbf{R}$ the tableau congruent to it and $w@$ its insertion tableau. It is well known (see **[Schu]**) that the involution $w \to w^{-1}$ on permutation words corresponds to the exchange of $w\mathbf{R}$ and $w@$; we shall not use this fact.

More explicitely, the insertion tableau (which is the Q-symbol of Schensted) of a word $w = x_1 x_2 ...$ describes the increasing sequence of the shapes of the tableaux $x_1\mathbf{R}$, $x_1x_2\mathbf{R}$, $x_1x_2x_3\mathbf{R}$,... . The particular choice of the alphabet being irrelevant, $\substack{2\\13}$, $\substack{6\\28}$ and $\substack{\beta\\\alpha\gamma}$, with $\alpha < \beta < \gamma$, must be considered as the same insertion tableau representing the sequence of shapes $\varnothing \to \diamond \to \substack{\diamond\\\diamond} \to \substack{\diamond\\\diamond\diamond}$.

More generally, any word congruent to $w@$ will be called an *insertion word* for w. Insertion words are compatible with restriction of alphabets (see **[L-S1]**):

LEMMA 2.5. *Given any word* $w = x_1 ... x_{m-1} x_m ... x_{m+r} x_{m+r+1} ...$, *then the word* $w@ \cap \{m, ..., m+r\}$ *is an insertion word for the factor* $x_m \cdots x_{m+r}$.

In particular, as pointed out by Schensted, $w@$ contains the subword $m\ m+1$ iff $x_m \le x_{m+1}$ and the subword $m+1\ m$ iff $x_m > x_{m+1}$. Call *file* of a permutation of $\{1, 2, 3 \dots\}$ any maximal subword of the type $(m+k) \cdots (m+1)m$. The shape $\|w\|$ corresponds to the files of any insertion word for w. More precisely, one has the following lemma:

LEMMA 2.6. *Let* $w = v_1 \cdot \ldots \cdot v_k$ *be a word,* μ *an insertion word for* w . *Then*

1) *The files of* μ *are the same as those of the composition word* $\|w\|\mathbf{M}$.
2) $\|w\| \le \|w\mathbf{R}\|$; *equality happens iff* $\|w\|\mathbf{M}$ *is an insertion word for* w.
3) *For each permutation* J *of the shape* $\|w\mathbf{R}\|$, *there exists one and only one word of shape* J *congruent to* w.

Proof. Assertion *1)* is a direct corollary of 2.5: the files of μ are the same as the files of $w@$ and they encode exactly the inequalities $x_i \le x_{i+1}$ or $x_i > x_{i+1}$ for all the pairs of adjacent letters in w . For what concerns *2)*, it is easy to check that the tableau $\mu\mathbf{R}$ has shape greater than the composition corresponding to the files of μ ; this composition being $\|w\|$ and the shape of $\mu\mathbf{R} = w@$ being equal to that of $w\mathbf{R}$, we get the required inequality. In the case of equality, the tableau $\mu\mathbf{R}$ is determined by its files : consecutive entries in a file must be in consecutive rows of $\mu\mathbf{R}$. A mild intimacy with the jeu de taquin shows that this last condition is equivalent to requiring that $\|w\|\mathrm{M} \equiv \mu\mathbf{R}$. Finally, condition *3)* is a rewriting of the case where $\|w\|$ is a permutation of $\|w\mathbf{R}\|$; we just saw that in this case the insertion tableau $w@$ is uniquely determined, which means, thanks to the bijection 2.4.*2* that w is uniquely determined. □

For example, $w = 53 \cdot 61 \cdot 2 \cdot 4$ has shape $2211 \leq 321 = \|w\mathbf{R}\|$; the sequence of tableaux congruent to the left factors of w : $\emptyset \longrightarrow 5 \longrightarrow \begin{smallmatrix}5\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&\\3&6\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&\\3&\\1&6\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&\\3&6\\1&2\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&&\\3&6&\\1&2&4\end{smallmatrix}$ shows that $w@ = \begin{smallmatrix}4&&\\2&5&\\1&3&6\end{smallmatrix}$; w admits the insertion word $\mu = 452361$, since $\mu \equiv w@$; the files of μ are 21, 43, 5, 6 and are identical to those of the composition word $\|w\|\mathrm{M} = 21\ 43\ 5\ 6$. On the other hand, in the same congruence class, we have a unique word w' of shape 213; it is determined by its insertion tableau congruent to the composition word $213\mathrm{M} = 21\ 3\ 654 \equiv \begin{smallmatrix}6&&\\2&5&\\1&3&4\end{smallmatrix}$. Indeed, $w' = 51\ 3\ 642$ as we can check from the sequence of tableaux congruent to its left factors : $\emptyset \longrightarrow 5 \longrightarrow \begin{smallmatrix}5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&\\1&3\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&&\\1&3&6\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&6&\\1&3&4\end{smallmatrix} \longrightarrow \begin{smallmatrix}5&&\\3&6&\\1&2&4\end{smallmatrix}$. The words corresponding to the other permutations of 321 are given next page.

The preceding lemma has detached in the congruence class of a tableau t, the set of those words w (among which the tableau and the contretableau) for which $\|w\|$ is a permutation of $\|t\|$:

DEFINITION 2.7. A word w is *frank* iff $\|w\|$ is a permutation of $\|w\mathbf{R}\|$.

Equivalently, thanks to 2.6.*2*, a word w is frank iff it admits the composition word $\|w\|\mathrm{M}$ as an insertion word.

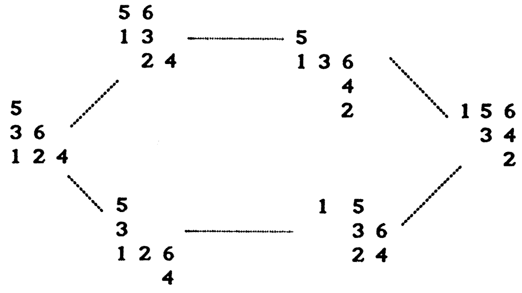
For a two-columns tableau t, finding its congruent contretableau t' can be considered as using the generator $\flat$ of the symmetric group $\mathfrak{S}(2)$ to transpose the two columns of t . This is best done with the jeu de taquin ([**L-S1**]) : $\boxed{\begin{smallmatrix}4\\3\\1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}6\\2\end{smallmatrix}} \Rightarrow \boxed{\begin{smallmatrix}4\\1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}6\\3\\2\end{smallmatrix}}$. We shall write $t' = t^\flat$ and $t = t'^\flat$. Notice that $t\$ = 62$ is a subword of $t'\$ = 632$ and that $t'\pounds = 41$ is a subword of $t\pounds = 431$.

More generally, on the set of k-columns words, one has an action (not everywhere defined; we use the symbol $\emptyset$ when it is not defined) of the symmetric group $\mathfrak{S}(k)$. First, if the factor $v_r v_{r+1}$ of $w = v_1 \cdot \ldots \cdot v_k$, $v_i \in \mathbf{V}$, is a tableau or a contretableau, then the image of w by the simple transposition σ_r , $1 \leq r < k$, is set equal to $v_1 \cdots v_{r-1}(v_r v_{r+1})^\flat v_{r+2} \cdots v_k$ if moreover this last word has still k columns. In all other cases, the image of w by σ_r is set equal to $\emptyset$. It is checked in section 6 that this extends to an action of the symmetric group for which frank words play a special rôle that we summarize in the following theorem (*1* and *2* being a rewriting of 2.6.*2* and 2.6.*3*):

THEOREM 2.8.

1) *For each word* w *, one has* $\|w\| \geq \|w\mathbf{R}\|$ *, with equality iff* w *is frank.*
2) *The set of frank words in the plactic class of a tableau* t *is in bijection with the set of permutations of the shape of* t *.*
3) *The product of two frank words* w *,* w' *is frank iff* $u\$.u'\pounds$ *is frank for any pair of frank words* u, u'*, with* $u \equiv w$ *and* $u' \equiv w'$ *.*

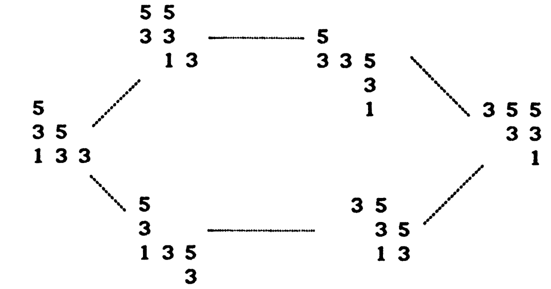
For example, the class of 531 62 4 contains the six frank words (read vertically!) which correspond to the six permutations of the shape 321 :



On the other hand, the product of the two frank words 31 42 and 4 51 is not frank: the insertion tableau of 31 42 4 51 is 721 43 5 6, which is not congruent to 21 43 5 76. Indeed, 4 51 $\equiv$ 41 5, and condition 3) is violated, because $42 \cdot 41$ is not frank.

We now come to the study of keys.

By definition, a *key* is a tableau such that its columns are pairwise comparable for the inclusion order. This condition implies that the action of the symmetric group giving the frank words is simply the permutation of the columns because this is true (and easily verified) in the special case of two-columns keys, where the operation $\flat$ reduces to just commutation. For example, 531 53 3 is a key and the frank words in its congruence class are



DEFINITION 2.9. The *right key* $t\mathbf{K}_+$ of a tableau t (or of any word congruent to t) is the tableau of the same shape as t whose columns belong to the set of columns $\{u\$\ ,\ u \equiv t \text{ and } u \text{ frank}\}$. The *left key* $t\mathbf{K}_-$ of t is the tableau of the same shape as t whose columns belong to the set $\{u\pounds,\ u \equiv t \text{ and } u \text{ frank}\}$.

In other words, the left key (resp. right key) of t is made of the left (resp. right) columns, repeated the appropriate number of times so as to fill the shape of t, of *the frank words in the class of* t . For instance, the above hexagon for the tableau 531 62 4 give the keys $t\mathbf{K}_- = \begin{smallmatrix}5\\3&5\\1&1&1\end{smallmatrix}$ and $t\mathbf{K}_+ = \begin{smallmatrix}6\\4&6\\2&4&4\end{smallmatrix}$. *Notice that a tableau is* a key iff it is equal to its right (resp. left) key. In other case, the keys of a tableau belong to different plactic classes.

Since the test that the product of two frank words w, w' is frank involves exactly the columns composing $w\mathbf{K}_+$ and $w'\mathbf{K}_-$, we can reformulate th.2.8:

THEOREM 2.10. *1) A word w is frank iff $\|w\mathbf{R}\|$ is a permutation of $\|w\|$.*
2) A product ww' of two frank words w, w' is frank iff the shape of $ww'\mathbf{R}$ is the union of the shapes $\|w\mathbf{R}\|$ and $\|w'\mathbf{R}\|$.
3) A product ww' of two frank words w, w' is frank iff $(w\mathbf{K}_+)(w'\mathbf{K}_-)$ is frank.

If a pair of columns satisfies $u \leq v$, then $u' \leq v'$ for any other pair of columns u', v' such that $\{u'\} \subseteq \{u\}, \{v'\} \supseteq \{v\}$; similarly, $u \triangleright v$ implies $u' \triangleright v'$ for any pair such that $\{u'\} \supseteq \{u\}$, $\{v'\} \subseteq \{v\}$. Thus, in the special case of two frank words w, w' having the same shape up to a reordering, condition *3)* can be restricted to the comparison of columns in $w\mathbf{K}_+$ and $w'\mathbf{K}_-$ of the same length instead of all pairs of columns (as required by 2.8.*3*). Recall that for columns of the same length, the order $\leq$ is the componentwise order on words of the same degree (that we can denote by the same symbol $\leq$). In short, one can replace in that case 2.10.*3* by:

THEOREM 2.11. *Assume that w, w' are two frank words such that $\|w\mathbf{R}\| = \|w'\mathbf{R}\|$, then ww' is frank iff $w\mathbf{K}_+ \leq w'\mathbf{K}_-$.*

For example, the product of the two tableaux $w =$ 421 41 3 and $w' =$ 432 32 4 is not frank ; condition *2)* is violated since (421 41 3)(432 32 4)$\mathbf{R}$ = 421 431 432 3 4 is a tableau of shape (= 33311) different from 332211. In fact, 421 41 3 $\cong$ 421 1 43 and 432 32 4 $\cong$ 32 432 4, but $(43\ 32)^\flat = \emptyset$, and thus condition *3)* of the theorem is violated. Condition 2.11 has the same fate, since 431 43 3 $(\ = (421\ 41\ 3)\mathbf{K}_+)$ is not smaller than 432 32 3 $(\ = (432\ 43\ 4)\mathbf{K}_-)$.

On the other hand, 421 31 3 432 32 4 $\mathbf{R}$ = 421 431 32 32 3 4 has shape 332211, as is insured by the inequality 431 31 3 $(= (421\ 31\ 3)\mathbf{K}_+) \leq$ 432 32 3 $(= (432\ 43\ 4)\mathbf{K}_-)$ required by 2.11.

Definition 2.12. *Key of a permutation*: to each pair consisting of a permutation $\zeta \in \mathfrak{S}(n)$, and a partition $I = (1I, 2I, \dots)$, Ehresmann [E] has associated a key, noted $\mathbf{K}(\zeta, I)$, by taking the sequence of left reordered factors of ζ (considered as a word) of successive degrees $1I, 2I, \dots$.

For example, $\zeta = 316452$ and $I = 532$ give the key 65431 631 31.

In case that $I = n...21$, we shall simply write $\mathbf{K}(\zeta)$ instead of $\mathbf{K}(\zeta, n...21)$; thus $\zeta \rightarrow \mathbf{K}(\zeta)$ is an embedding of $\mathfrak{S}(n)$ into the set of tableaux of shape $n...21$. The reader may notice that the so-called "strong", "Bruhat order" on permutations (see **[Bj]**) is a special case of the pervading order on words (componentwise) which we have been repeatedly using, by way of the equivalence due to Ehresmann :

$$\eta \leq \zeta \quad \Leftrightarrow \quad \mathbf{K}(\eta) \ \leq \ \mathbf{K}(\zeta) \tag{2.13}$$

For example, the keys associated to $\zeta = 3241, \eta = 2143$ and $\mu = 1423$ are $\mathbf{K}(\zeta) = 4321\ 432\ 32\ 3$, $\mathbf{K}(\eta) = 4321\ 421\ 21\ 1$, $\mathbf{K}(\mu) = 4321\ 421\ 41\ 1$; thus $\zeta \geq \eta$, but ζ and μ are not comparable since the two columns 32 and 41 are not comparable.

3. Symmetrizations. The definition of tableaux is strictly dependent upon a chosen total order on $\mathbb{A}$. It is remarkable that nonetheless the commutative image of the sum S_I of all tableaux of a given shape I be a symmetrical function: this is the most constructive definition of the *Schur Function* of index I. To understand this phenomenon (see nevertheless Knuth's proof **[B-K]**), one must define an action of the symmetric group on the free algebra such that the S_I are invariant under this action. Further, this action must induce the usual action of the symmetric group when projected by $w \rightarrow \underline{w}$ on the commutative algebra. By the duality $w \rightarrow w^{-1}$ for permutation words, the new action we shall define now can be specialized to give the action that we have been using in our study of frank words.

Consider first the case of a two-letters alphabet $\mathbb{A} = \{a, b\}$. It is clear that the image by the transposition $\sigma = \sigma_{ab}$ of the tableau $t = (ba)^h a^k b^m$ must be $t^\sigma = (ba)^h a^m b^k$, since t^σ is the only tableau of the same shape as t whose commutative image is the monomial $a^{m+h} b^{h+k}$.

More generally, because words w in a, b are determined by their insertion tableau $w@$ and their commutative image $\underline{w}$ (we recover $w\mathbf{R}$ from its content $\underline{w}$ and its shape, equal to that of $w@$), one defines w^σ to be the word:

$$(w^\sigma)@ = w@ \quad \& \quad (\underline{w})^\sigma \ = \underline{(w^\sigma)} \tag{3.1}$$

In other terms σ , as it has been defined, preserves the insertion tableau and commutes with the projection $\mathbb{Z} < a, b > \rightarrow \mathbb{Z}[a, b]$.

For example, the image by σ of the word *baa* **a** *bbaa* **aa** *b* is *baa* **b** *bbaa* **bb** *b* (we have marked the letters which change) because these two words have the same insertion tableau, equal to $\begin{smallmatrix}1 & 6 & 7 \\ 0 & 2 & 3 & 4 & 5 & 8 & 9 & X\end{smallmatrix}$, and they project onto $a^7 b^4$ and $a^4 b^7$ in $\mathbb{Z}[a, b]$.

Since the column ba commutes (plactically) with a and b , shifting the factors ba of a word w in a, b generates the congruence class of w. This remark implies the following easy algorithm to compute $w \rightarrow w^\sigma$:

$$\left\{ \begin{array}{l} \text{fix the successive factors } ba \text{ of } w, \text{ then change the remaining} \\ \text{subword } a^k b^m \text{ into } a^m b^k. \end{array} \right. \tag{3.2}$$

For instance the preceding word gives $(ba)\ aa\ \big(b(ba)a\big)\ aab$ and we have to change the remaining word $aa\ aab$ into $ab\ bbb$ to get $(ba)\ ab\ \big(b(ba)a\big)\ bbb\ =\ w^{\sigma}$.

Consider now the more general case of a simple transposition σ_i of consecutive letters a_i, a_{i+1} . One defines w^{σ_i} to be the word in which the subword $w \cap \{a_i, a_{i+1}\}$ has been modified according to 3.1 or 3.2, the other letters being left unchanged. For example, denoting by $x...x$ any word in letters different from a and b, the above computation shows that the image of $bxaxaxaxbxbxaxaxaxaxb$ is $bxaxax\mathbf{b}xbxbxaxax\mathbf{b}x\mathbf{b}xb$.

It is proven in **[L-S1]** that $w \longrightarrow w^{\sigma_i}$ extends to an action of the symmetric group on $\mathbf{Z}\langle\mathbb{A}\rangle$, i.e. that given a permutation μ and a word w, all factorizations of $\mu = \sigma\ \sigma' \dots\ \sigma''$ into simple transpositions produce the same word $\big((w^{\sigma})^{\sigma'} \dots\big)^{\sigma''}$ denoted w^{μ}.

One can note in reference to a previous remark that in section 2, we have acted on the insertion words to generate from a tableau the frank words which are congruent to it, thus preserving $w\mathbf{R}$, and that the action described here preserves $w@$.

At the commutative level, on $\mathbf{Z}[\mathbb{A}]$, we have at our disposal other actions of the symmetric group $\mathfrak{S}(\mathbb{A})$ (see **[L-S 2]**).

In particular, two operators $\overline{\pi}_{\mu}$ and π_{μ} on $\mathbf{Z}[\mathbb{A}]$ are associated to each permutation μ. For a simple transposition σ_i the operator $\overline{\pi}_{\sigma_i}$ (abbreviated $\overline{\pi}_i$, and acting as always on its left) is

$$(3.3)\qquad f \longrightarrow \left(f^{\sigma_i} - f\right) \Big/ \left(1 - a_i/a_{i+1}\right) = f\overline{\pi}_i$$

and the operator π_i is just the sum of $\overline{\pi}_i$ and the identity :

$$(3.4)\qquad \pi_i = \overline{\pi}_i + 1$$

Let $\underline{w} = \underline{v}a_i^k\ \in\ \mathbf{Z}[\mathbb{A}]$, with $\underline{v}$ symmetrical in a_i and a_{i+1}.

Then direct computation gives

$$\underline{w}\pi_i\ =\ \underline{v}a_i^k + \underline{v}a_i^{k-1}a_{i+1} + \cdots + \underline{v}a_{i+1}^k$$

i.e. $\underline{w}\pi_i$ is the sum of all monomials between $\underline{w}$ and $\underline{w}^{\sigma_i}$, and $\underline{w}\overline{\pi}_i$ is the same sum apart from the first term ($=w$) missing.

This indicates how we can lift $\overline{\pi}_i$ into an operator, denoted θ_i, on the free algebra. Given i and a word w, let its degree in a_{i+1} be m and its degree in a_i be $m+k$. Then w and w^{σ_i} differ by the exchange of a subword $\mathbf{a}_i^k$ into $\mathbf{a}_{i+1}^k$ if $k \geq 0$, or of $\mathbf{a}_{i+1}^{-k}$ into $\mathbf{a}_i^{-k}$ if $k \leq 0$.

In the first case, we define $w\theta_i$ to be the sum of all words in which the subword $\mathbf{a}_i^k$ has been changed respectively into $\mathbf{a}_i^{k-1}\mathbf{a}_{i+1}, \mathbf{a}_i^{k-2}\mathbf{a}_{i+1}^2, \cdots, \mathbf{a}_{i+1}^k$; in the second, we put $w\theta_i = -\big(w^{\sigma_i}\big)\theta_i$ as in the commutative case. In other terms, θ_i interpolates between the identity and σ_i for the words having more occurences of a_i than of a_{i+1}. The corresponding algorithm is in this case ($k \geq 0$), putting $a_i = a, a_{i+1} = b, \theta = \theta_i$:

$$(3.5)\quad \begin{cases} \text{fix the successive factors } ba \text{ of } w \text{ , then change the remaining} \\ \text{subword } a^{m+k}b^m \text{ into successively } a^{m+k-1}b^{m+1}, a^{m+k-2}b^{m+2}, \dots, \\ a^m b^{m+k} \text{ and take the sum of all the words so obtained.} \end{cases}$$

For instance, for the word studied in 3.2, we have $(ba)aa\big(b(ba)a\big)aab\theta = (ba)aa \big(b(ba)a\big)abb + (ba)aa\big(b(ba)a\big)bbb + (ba)ab\big(b(ba)a\big)bbb$.

More generally, we can transform w by changing its subword $a^{m+k}b^m$, $k \in \mathbf{Z}$, into any row $a^r b^{2m+k-r}$ of the same degree. This operation will preserve the insertion tableau, as does σ_i (which is a special case). In particular, we shall need the projection of $a^{m+k}b^m$ onto a^{2m+k}, $k \in \mathbf{Z}$, that we shall denote λ (and λ_i for the pair of letters a_i, a_{i+1}):

$$(3.6)\quad \begin{cases} \text{fix the successive factors } ba \text{ of } w \text{, then change the remaining} \\ a^{m+k}b^m \text{ into } a^{2m+k} \text{ to obtain } w\lambda. \end{cases}$$

Since $\sigma_i = \sigma, \theta_i = \theta, \lambda_i = \lambda$ preserve the insertion tableau, they are also compatible with the right and left keys : if w is a frank word congruent to t, then w^σ and $w\lambda$ are also frank, and $w\theta$ is a sum of frank words; $w^\sigma\$$ and $w\lambda\$$ are equal to $w\$^\sigma$ or $w\$$. Thus $t^\sigma\mathbf{K}_+$, $t\lambda\mathbf{K}_+$ and $t'\mathbf{K}_+$, with any t' in the sum $t\theta$, are equal to $t\mathbf{K}_+$ or $(t\mathbf{K}_+)^\sigma$. We shall give a more precise statement in theorem 3.8.

The operators θ_i do not satisfy the Coxeter relations $\theta_i\theta_{i+1}\theta_i = \theta_{i+1}\theta_i\theta_{i+1}$, contrary to the operators $\overline{\pi}_i$, π_i and λ_i; thus, if $\sigma_i \cdots \sigma_j$ and $\sigma_h \cdots \sigma_k$ are two reduced decompositions of the same permutation, the operators $\theta_i \cdots \theta_j$ and $\theta_h \cdots \theta_k$ will in general be different and there is no canonical way of defining operators θ_μ by products of operators θ_i.

Nevertheless, we recover this lost Coxeter relation when acting on dominant monomials, as we shall see in 3.8.

DEFINITION 3.7. The *standard basis* $\mathfrak{U}(\mu, I)$ associated to the pair μ, I (μ permutation, I partition) is the sum in the free algebra of all tableaux having right key $\mathbf{K}(\mu, I)$. The *costandard basis* $\mathfrak{V}(\mu, I)$ is the sum of all contretableaux having right key $\mathbf{K}(\mu, I)$.

Since by definition all the elements in a plactic class have the same right key, it is clear that $\mathfrak{U}(\mu, I) \equiv \mathfrak{V}(\mu, I)$, and more precisely, that $\mathfrak{V}(\mu, I)\mathbf{R} = \mathfrak{U}(\mu, I)$.

To any partition $I = (1I, 2I, \ldots)$ on associates the *dominant* monomial $a^I = (a_{1I} \ldots a_2a_1)(a_{2I} \ldots a_2a_1)(a_{3I} \ldots a_2a_1) \ldots$.

THEOREM 3.8. *Let* a^I be a dominant monomial and $\sigma_i\sigma_j \ldots \sigma_k$ *be any reduced decomposition of a permutation* μ. Then

$$\mathfrak{U}(\mu, I) = a^I\theta_i\theta_j \cdots \theta_k.$$

Proof. Let μ and i be such that $\ell(\mu\sigma) > \ell(\mu)$, with $\sigma = \sigma_i$, $a_i = a, a_{i+1} = b$. If w is a frank word such that $w\mathbf{K}_+ = \mathbf{K}(\mu, I)$ or $\mathbf{K}(\mu\sigma, I)$, then $w\lambda\mathbf{K}_+ = \mathbf{K}(\mu, I)$. Let t be a tableau such that $t\mathbf{K}_+ = \mathbf{K}(\mu, I)$, $t^\sigma\mathbf{K}_+ \neq \mathbf{K}(\mu, I)$ (this implies that $t^\sigma\mathbf{K}_+ = \mathbf{K}(\mu\sigma, I)$). Then there exists a frank word $w \equiv t$ such that the right factor $w^\sigma\$$ of w^σ contains the letter b and not the letter a; thus w contains a and not b; this implies that $t\lambda = t$. One checks moreover that all the tableaux (not only t^σ) in the sum $t\theta$ have the same right key $\mathbf{K}(\mu, I)$.

Conversely, if t is such that $t\mathbf{K}_+ = t^\sigma\mathbf{K}_+ = \mathbf{K}(\mu, I)$, then $(t + t^\sigma)\theta = 0$. Supposing the theorem true for μ, it is also true for $\mu\sigma$. □

For instance, suppose that we already know $\mathfrak{B}(426135, 321)$; we compute $\mathfrak{B}(436125, 321)$ by using the operator $\theta = \theta_2$, the contretableaux t such that t^σ also belong to $\mathfrak{B}(426135, 321)$ give a zero contribution; the others are of the type $t = t\lambda_2$:

4 42 642	$\longrightarrow$	4 42 643 + 4 43 643
4 41 642	$\longrightarrow$	4 41 643
3 42 642	$\longrightarrow$	3 42 643
2 41 642	$\longrightarrow$	2 41 643 + 3 41 643
3 41 642	$\longrightarrow$	0
3 32 642	$\longrightarrow$	0
3 31 642 + 2 31 642	$\longrightarrow$	0

All the contretableaux belonging to a costandard basis $\mathfrak{B}(\mu, I)$ having the same right column (since it is the reordering of the factor of μ of length $1I$), we have a faster way to compute the costandard bases, by induction on the number of parts of I :

LEMMA 3.9. *Let p be a positive integer, $I = (1I, ..., rI)$ be a partition with $r \geq p$, I' the resulting partition after deletion of the part pI, μ a permutation , v the column such that $\{v\} = \{1\mu, ..., (pI)\mu\}$. Then there exist permutations $\nu, \eta \ldots$ such that*

$$\mathfrak{B}(\mu, I) = \left[\mathfrak{B}(\nu, I') + \mathfrak{B}(\eta, I') + \cdots\right]v$$

Proof. Two congruent frank words w, w' have the same right column $w\$ = w'\$$ iff $|w\$| = |w'\$|$. Thus, to compute the right key of a tableau, we need only to generate a set of frank words $w^{(1)}, w^{(2)}, \ldots$ such that $\{|w^{(1)}\$|, |w^{(2)}\$|, \ldots\} = \{1I, 2I, \ldots\}$. We can require that the shapes of these frank words be $(rI, \ldots, 2I, pI, 1I)$, $(rI, ..., 1I, pI, 2I), \ldots, ((r-1)I, ..., 1I, pI, rI)$. The images of $w^{(1)}, \ldots, w^{(r)}$ by the transposition (of columns) σ_{r-1} will be frank words with right column *of degree pI. Thus the right key of any frank word $w = v_1 \cdot \ldots \cdot v_r$ is equal* to that of the frank word $(v_1 \cdot \ldots \cdot v_{r-1}\mathbf{K}_+) \cdot v_r$. To describe a standard basis, we need only to look for frank words of the type $w = w' \cdot v$, w' being a key of shape I' and v the column: $\{v_r\} = \{1\mu, ..., (pI)\mu\}$, such that $w\mathbf{K}_+ = \mathbf{K}(\mu, I)$. □

This lemma gives a fast induction when we take $p = 1$ to factorize the column of maximal length. For example, let $\mu = 32514$, $I = 4321$. Then $v = 5321$, $I' = 321$; $\mathfrak{B}(32514, 4321) = (32\ 532\ v + 3\ 31\ 532\ v + 2\ 31\ 532\ v) + (3\ 32\ 432\ v + 3\ 31\ 432\ v + 2\ 31\ 432\ v)$ decomposes into $\left[\mathfrak{B}(32514, 321) + \mathfrak{B}(32415, 321)\right]\ v$.

4 . Postulation. Let $\mathcal{A}$ be a vector bundle on any variety $\mathcal{M}$, $\mathcal{F}(\mathcal{A}) \to \mathcal{M}$ the relative flag manifold of complete flags of quotient bundles of $\mathcal{A}$. If $\mathcal{A}$ is of rank n, one has from definition (see **[Gr]**) n tautological line bundles $L_1, \dots, L_n$ on $\mathcal{F}(\mathcal{A})$. The Grothendieck ring $\mathcal{K}(\mathcal{F}(\mathcal{A}))$ of classes of vector bundles is a quotient of the ring of polynomials $\mathcal{K}(\mathcal{M})[\mathbb{A}]$, $\mathbb{A}$ being an alphabet of cardinal n, by a certain ideal $\mathcal{I}$, the images of $a_1, \dots, a_n$ being respectively the classes of $L_1, \dots, L_n$.

Since all constructions given here are compatible with $\mathcal{I}$, we can replace $\mathcal{K}(\mathcal{F}(\mathcal{A}))$ by $\mathbf{Z}[\mathbb{A}]$ and $K(\mathcal{M})$ by the ring of symmetric polynomials $\mathbf{Z}[\mathbb{A}]^{\mathfrak{S}(\mathbb{A})}$. The projection $p : \mathcal{F}(\mathbb{A}) \to \mathcal{M}$ induces a morphism $p_* : \mathcal{K}(\mathcal{F}(\mathbb{A})) \to \mathcal{K}(\mathcal{M})$ which corresponds in fact to the operator $\pi_\omega : \mathbf{Z}[\mathbb{A}] \to \mathbf{Z}[\mathbb{A}]^{\mathfrak{S}(\mathbb{A})}$ associated to the maximal permutation of $\mathfrak{S}(\mathbb{A})$. We can express π_ω as a product of simple operators 3.4, but it can be directly defined by the following global expression (see **[L-S2]**):

$$\text{(4.1)} \qquad \mathbf{Z}[\mathbb{A}] \ni f \longrightarrow \sum_{\mu \in \mathfrak{S}(\mathbb{A})} \Big[f / \prod_{i<j} (1 - a_j/a_i)\Big]^\mu$$

In case that $\mathcal{M}$ is a point, the morphism p_* associates to any vector bundle $\mathcal{B}$ the Euler-Poincaré characteristics : $\sum_i (-1)^i \dim \mathcal{H}^i(\mathcal{B})$; in terms of polynomials, this should be interpreted as $\mathbf{Z}[\mathbb{A}] \ni f \to f\pi_\omega \varepsilon_{\mathbb{A}}$, f being any polynomial lifting the class of $\mathcal{B}$ and $\varepsilon_{\mathbb{A}}$ being the specialisation $a_1 \to 1, \dots, a_n \to 1$.

Let J be a partition, I its conjugate, L the line bundle $L = L_1^{1J} \otimes L_2^{2J} \otimes \cdots$. From Demazure's construction, **[D1]** **[D2]** **[L–S5]** we have that the number $a^I \pi_\mu \varepsilon_A$ is the postulation (that is to say, the dimension of the cohomology $\mathcal{H}^0$; the other spaces $\mathcal{H}^i$ being null, the postulation coincide in that case with the Euler-Poincaré characteristics) of the line bundle L on the Schubert variety of index $\omega\mu^{-1}$.

More generally, considering simultaneously all the powers of L together, we have the ***Hilbert series*** $\mathcal{H}_{I,\mu}(z) = (1 - za^I)^{-1} \pi_\mu \varepsilon$ relative to L of the Schubert variety $Schub_{\omega\mu^{-1}}$ (L defines an embedding of the flag variety into a projective space if $1I > 2I > \dots$).

From considerations of dimension, we know that the series $\mathcal{H}_{I,\mu}(z)$ is rational of the type $\mathcal{N}_{I,\mu}(z)/(1 - z)^{\ell(\mu)+1}$, $\mathcal{N}_{I,\mu}(z)$ being a polynomial of degree $\le \ell(\mu)$. However $(1 - za^I)^{-1}\pi_\mu$ has in general a denominator of degree greater than $\ell(\mu)+1$. Raising up to the free algebra, we shall get a combinatorial interpretation (4.4) of the Hilbert series and clarify in particular this drop in the degrees.

From 3.8 , given any reduced decomposition $\sigma_i \dots \sigma_j$ of μ , then $\underbrace{a^I \dots a^I}_k(\theta_i + 1) \cdots (\theta_j + 1)$ is a sum of words having the same insertion tableau as $a^I \dots \dots a^I$, thus it is a product of k tableaux of shape I . On the other hand, again according to 3.8, $(a_{1I} \dots a_1)^k (a_{2I} \dots a_1)^k \cdots (\theta_i + 1) \dots (\theta_j + 1)$ is the sum of tableaux $\mathcal{T}(\mu, I^k)$, I^k denoting the partition $\underbrace{1I \dots 1I}_k \; \underbrace{1I \dots 1I}_k \cdots$.

Since the operators θ_i are compatible with the plactic congruences, comparing the two sums gives that each tableau t in $\mathcal{T}(\mu, I^k)$ is congruent to a frank word which is a product $t_1 \cdots t_k$ of tableaux of shape I.

Conversely, from 2.11, we see that a product $t^{(1)} \cdots t^{(k)}$ of tableaux belonging to $\mathcal{T}(\mu, I)$ is congruent to a tableau $t \in \mathcal{T}(\mu, I^k)$ iff the following inequalities are satisfied:

$$(4.2) \qquad t^{(1)}\mathbf{K}_+ \le t^{(2)}\mathbf{K}_- \ ; \ t^{(2)}\mathbf{K}_+ \le t^{(3)}\mathbf{K}_- \ ; \dots ; \ t^{(k-1)}\mathbf{K}_+ \le t^{(k)}\mathbf{K}_-$$

Moreover, in such a case, if $v_1 \cdots v_r$ is the right key of $t^{(k)}$, then $v_1^k \cdots v_r^k$ is the right key of $t^{(1)} \dots t^{(k)}$ because each frank word in the class of $t^{(1)} \cdots t^{(k)}$ has a right column which is one the columns $v_1, \dots, v_r$.

Let us call *I-chain of length k* a product of tableaux of the same shape I satisfying the inequalities 4.2; the *right key* of a chain will be the right key of its last tableau, *the left key* of a chain being the left key of its first tableau.

The preceding results may be summarized in the following theorem:

THEOREM 4.3. *Let I be a partition, μ a permutation in $\mathfrak{S}(\mathbb{A})$, $\sigma_i \dots \sigma_j$ any reduced decomposition of μ . Then*

$$(1-a^I)^{-1}\theta_i \cdots \theta_j \ = \ \sum_{\Gamma} \{\Gamma : \mathbf{K}_+(\Gamma) < \mathbf{K}(I,\mu)\}$$

sum of all I- chains Γ of right key $\mathbf{K}(I,\mu)$ and

$$(1-a^I)^{-1}(\theta_i+1) \cdots (\theta_j+1) \ = \ \sum_{\Gamma} \{\Gamma : \mathbf{K}_+(\Gamma) \le \mathbf{K}(I,\mu)\}$$

sum of all I- chains Γ of right key less or equal to $\mathbf{K}(I,\mu)$.

For instance, the 21 chains of length 2 for $\mathfrak{S}(3)$ are all the 27 products $\neq \emptyset$ of two tableaux of shape 21 described below, and correspond bijectively to the 27 tableaux of shape 42. There are 8 tableaux of shape 21, only two being not keys; for them, one has $\left({}^{2}_{1\,3}\right)\mathbf{K}_- = {}^{2}_{1\,2}$ and $\left({}^{2}_{1\,3}\right)\mathbf{K}_+ = {}^{3}_{1\,3}$, $\left({}^{3}_{1\,2}\right)\mathbf{K}_- = {}^{3}_{1\,1}$ and $\left({}^{3}_{1\,2}\right)\mathbf{K}_+ = {}^{3}_{2\,2}$. On the second row, for example, one reads that the chain ${}^{2}_{1\,2} \cdot {}^{2}_{1\,3}$ factorizes the tableau ${}^{2\,2}_{1\,1\,2\,3}$, its left key being $\left({}^{2}_{1\,2}\right)\mathbf{K}_-$ and its right key being $\left({}^{2}_{1\,3}\right)\mathbf{K}_+$.

	2 11	2 12	3 11	2 13	3 13	3 12	3 22	3 23
2 11	22 1111	22 1112	23 1111	22 1113	23 1113	23 1112	23 1122	23 1123
2 12	∅	22 1122	∅	22 1123	22 1133	∅	23 1222	23 1223
3 11	∅	∅	33 1111	∅	33 1113	33 1112	33 1122	33 1123
2 13	∅	∅	∅	∅	23 1133	∅	∅	23 1233
3 13	∅	∅	∅	∅	33 1133	∅	∅	33 1233
3 12	∅	∅	∅	∅	∅	∅	33 1222	33 1223
3 22	∅	∅	∅	∅	∅	∅	33 2222	33 2223
3 23	∅	∅	∅	∅	∅	∅	∅	33 2233

Reintroducing a parameter z , projecting to $\mathbf{Z}[\mathfrak{A}]$ and using the specialization $\varepsilon_{\mathfrak{A}} : \mathfrak{A} \to \{1, \dots, 1\}$, we get:

COROLLARY 4.4. *Let I be a partition, μ a permutation in $\mathfrak{S}(\mathfrak{A})$. Then the postulation $(1 - za^I)^{-1}\overline{\pi_\mu}\varepsilon_{\mathfrak{A}}$ (resp. $(1 - za^I)^{-1}\pi_\mu\varepsilon_{\mathfrak{A}}$) is equal to the generating function of the number of I-chains Γ having right key $\mathbf{K}(I,\mu)$ (resp. having right key less or equal to $\mathbf{K}(I,\mu)$) , i.e.*

$$(1 - za^I)^{-1}\, \overline{\pi_\mu}\, \varepsilon \;=\; \sum_{\Gamma} z^{\mathrm{length}\Gamma}\, \Gamma\, \varepsilon_A$$

sum on all I-chains Γ having right key $\mathbf{K}(I,\mu)$.

5. Avatars of standard bases. According to theorem 3.8, if μ and σ_k are such that $\ell(\mu\sigma_k) > \ell(\mu)$, then for any partition I, $\mathfrak{U}(\mu, I)\theta_k = \mathfrak{U}(\mu\sigma_k, I)$. Thus the operators θ_k allow to connect the standard bases corresponding to different permutations. Using the same induction $\mu \to \mu\sigma_k$, it is not too difficult, but we shall abstain from doing it, to check that standard bases can also be defined in the following two other manners 5.2 and 5.8.

First, according to [L-M-S], a tableau can be considered as an increasing chain of permutations (with respect to the Ehresmann order 2.13). One says that a chain of permutations $\mu^{(1)} \le \mu^{(2)} \le \dots \le \mu^{(r)}$ *lifts* a tableau $t = v_1 \cdot \ldots \cdot v_r$ if $\widetilde{v}_1, \dots, \widetilde{v}_r$ are respective left factors of $\mu^{(1)}, \dots, \mu^{(r)}$, where, for a word $v = x_1 \cdots x_m$, the notation $\widetilde{v}$ stands for the reverse word $x_m \cdots x_1$.

It is clear that given a tableau, there exists a unique minimal lift of it. Indeed, putting $\mu^{(0)} = \mathit{identity}$ and having found the minimal chain $\mu^{(0)} \le \mu^{(1)} \le \dots \le$

$\mu^{(p-1)}$ with respective left factors $\widetilde{v}_1, \ldots, \widetilde{v}_{p-1}$, given moreover $v_p = x_1 \cdots x_m$, we see that the set of permutations μ such that $\mu \geq \mu^{(p-1)}$ and $1\mu = x_m, \ldots, m\mu = x_1$ admits a unique minimal element $\mu^{(p)}$. An induction on p thus gives a lift $\mu^{(1)}(t) \leq \cdots \mu^{(r)}(t)$ that we shall call the *canonical lift of t*. From the construction, for any other lift $\zeta^{(1)} \leq \cdots \leq \zeta^{(r)}$, one has $\mu^{(1)} \leq \zeta^{(1)}, \ldots, \mu^{(r)} \leq \zeta^{(r)}$, i.e. the canonical lift is minimal with respect to the Ehresmann order.

For example, the canonical lift of the tableau 531 62 4 is **135** 246 $\leq$ **26** 3145 $\leq$ **4** 62135. Let us illustrate on this example how to pass from $\mu^{(p-1)}$ to $\mu^{(p)}$, say for $p = 2$. The left reordered factors of $\mu^{(1)}$ are $1, 13, 135, 1235, 12345$; 236 is the minimum word having subword 26 bigger than 135, 1236 is the minimum word containing 236 bigger than 1235, and finally, 12346 is the minimum word containing 1235 bigger than 12345. These minimum words are the left reoredered factors of $\mu^{(2)} = 263145$ which therefore is the minimum permutation bigger than $\mu^{(1)}$ and beginning by **26**.

DEFINITION 5.1. *Given a partition I and a permutation μ , the L-M-S standard basis $\mathfrak{U}'(\mu, I)$ is the set of tableaux t such that the last permutation of their canonical lift is equal to μ.*

When $\mu = identity$, the set $\mathfrak{U}(\mu, I)$ reduces to the tableau $(1I \cdots 1)(2I \cdots 1) \times \cdots (rI \cdots 1)$ as well as $\mathfrak{U}(\mu, I)$; the induction $\mu \to \mu\sigma$ proves, as claimed in the beginning of this section, that $\mathfrak{U}(\mu, I)$ is the sum of the tableaux belonging to $\mathfrak{U}'(\mu, I)$. In other words, one has the following property showing that the L-M-S standard bases coincide with the one defined in 3.5, up to the change of the alphabet A with $\mathbf{N}$.

PROPOSITION 5.2. *A key $\mathbf{K} = \mathbf{K}(\mu, I)$ is the right key of a tableau t iff t has shape I and μ is the last permutation in the canonical lift of t.*

For example, the last permutation 462135 in the canonical lift of the tableau $t = 531\ 62\ 4$ gives the key 642 64 4, which is the right key of t as seen in 2.9.

One may favor horizontals rather than verticals. Reading the successive horizontals of a tableau t , one gets a word which is a product of rows (as defined in sect.2) and which is congruent to t; we shall call this word the *row-word* of t.

For example, the row-word of $\begin{matrix} 5 & & \\ 3 & 6 & \\ 1 & 2 & 4 \end{matrix}$ is 5 36 124. Row-words are characterized by their insertion tableau, as seen from property 2.5 :

LEMMA 5.3. *A word w is the row-word of a tableau t iff there exists a partition $J = 1J \geq 2J \geq \cdots \geq pJ$ such that $\big[(pJ + \ldots + 2J + 1) \cdots (pJ + \ldots + 1J)\big] \cdots \big[(pJ + 1) \cdots (pJ + (p-1)J)\big]\big[1 \cdots pJ\big]$ is an insertion word for w. In that case, J is the partition conjugate to the shape of t.*

For example, (5 36 124)@ $= \begin{matrix} 4 & & \\ 2 & 5 & \\ 1 & 3 & 6 \end{matrix}$, and this tableau is congruent to the word [456] [23] [1].

Apart from the symmetry between rows and columns, which means taking instead of composition words their reverse, the same property as 2.8.*2* holds : in the class of any tableau t of shape conjugate to a partition $J = (1J, \ldots, pJ)$, for any permutation H of J, there exists a unique word w congruent to t, which admits $\big[(pH + \ldots + 2H + 1) \cdots (pH + \ldots + 1H)\big] \cdots \big[1 \cdots pH\big]$ as an insertion word. This forces w to be a product $w_p \cdots w_1$ of rows of respective degrees $pH, \ldots, 2H, 1H$. For standard tableaux, transposition (i.e. exchange of the two axes of coordinates) commutes this construction with the one given in section 2 The hexagon generated by the action of the symmetric group on the row-word 5 36 124 is now

56 3 124 ——— 56 134 2

5 36 124   156 34 2

5 136 24 ——— 156 3 24

The corresponding insertion words are respectively

456 3 12 ——— 6 345 12

456 23 1   6 45 123

56 234 1 ——— 56 4 123

Given $H \in \mathbb{N}^p$, let T_H be the sum of words w such that:
(5.4) w has the insertion word

$$\varphi = \big[(pH + \ldots + 2H + 1) \cdots (pH + \ldots + 1H)\big] \cdots \big[1 \cdots pH\big]$$

(5.5) For the factorization $w = w_p \cdots w_1$ corresponding to φ , every w_j, $1 \leq j \leq p$, belongs to the monoid generated by $\mathsf{A}_j = \{a_1, \ldots, a_j\}$, i.e. $w_1 \in \mathsf{A}_1^*, w_2 \in \mathsf{A}_2^*, \ldots, w_p \in \mathsf{A}_p^*$.

Because of the explicit value of φ, the above factorization is the row-factorization of w, apart from *void* factors that we must specify in order to fix the flag conditions 5.5.

For example, $T_{1302} = (44\ 222\ 1 + 34\ 222\ 1 + 44\ 122\ 1 + 34\ 122\ 1) + (44\ 112\ 1 + 34\ 112\ 1) + (24\ 122\ 1) + (33\ 222\ 1 + 33\ 122\ 1) + (24\ 112\ 1) + (33\ 112\ 1) + (23\ 122\ 1) + (23\ 112\ 1)$ is the sum of words $w = w_3 w_2 w_1$such that $w@ \equiv 6\ 345\ 12$ $\left(w@ = \begin{smallmatrix} 6 & & \\ 3 & 4 & \\ 1 & 2 & 5 \end{smallmatrix}\right)$ and $w_1 \in \{1\}^*, w_2 \in \{1,2\}^*, w_3 \in \{1,2,3\}^*$.

As we already said, the induction $\mu \to \mu\sigma$, starting from the case T_J with J partition (T_J is the single word $\cdots 2^{2J} 1^{1J}$), allows to obtain the general case, summarized in the following proposition:

PROPOSITION 5.6. *Let J be a partition, μ a permutation, $H = J^\mu$, I the partition conjugate to J . Then T_H is congruent to $\sum_{\nu \leq \mu} \mathfrak{U}(I, \mu)$.*

In the preceding example, $J = 321$, $\mu = 2413$. One has 8 permutations in the interval [1234, 2413]. According to the proposition, $T_{1302} \equiv \mathfrak{U}(321, 2413) +$

$\mathfrak{U}(321,1423)+\mathfrak{U}(321,2143)+\mathfrak{U}(321,2314)+\mathfrak{U}(321,1243)+\mathfrak{U}(321,1324)+\mathfrak{U}(321,2134)+\mathfrak{U}(321,1234) =$

$\{421422+321422+421412+321412\}+\{421411+321411\}+\{421212\}+\{321322+321312\}+\{421211\}+\{321311\}+\{321212\}+\{321211\}.$

These tableaux are respectively congruent to the words enumerated in the same order above.

Flags of alphabets or of modules naturally occur in the study of Schubert polynomials **[L-S2]** or of Schubert subvarieties of a flag manifold.

One can restrict the sum T_H to its component T'_H congruent to $\mathfrak{U}(\mu, I)$. Indeed, one has the following property, which is also proved through the induction $\mu \to \mu\sigma$:

LEMMA 5.7. *Let t be a tableau of shape I conjugate to J, $\mathbf{K} = \mathbf{K}(\mu, I)$ its right key, ζ a permutation, $H = J^\zeta$. Then there exists a word in T_H congruent to t iff $\zeta \geq \mu$.*

In other terms, for any word w, the set of H such that w is congruent to a word in T_H is either void or admits a unique minimum element (i.e. an $H = J^\mu$ such that μ is minimal for the Ehresmann order, J being the partition conjugate to the shape of $w\mathbf{R}$). One can now define T'_H to be the restriction of T_H to such words. For example, the tableau 4321 321 31 41 3 is congruent to the words 3344 11233 2 11 $\in T_{2154}$, 3344 11233 12 1 $\in T_{1254}$, 13344 1233 2 11 $\in T_{2145}$, 13344 1233 12 1 $\in T_{1245}$ which correspond to all the permutations above 3412 ; it is also congruent to the words 3344 23 2 11113 , 34 12334 2 1113, 3344 3 11223 11, 4 13334 1223 11 but these words do not belong to respectively $T_{5124}, T_{4152}, T_{2514}, T_{2451}$ (which are just below in the Ehresmann order) because the flag condition 5.5 is violated. Thus t is congruent to a word in T'_{2154}. Proposition 5.6 can now be reformulated:

PROPOSITION 5.8. *Let J be a partition, μ a permutation, $H = J^\mu$, I the partition conjugate to J. Then T'_H is congruent $\mathfrak{U}(I, \mu)$.*

The key of the preceding tableau is $\begin{smallmatrix} 4 \\ 3\,4 \\ 2\,3\,4\,4 \\ 1\,1\,3\,3\,3 \end{smallmatrix}$, i.e. is equal to $\mathbf{K}(3412, 5421)$, in accordance with the fact that $(5421)^{3412}$ is equal to 2154.

6. Appendix. Let U, Ξ be two sets, Ξ^* the free monoid generated by Ξ. An *action* of Ξ^* on U is a *function* (not everywhere defined; we use the symbol $\varnothing$ for the points of indeterminacy) : $U \times \Xi^* \to U \cup \{\varnothing\}$ such that $u(\xi\xi') = (u\xi)\xi'$ and $u\xi = \varnothing \Rightarrow u\xi\xi' = \varnothing$ for any $u \in U, \xi, \xi' \in \Xi$.

Let Ξ be finite and totally ordered: $\Xi = \{\xi_1, \dots, \xi_{p+1}\}$. Suppose that "Moore-Coxeter" relations hold, i.e. that for any pair $\xi_i = \sigma, \xi_j = \tau$ and any u in U, one has identically:

(6.1) $\quad u\xi \neq \varnothing \;\Rightarrow\; u\xi\xi = u$

(6.2) $\quad$ if $|i-j| \geq 2$, $u\sigma\tau = u\tau\sigma$

(6.3) $\quad$ if $|i-j| = 1$, $u\sigma\tau\sigma = u\tau\sigma\tau$.

Remark 6.4. 1) Let $|i-j|=1$ and $u\sigma, u\tau, u\sigma\tau \neq \emptyset$, then $u\sigma\tau\sigma = u\tau\sigma\tau \neq \emptyset$. 2) Let $|i-j| \geq 2$ and $u\sigma, u\tau \neq \emptyset$. Then $u\sigma\tau = u\tau\sigma \neq \emptyset$.

Proof of 1): $u\tau \neq \emptyset$ implies $(u\tau)\tau = u$ according to 6.1; the hypothesis becomes $(u\tau)\tau, (u\tau)\sigma\tau, (u\tau)\tau\sigma\tau \neq \emptyset$ showing that $(u\tau)\sigma\tau\sigma = (u\tau)\tau\sigma\tau \neq \emptyset$ by 6.3. Therefore, $\emptyset \neq (u\tau)\sigma\tau\sigma\sigma = (u\tau)\tau\sigma\tau\sigma = u\sigma\tau\sigma$ as required.

Proof of 2): As above, we use 6.1 to write $u\sigma = (u\tau)\tau\sigma$; according to 6.2, $(u\tau)\tau\sigma = (u\tau)\sigma\tau$; since $u\sigma \neq \emptyset$, $u\tau\sigma\tau$ is different from $\emptyset$ as well as its factor $u\tau\sigma$, and $u\sigma\tau$ by symmetry. □

Choose any $u = u_0$ in U. The three preceding axioms allow to identify the orbit $\Omega = \{u\xi : u\xi \neq \emptyset\}$ to a quotient (the $u\xi$ need not to be all different) of a subset of $\mathcal{S}(p+1)$, u being sent to the identity element of $S(p+1)$. The following proposition gives a necessary and sufficient condition for the orbit to be a quotient of the full symmetric group.

PROPOSITION 6.5. *Let $n, m \geq 1$, $p = n+m$ and set $\rho = \xi_n$, $\Xi_1 = \{\xi_1, \dots, \xi_{n-1}\}$, $\Xi_2 = \{\xi_{n+1}, \dots, \xi_{p-1}\}$. Assume that both $u\Xi_1^*$ and $u\Xi_2^*$ do not contain $\emptyset$ and that $\xi_1 \in \Xi_1^*, \xi_2 \in \Xi_2^* \Rightarrow u\xi_1\xi_2\rho \neq \emptyset$.*

Then $u\Xi^*$ does not contain $\emptyset$.

Proof. We can suppose $n \leq m$ by symmetry, and deduce the general case from the case where all the points $\neq \emptyset$ in $u\Xi^*$ are different. Thus the orbit Ω is a subset of the symmetric group and we write its elements as permutations. If $n = m = 1$, there is nothing to prove. Consider the case where $n = 1, m = 2$. Then Ξ_1 is void, $\Xi_2 = \{\xi_2\}, \rho = \xi_1$. By hypothesis, $u = 123, u\rho = 213$, $u\xi_2 = 312$ and $u\xi_2\rho = 312$ are all different from $\emptyset$. Thus taking $u' = 213, \sigma = \rho, \xi_2 = \tau$ in 6.3, we get that $u'\tau = u\rho\xi_2 = 231$, $u'\tau\sigma = u\rho\xi_2\rho = 321$ are different from $\emptyset$; this proves the proposition in this case.

Let again $n = 1$ and $m \geq 3$. As above, $\rho = \xi_1$ and Ξ_1 is void. Using induction on m, we have that Ω contains all the permutations such that their rightmost letter is $\neq 1$. In particular, for any $i, j > 1$, Ω contains all the permutations such that their restriction to the third rightmost letters is $1ij, i1j, 1ji$ or $j1i$. Repeating the same argument with $\sigma = \xi_{p-2}$ and $\tau = \xi_{p-1}$, we conclude that Ω contains all the permutations such that their right factor of length 3 is $ij1$ or $ji1$, concluding the proof of the proposition for $n = 1$.

Consider now the general case where $n \geq 2, m \geq 1$. For any $k \leq n$, we can find some ξ in Ξ_1^* such that the first (ie. left) letter of $u\xi$ is k. Thus by induction on n, i.e. by considering the restriction of $u\xi$ to all its letters except the first, we have that Ω contains all the permutations such that their first letter is $h \leq k$. Considering now the first three letters on the left and applying the same argument as for the case of $n = 1$, we conclude that Ω is the full symmetric group. □

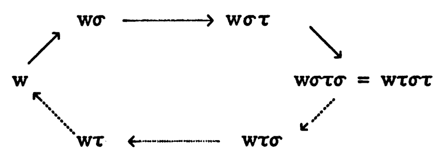
The action of "commutation" of columns seen in section 2 satisfy the axioms 6.1, 6.2 and 6.3 . Only 6.3 is not straightforward. Since it involves only triples of consecutive columns in a word, it needs to be checked only for 3-columns words. This we do just now.

LEMMA 6.6. *Let w be a 3-columns word, σ and τ be the two generators of $\mathbb{S}(3)$. Then $\{w\sigma, w\sigma\tau, w\sigma\tau\sigma \neq \emptyset\} \Rightarrow \{w\tau, w\tau\sigma \neq \emptyset \;\&\; w\sigma\tau\sigma = w\tau\sigma\tau\}$.*

Proof. One of the four words w , $w\sigma$, $w\sigma\tau$, $w\sigma\tau\sigma$ has its shape decreasing or increasing. Let it be $w\sigma = v_1 \cdot v_2 \cdot v_3$. Recall that a 2-columns word w is a tableau or a contretableau iff $w^\flat \neq \emptyset$, i.e. iff w is frank, and that a word is a tableau (resp. a contretableau) if each factor made of two consecutive columns is such. The two factors $v_1 v_2$ and $v_2 v_3$ being frank, $w\sigma$ is a tableau or a contretableau. According to 2.6.$\mathcal{S}$, the action of permutation of columns on a tableau or a contretableau generate the frank words in its class: thus $w, w\sigma, w\sigma\tau, w\sigma\tau\sigma = w\tau\sigma\tau, w\tau\sigma, w\tau$ are the frank words in the class of $w\sigma$.

Suppose now that this is $w = v_1 \cdot v_2 \cdot v_3$ which has a decreasing shape $\|w\| = ijh$, and let t be the insertion tableau of w and σ be the first generator of $\mathbb{S}(3)$. Since $v_1 v_2 \sigma \neq \emptyset$, the word $v_1 \cdot v_2$ is a tableau and this determines $t \cap \{1, \ldots, i+j\}$. Since $v_1 \cdot v_2 \cdot v_3 \sigma\tau \neq \emptyset$, the digit $i+j+h$ cannot be in the first column of t; since $v_1 \cdot v_2 \cdot v_3 \sigma\tau\sigma \neq \emptyset$, it cannot be either in the second column of t. It must be in the third, which means that t is equal to $\big[i \cdots 1\big]\big[(i+j) \cdots (i+1)\big] \times \big[(i+j+h) \cdots (i+j+1)\big] = ijh\mathrm{M}$. Thus w is a tableau and we conclude as before. This reasoning also applies to the case where $\|w\|$ is increasing, since then $\|w\sigma\tau\sigma\|$ is decreasing and we can exchange the rôle of w and $w\sigma\tau\sigma$. □

Pictorially, hypothesis 6.6 is that if the four consecutive words $w \to w\sigma \to w\sigma\tau \to w\sigma\tau\sigma$ are different from $\emptyset$, then we can "close the hexagon":



Let us finish with an example of a word whose orbit (under commutation of columns) is not a quotient of the full symmetric group.

Let $w = 31 \cdot 42 \cdot 4 \cdot 51$, and σ, ρ, τ be the three generators of $\mathbb{S}(4)$ We get four double points: $w = w\sigma, w\rho = 31\ 4\ 42\ 51 = w\sigma\rho$, $w\rho\sigma = \omega\rho\sigma\rho = 3 \cdot 41 \cdot 42 \cdot 51, w\tau = w\tau\sigma = 31 \cdot 42 \cdot 41 \cdot 5$. Since the words $42 \cdot 41$ and $42 \cdot 51$ are not frank, all the neighbours $w\rho\tau, w\tau\rho, w\sigma\tau\rho, w\sigma\rho\tau, w\rho\sigma\tau$ and $w\sigma\rho\sigma\tau$ are $\emptyset$ and thus the orbit Ω of w is restricted to the enumerated four double points. Indeed, condition 2.8.$\mathcal{S}$ to ensure that w be frank is exactly that $42 \cdot 4$ (central factor of w) and $42 \cdot 41$ (central factor of $w\sigma$, $w\tau$ and $w\tau\sigma$) should be frank. Since this not the case for the last word, we already knew from th.2.8 that Ω could not be a quotient of the full symmetric group.

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