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Planarity Properties of the Schensted Correspondence

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We give a natural decomposition of the set of standard Young tableaux of a given shape into intervals with respect to the weak Bruhat order; each class is completely determined by a partial order on letters which admits a remarkable planar representation. © 1993 Academic Press, Inc.

1. INTRODUCTION

Tableaux were introduced by A. Young in his fundamental papers on the representation of the symmetric groups; a slight modification was soon used by Richardson and Littlewood to provide a combinatorial definition of Schur functions. Since then tableaux have become an essential tool in related fields such as invariant theory or the representation of classical groups. The papers collected under the title "Invariant Theory and Young Tableaux" [8] provide an outlook of current research in this field. In the present paper we restrict our attention to the basic case in which the tableaux are standard in the sense of A. Young, meaning they are defined as "fillings" of the Ferrers diagram of a partition J of n with the letters of the totally ordered set $\mathscr{A} = \{1, 2, ..., n\}$ in such a way that letters appear in increasing order along each row and each column and that each letter of $\mathscr{A}$ appears exactly once. The partition J is the *shape* of the tableau. From Young's point of view tableaux are a convenient way of dealing with the chains of nested partitions of 1, 2, ... which are associated with the sequence of symmetric groups $\mathscr{S}_1 \subsetneq \mathscr{S}_2 \subsetneq \cdots \subsetneq \mathscr{S}_n$. At the same time tableaux summarize the systems of cosets linked with the Frobenius subgroups of $\mathscr{S}_n$ characterized by the shape J.

A different approach has been initiated by Schensted who discovered a bijection between permutations and pairs of tableaux of same shape, called

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respectively the P-symbol and the Q-symbol in Schensted [5]. Here permutations and tableaux are seen as words of the free monoid $\mathscr{A}^*$ generated by the alphabet $\mathscr{A}$. For instance the tableau t with rows 3, 5 and 1, 2, 4 is identified with the word 35124 and it is associated with the words 31524 (which is the reading of t by columns), 31254, 13254, and 13524, all of which have the same P-symbol. Knuth discovered a simple congruence $\cong$ on $\mathscr{A}^*$ which underlies Schensted's correspondence in the sense that two words are congruent if and only if they have the same P-symbol. The quotient $\mathscr{A}^*/\cong$ is the plactic monoid and the (semistandard) tableaux t make up a set of representatives of the classes W_t of the plactic congruence $\cong$; our (standard) tableaux correspond to the words in which each letter of A appears exactly once. A theorem due to C. Greene gives a very useful characterization of the shape of t in terms of families of increasing subwords of the words of W_t. These considerations bring us back to Littlewood's use of tableaux. The plactic algebra $Z(\mathscr{A}^*/\cong)$ turns out to be the proper set up for dealing with Schur functions and their q-generalizations, i.e., the Hall–Littlewood polynomials. In fact the plactic congruence can be defined directly as one of the two extremal congruences $\cong$ on $\mathscr{A}^*$ such that the symmetric polynomials make up a commutative subalgebra of $Z(\mathscr{A}^*/\cong)$.

Returning to the standard case, Schensted's construction reveals an interplay between the tableaux corresponding to a permutation w and to its inverse and it involves in an essential manner the descent set of $w = x_1 x_2 \cdots x_n$ $(x_i \in \mathscr{A})$, i.e., the set $\mathrm{Des}(w)$ of indices j for which $x_j > x_{j+1}$. Both notions are ingredients of the Kazhdan–Lusztig theory [2] which studies properties of the decomposition of $\mathscr{S}_n$ into the cells W_t. Several conjectures concerning the Kazhdan–Lusztig polynomials in the case of the symmetric group require a better understanding of the combinatorial properties of the plactic congruence $\cong$. It remains somewhat of a mystery why these purely algebraic concepts should have anything to do with several properties of the tableaux which derive from their interpretation as a planar disposition of letters. The present paper is an attempt to further analyze these relationships.

Say that a permutation σ acting on $\mathscr{A}$ is *admissible* for a tableau t if w and σw have the same descent set for every w in the plactic class W_t of t. This implies that σt is a tableau of same shape as t and we prove that when σ is admissible for t, then it is a bijection from W_t onto $W_{\sigma t}$. This defines a new equivalence relation on the set of tableaux of same shape; we call *plaques* its equivalence classes. It is remarkable that all tableaux of shape J are in the same plaque if and only if J is rectangular (i.e., all part of J are equal), a case which appears often in the study of tableaux. In this case we give a formula which links Schensted's correspondence to the product in the symmetric group. In the opposite direction every plaque reduces to a single tableau if and only if the shape is a hook (i.e., $J = m1^k$). Our main result is that in the set of all permutations provided with the so-called weak Bruhat order, every plaque is an interval having unique extremal elements. Our second main claim is a construction which associates to each plaque a planar configuration describing a partial order on $\mathscr{A}$ on which the set of admissible permutations can be read in a direct manner.

In view of the elementary nature of our considerations we have preferred to give a complete exposition from scratch. Thus our paper includes a (partly) new presentation of Schensted's theory.

2. Jeu de Taquin

In this section we introduce the vocabulary of the *plactic classes* and we prove some basic properties which will be needed in the following sections. We also give an account of the equivalence between plactic classes and *Knuth classes*, although this would not be essential to the comprehension of the paper.

Consider two points $p = (i, j)$ and $p' = (i', j')$ on the discrete plane $Z \times Z$. We write $p \nearrow p'$ if p precedes p' in the natural order, that is to say if $i \leqslant i'$ and $j \leqslant j'$ and we say then that p' lies on the nort-east of p (or equivalently that p lies on the south-west of p'). An *interval* is a subset $\mathscr{I}$ of $Z \times Z$ such that the relation $p \nearrow p' \nearrow p''$ entails $p' \in \mathscr{I}$ when $p, p'' \in \mathscr{I}$. Remark that if an interval has a unique minimal element it may be viewed as the Ferrers diagram of a partition. We also need the similar relation $\searrow$ which is defined in the same manner, except for the exchange of north and south. Thus any two points of $Z \times Z$ are comparable with respect to either $\nearrow$ or $\searrow$ and they are comparable with respect to both if and only if they lie on the same row or on the same column.

If $\mathscr{I}$ is an interval and if $\mathscr{B}$ is a set of integers, an *inscription* s with *content* $\mathscr{B}$ and *domain* $\mathscr{I}$ is an order preserving bijection from $\mathscr{I}$ to $\mathscr{B}$:

$$a \nearrow b \Rightarrow s(a) < s(b).$$

Two inscriptions which differ only by a translation of their domain will be said to represent the same *skew tableau*. If the domain of an inscription is a Ferrers diagram, then the inscription represents a *Young tableau*. Similarly, if the domain of the inscription has a unique maximal element, then the inscription represents a *contretableau*. Trivially, reversing the order $\nearrow$ as well as the natural order on contents exchanges tableaux and contretableaux.

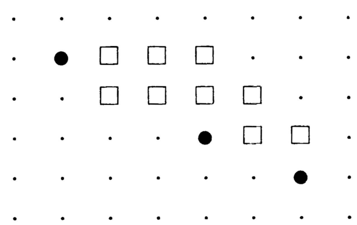
We now define a family of operations τ_p indexed by the points $p \in Z \times Z$ that act injectively on the set of all inscriptions. Since this construction is basic to all what follows we describe it with more details than would be strictly necessary. Let w be an inscription with domain $\mathscr{I}$. We set $w\tau_p = w$ unless p satisfies the following three conditions, which characterize what we call a *starting point*:

(1) $p \notin \mathscr{I}$;

(2) $\{p\} \cup \mathscr{I}$ is an interval;

(3) $p \nearrow p'$ for at least one $p' \in \mathscr{I}$.

For instance, if $\mathscr{I}$ consists of the points marked $\square$ in the figure below, $\mathscr{I}$ admits exactly three starting points indicated by the symbol $\bullet$



Consider now a starting point p and define its *trail* as the unique maximal sequence $(p_0, .., p_s)$ of $\nearrow$-consecutive points of $\mathscr{I}$ such that, letting $p_0 = p$, the letter x_i lying on p_i in w is the least of the letters lying on the points on the north-east of p_{i-1} $(i = 1, ...)$. By construction, the last point p_s is a $\nearrow$-maximal point of $\mathscr{I}$. Finally we define $w\tau_p$ by moving each letter x_r from p_r to p_{r-1} for $r = 1, ..., s$, with the result that the domain of $w\tau_p$ is the union of $\{p_0\}$ with $\mathscr{I} \setminus \{p_s\}$. In the sequel τ_p will be called the *jeu de taquin* of starting point p, and the moving of x_r will be called a *switch*.

The proofs of the following three remarks are left to the reader.

Remark 1. If $w\tau_p \neq w$, then $w\tau_p$ is the unique inscription w' such that:

(1) its domain contains p and is contained in $\{p\} \cup \mathscr{I}$.

(2) the location of every letter in w' is on the south-west of its location in w.

Remark 2. If $w\tau_p \neq w$, then the inscription w and the point p are completely determined by the data of $w' = w\tau_p$ and of the last point $\bar{p}$ of the trail (which by construction does not belong to the domain of w').

We now have at hand two different jeux de taquin, allowing to move inscriptions towards the south-west or towards the north-east. For the dual version we will use the letter γ instead of τ.

Remark 3. If $\mathscr{A}$ is an interval of the content $\mathscr{B}$ of the inscription w, then there is a point p' such that the restriction of $w\tau_p$ to $\mathscr{A}$ is equal to $w'\tau_{p'}$, where w' is the "restriction" of w to A.

PROPOSITION 1. *If p is a starting point of w and if p' is a starting point of $w\tau_p$ with corresponding final points $\bar{p}$ and $\bar{p}'$ and if one has $p \searrow p'$ (respectively $p' \searrow p$), then one has also $\bar{p} \searrow \bar{p}'$ (respectively $\bar{p}' \searrow \bar{p}$).*

Proof. If the points (i, j) and $(i, j+1)$ (respectively (i, j) and $(i+1, j)$) are on the trail of p, then in case the *cell* $(i-1, j+1)$ (respectively $(i+1, j-1)$) is not empty in $w\tau_p$, the letter occupying the cell (i, j) in $w\tau_p$ is greater than that occupying the cell $(i-1, j+1)$ (respectively $(i+1, j-1)$). This ensures that the trail of p and that of p' do not "properly" cross, whence the result. ▮

Remark that by the same argument if p and p' are starting points of w and if the trails of p and p' (in w) are disjoint, then $w\tau_p\tau_{p'} = w\tau_{p'}\tau_p$.

We also need the following remarks whose proofs are left to the reader.

PROPOSITION 2. *The starting points of any inscription w are completely ordered with respect to $\searrow$. Moreover, if $p \searrow p'$ are starting points of w, then there is at least one point p'' which is a starting point of both $w\tau_p\tau_{p'}$ and $w\tau_{p'}\tau_p$, with $p \searrow p'' \searrow p'$.*

PROPOSITION 3. *By using the jeu de taquin, it is possible to translate any inscription w by one unit, horizontally as well as vertically.*

We consider in more details the case when w, p', and p'' satisfy the conditions of Proposition 2 and the additional condition that the trails corresponding to p and p' are disjoint except for their last point $\bar{p} = \bar{p}'$.

LEMMA 1. *Under the above conditions we have*

$$w\tau_p\tau_{p'}\tau_{p''} = w\tau_{p'}\tau_p\tau_{p''}.$$

Proof. Let (i, j) be the coordinates of $\bar{p} = \bar{p}'$ and let P and P' be the respective trails of p and p'. We may suppose that $(i-1, j)$ is on P and $(i, j-1)$ is on P'. If we erase the letter z in w, obtaining the inscription $\underline{w}$, the endpoints of the trails of p and p' are $(i-1, j)$ and $(i, j-1)$, and τ_p and $\tau_{p'}$ commute, by the remark following Proposition 1. By Proposition 1, the endpoint q of the trail of p'' (in $\underline{w}\tau_p\tau_{p'}$) is such that $(i-1, j) \searrow q$ and $q \searrow (i, j-1)$, whence $q = (i-1, j-1)$ and $w\tau_p\tau_{p'}\tau_{p''} = w\tau_{p'}\tau_p\tau_{p''}$. ▮

THEOREM 1. *If w is an inscription and if $p \searrow p'$ are two starting points of w, then for any starting point p'' of $w\tau_p\tau_{p'}$ lying between p and p' with respect to the order $\searrow$ we have*

$$w\tau_p\tau_{p'}\tau_{p''} = w\tau_{p'}\tau_p\tau_{p''}.$$

Proof. We reason by induction on the cardinality of the domain $\mathscr{I}$ of w. We may suppose that the trails of p and p' intersect. Suppose that the letter z occupies the first intersection (i, j) of the trails of p and p'. We may suppose that the trail of p contains $(i-1, j)=a$ and that the trail of p' contains $(i, j-1)=b$. Let $\mathscr{J}$ be the interval consisting of the points which are (strictly) on the north-east of a or b. Still, by the above remark and Lemma 1, we may suppose that $\mathscr{I}=\mathscr{J}$, so z is the smallest element of the content of w. If we erase z in $w\tau_a$ or $w\tau_b$, we obtain the same inscription $\underline{w}$. By induction we have $\underline{w}\tau_a\tau_b\tau_c=\underline{w}\tau_b\tau_a\tau_c$, with $c=(i-1, j-1)$. Remark that $\underline{w}\tau_a\tau_b\tau_c$ may be obtained by erasing z in $w\tau_b\tau_a\tau_c$, then applying τ_c. It follows (by induction) that if $w\tau_a\tau_b\tau_c \neq w\tau_b\tau_a\tau_c$, then we may suppose that all integers of $\mathscr{I}$ occupy the same cell in $w\tau_a\tau_b\tau_c$ and $w\tau_b\tau_a\tau_c$ except for the largest element of $\mathscr{I}$, say n, with n occupying the cell $(k-1, l)$ in $w\tau_a\tau_b\tau_c$ and the cell $(k, l-1)$ in $w\tau_b\tau_a\tau_c$. Then (by considering the dual jeu de taquin) n occupies the cell (k, l) in $w\tau_a$ and $w\tau_b$ and the endpoints of the trails of b and a in $w\tau_a$ and $w\tau_b$ are both equal to (k, l). This contradicts Proposition 1, since the endpoints of the trails of a and b in w are also equal. ∎

THEOREM 2 [7]. *For any inscription w, the set of inscriptions that may be deduced from w by a succession of jeux de taquin (respectively dual jeux de taquin) contains one and only one tableau (respectively contretableau).*

Proof. By a preliminary translation we may suppose that w is in the quadrant $N \times N$. It is then easy to "push" w towards the south-west in order to obtain an inscription with the unique minimal point $(0, 0)$. As for the unicity, let us suppose to the contrary that w may be transformed into two different tableaux t and t' by pushing it towards the "corner" of $N \times N$. By considering the sets of inscriptions leading from w to t and to t', we may suppose that these "paths" have no common point except w. We put $t=w\tau_p\Theta$ and $t'=w\tau_{p'}\Theta'$, where p and p' are starting points of w and Θ and Θ' are products of jeux de taquin. Now by the above theorem for some p'' we have $w\tau_p\tau_{p'}\tau_{p''}=w\tau_{p'}\tau_p\tau_{p''}=w'$, unless the distance from w to t is less than 3, in which case there is only one way to push w. It is now easy to reason by induction on the distance from w to t or t': Θ joins $w\tau_p$ to t, so there must be some path joining w' to t; in the same manner there must be some path joining w' to t', a contradiction. The rest of the theorem follows by duality. ∎

We define the *reading* of any inscription w as the word obtained by reading the consecutive rows of w from north to south and from west to east. Similarly, the *column reading* of w is obtained by reading the consecutive columns of w from east to west and from north to south. For instance, the reading and the column reading of ${}^{2}_{1}\;{}^{4}_{3}\;{}_{5}$ are respectively 2, 4, 1, 3, 5 and 2, 1, 4, 3, 5. We will need the following properties of readings extensively; here $a<b<c$ are consecutive letters of the content of the inscription s, and p is a point in $Z \times Z$.

LEMMA 2. (1) *If ab (or ba) is a subword of the reading w of s, then the same holds for the reading of* $s\gamma_p$.

(2) *If bca is a subword of w, then the same holds for the reading of* $s\gamma_p$.

(3) *If bac is a subword of w and if* $s' = s\gamma_{p_1} \cdots \gamma_{p_k}$, *the following properties are equivalent*:

- *bca is a subword of the reading w′ of s′.*
- *bc is a factor of the reading of* $s\gamma_{p_1} \cdots \gamma_{p_l}$ *for some* $l \leqslant k$.

Proof. Part (1) is left to the reader. If *bca* is a subword of *w*, then by (1), *bca* or *bac* is a subword of $s\gamma_p$. In the latter case we have necessarily $\begin{smallmatrix} c & \cdot \\ a & b' \end{smallmatrix} \to \begin{smallmatrix} \cdot & c \\ a & b' \end{smallmatrix} \to \begin{smallmatrix} a & c \\ \cdot & b' \end{smallmatrix}$ with $a < b' < c$, a contradiction. We suppose now that *bac* is a subword of *w*. If *bca* is a subword of the reading *w′* of $s' = s\gamma_{p_1} \cdots \gamma_{p_k}$, then at some point along the path *s*, $s\gamma_{p_1}$, ..., *s′* we have $bac \to bca$ and $\begin{smallmatrix} b' & \cdot \\ a & c \end{smallmatrix} \to \begin{smallmatrix} b' & c \\ a & \cdot \end{smallmatrix} \to \begin{smallmatrix} b' & c \\ \cdot & a \end{smallmatrix}$, with $a < b' < c$; consequently $b' = b$ and *bc* is a factor of the reading of $s\gamma_{p_1} \cdots \gamma_{p_l}$ for some $l \leqslant k$. Inversely, if *b* and *c* "meet" somewhere along the path joining *s* and *s′*, then at this point, say *t*, *bca* is a subword of the reading of *t*; then by (2), *bca* is a subword of the reading of *s′*. ∎

Remark that a dual version of this lemma exists, since the jeu de taquin is symmetrical with respect to the *x*- and *y*-axis.

Considering both the jeu de taquin and its dual version allows us to define an equivalence relation on inscriptions; this relation extends trivially to readings: by definition, two permutations *w* and *w′* belong to the same *plactic class* if they are the respective readings of inscriptions *s* and *s′* which are themselves equivalent. In the sequel this relation will be denoted by the symbol $\equiv$. Remark that the reading and the column reading of any inscription are equivalent, since it is always possible to transform any inscription by sliding its consecutive columns in such a way that the new domain contains at most one point in each row of $Z \times Z$ and that accordingly the new reading coincides with the former column reading. Another classical relation is defined by the equivalences $wbacw' \cong wbcaw'$ and $wacbw' \cong wcabw'$ for any permutation $wbacw'$ or $wacbw'$ where $a < b < c$ and *w*, *w′* represent factors [4]. Now if *w* is a permutation, let $w\Pi$ and $w\text{П}$ represent Schensted's *P*- and *Q*-symbols of *w*, respectively [5]. The interesting fact about this equivalence is that we have $w \cong w' \Leftrightarrow w\Pi = w'\Pi$ [4].

PROPOSITION 4 [7]. *We have* $w \cong w' \Leftrightarrow w \equiv w'$.

Proof. We will use the following lemma:

LEMMA 3. *The reading and the column reading of any inscription are equivalent with respect to* $\cong$.

Proof. Let $b_l, ..., b_2, b_1$ be the first column of an inscription w. If $b_2, a_1, ..., a_k$ is the row containing b_2, the reading of w contains the factor $b_2, a_1, ..., a_k, b_1$; by using the equivalences defined above, we can switch the letter b_1 to the "left" until it lies next to b_2. In other words, it is possible to "pull out" the first column of the reading of any inscription with two rows. The lemma follows by an easy induction, remarking that it is then possible to pull out b_3, b_2, b_1 in the same way, etc. ▮

We can now make the following useful remarks:

(1) Suppose there is a row immediately "below" that containing b_1 whose first letter c is smaller than b_1; then it is possible to pull out not only the first column of w, but the whole sequence $b_l, ..., b_1, c$ in exactly the same manner.

(2) A dual operation consists in pulling out (from the right side) the last row of the column reading of w, and as above, if the last row of w is $c_1, ..., c_m$ and if the row above that starts with the letter $c < c_1$, then it is possible to pull out the sequence $c, c_1, ..., c_m$ from the column reading of w.

We wish to prove that $w\tau_p \cong w$. Suppose that p is a starting point of w. If p does not precede the whole domain of w (with respect to $\nearrow$), it is easy to conclude by induction on the number of letters in w, using the above lemma. In the other case we may suppose that the first column of w is $b_l, ..., b_1$ and that the last row of w is $c_1, ..., c_m$, with the cell p just below b_1 and just on the left of c_1. If $c_1 < b_1$, then we can pull out the sequence $b_l, ..., b_1, c_1$ from the reading of w. Remarking that this sequence is exactly the first column of $w\tau_p$, we conclude by induction, still using the above lemma. If $b_1 < c_1$, it suffices to use a dual reasoning. It follows that we have $w \equiv w' \Rightarrow w \cong w'$. Now let $w = w_0, ..., w_n$ be a permutation containing the factor bac with $a < b < c$. By placing the letter w_0 in the cell $(0, n)$, the letter w_1 in the cell $(1, n-1)$, ..., the letter w_n in the cell $(n, 0)$ we obtain a inscription whose reading (or column reading) is w (this inscription will be said to be *canonical*). By considering the canonical inscription of w we see that the relation $\equiv$ applied to any factor of w extends to w. It follows that the easily checked relation $bac \equiv bca$ implies $w \equiv (a, c)w$ where $(a, c)w$ denotes the permutation obtained by transposing the letters a and c in w. In the same way, if w contains the factor acb, then $w \equiv (a, c)w$. It follows that $w \cong w' \Rightarrow w \equiv w'$, which concludes the proof of the proposition. ▮

3. Admissible Permutations, Plaques, Plaque Order

We remind the reader that the *descent set* of the permutation w in $\mathscr{S}_n$ is the set $\{i, 1 \leqslant i \leqslant n-1, w(i) > w(i+1)\}$. If W is a subset of the symmetric group $\mathscr{S}_n$ we say that a permutation σ is *admissible* for W if and only if w and σw have the same descent set for every w in W. In what follows we consider the elements of $\mathscr{S}_n$ as (standard) words in the letters of $\mathscr{A} = \{1 < 2 < \cdots < n\} = [1, n]$.

DEFINITION 1. The *plaque order induced by* W is the least order $\leqslant_W$ on $\mathscr{A}$ such that for all a, b in $\mathscr{A}$ one has $a \leqslant_W b$ if $a < b$ and if ab or ba is a factor of a least one w in W.

We let $\mathscr{G}_W$ be the graph of $\leqslant_W$ and we denote by $\sigma\mathscr{G}_W$ the image of $\mathscr{G}_W$ by any permutation σ acting on A.

PROPOSITION 5. *For any* $W \subseteq \mathscr{S}_n$, *the permutation* σ *is admissible for* W *if and only if we have* $\mathscr{G}_{\sigma W} = \sigma\mathscr{G}_W$.

Proof. Let $\sigma = s_p s_{p-1} \cdots s_1$ be a reduced expression, meaning that the s_i's are of the form $s_i = (k, k+1)$, $1 \leqslant k \leqslant n-1$, with p minimum. By reasoning by induction on p, it is easy to see that for i, j in $[1, n]$, if $w(i) < w(j)$ and $\sigma w(i) < \sigma w(j)$, then for any k $(1 \leqslant k \leqslant p)$ we have also $s_k \cdots s_1 w(i) < s_k \cdots s_1 w(j)$, a property that holds for any Coxeter group [9]. The following two lemmas are direct consequences of this fact.

LEMMA 4. *With* σ *as above,* σ *is admissible for* W *if and only if* s_1 (*respectively* $s_2, ..., s_p$) *is admissible for* W (*respectively* $s_1 W, s_2 s_1 W, ..., s_{p-1} \cdots s_1 W$).

LEMMA 5. *If* G *is a graph whose vertices are the letters* $1, 2, ..., n$, *then* σ *preserves the orientation of* G (*meaning that for any edge* (i, j) *we have* $i < j \Leftrightarrow \sigma(i) < \sigma(j)$) *if and only if* s_1 (*respectively* $s_2, ..., s_p$) *preserves the orientation of* G (*respectively* $s_1 G$, $s_2 s_1 G, ..., s_{p-1} \cdots s_1 G$).

If $\sigma = (i, i+1)$, then w and σw have the same descent set for any w in W if and only if $i \preccurlyeq_W i+1$ does not hold, so the result is verified in this case. By Lemmas 4 and 5 the proposition follows by induction on p. ∎

In the same way we have:

PROPOSITION 6. *For any* W *in* $\mathscr{S}_n$, σ *is admissible for* W *if and only if* σ *preserves the orientation of* $\mathscr{G}_W$.

THEOREM 3. *If* W *is a plactic class, then* σ *is admissible for* W *if and only if for any* w *in* W *we have*

$$(\sigma w)\Pi = \sigma(w\Pi); \qquad (\sigma w)\amalg = w\amalg.$$

Proof. By induction we may suppose that $\sigma = (i, i+1)$. In this case the first formula is trivially verified. For any w in W, the letters i and $i+1$ cannot become adjacent along the bumping process of w, whence the second formula. If σ is not admissible for W, then there exists at least one element of W containing the factor $i, i+1$ or $i+1, i$. Suppose for instance that w contains the factor $i, i+1$, with $w(j) = i$ and $w(j+1) = i+1$. If the formulas are verified, then we have $w^{-1}\Pi = (w^{-1}\sigma)\Pi$, but since w^{-1} contains the factor $j, j+1$ and $w^{-1}\sigma$ contains the factor $j+1$, j, this contradicts Lemma 2. ▮

Remark that by Theorem 3, if W is a plactic class with σ admissible for W, then σW is a plactic class.

4. Plaques and Weak Bruhat Order

Definition 2. If W is a plactic class and if σ is admissible for W, then W and σW belong to the same plaque.

We will prove now that the plaques are intervals with respect to the weak Bruhat order $\leqslant_B$. We remind the reader that this order is defined tansitively by the relation $\forall w \in \mathscr{S}_n$, $w \leqslant_B (i, i+1)w \Leftrightarrow w^{-1}(i) < w^{-1}(i+1)$ (cf. [1]). Let $\mathscr{I}(w)$ be the set $\{(i, j),\ 1 \leqslant i < j \leqslant n,\ w(i) < w(j)\}$; w is completely determined by $\mathscr{I}(w)$, and we have the characterization $w \leqslant_B w' \Leftrightarrow \mathscr{I}(w) \supseteq \mathscr{I}(w')$ (cf. [1]), which implies that the set of Young tableaux of a given shape is an interval with respect to $\leqslant_B$ (identifying tableaux with their readings). In the sequel this identification is left to the reader, whenever necessary. We suppose that W is a plactic class of tableau T and contretableau T^c.

Theorem 4. *With respect to the weak Bruhat order every plaque is an interval having a unique maximal element and a unique minimal element.*

Proof. If $s = (i, i+1)$ is admissible for the plactic class W, then for every w in W we have $sw \leqslant_B w$ or else for every w in W we have $w \leqslant_B sw$, so we can write $W \leqslant_B sW$ or $sW \leqslant_B W$.

Lemma 6. *If $sW \leqslant_B W$ and $tW \leqslant_B W$ with $s = (i, i+1)$, $t = (j, j+1)$, $i < j$, then*

- *if $j > i+1$, s and t commute and we have $tsW \leqslant_B sW$ and $stW \leqslant_B tW$.*
- *if $j = i+1$, $sts = tst$ and we have $stsW \leqslant_B tsW \leqslant_B sW$ and $tstW \leqslant_B stW \leqslant_B tW$.*

Proof. Left to the reader. ▮

Now let us suppose that σ is admissible for W. By Lemma 4, it is possible to construct a path from W to σW that stays in the same plaque. Suppose such a path contains the factor $sW' \leqslant_B W' \geqslant_B tW$. By the above lemma it is always possible to "lower" that portion of the path, and by iterating this process we conclude that there exists a plactic class W'' in the plaque of W with $W'' \leqslant_B W$ and $W'' \leqslant_B \sigma W$. ▮

We will now study the order $\preccurlyeq_W$, and particularly its graph $\mathscr{G}_W$. The study will lead to an alternative proof of Theorem 4. A byproduct will be an efficient algorithm for the computation of the plaque of a given plactic class. The following reduction theorem is the first step towards this end. Remark that since every plactic class contains one and only one tableau, we may as well adopt the notation $\mathscr{G}(T)$ for $\mathscr{G}_W$.

THEOREM 5. *For any path $C = t, t\gamma_1, ..., t\gamma_p = u$ joining an inscription t representing T to an inscription u representing T^c, the orders $\preccurlyeq_W$ and $\preccurlyeq_C$ are identical (identifying C with the set of readings of the inscriptions contained in C).*

Proof. In turns we will need the two lemmas below.

LEMMA 7. *With the notations above, we have*

$$i \preccurlyeq_W i+1 \Leftrightarrow i \preccurlyeq_C i+1 \qquad (1 \leqslant i \leqslant n-1).$$

Proof. We may suppose that i and $i+1$ are adjacent for neither t nor u. By symmetry around the axes we may suppose that $i, i+1$ is a subword of the reading of t. Let us erase from t the letters $1, 2, ..., i+1$. There exists an inscription t' with content $\{i-1, i, ..., n\}$ such that the restriction of t' to $i, i+1, ..., n$ is t and the reading of t' has $i, i-1, i+1$ as a subword. Now the jeux de taquin used along C provide a path C' joining t' to u', with t' representing some tableau T' and u' representing T'^c. From Lemma 2, $i, i+1$ is a factor of some inscription in C' if and only if $i, i+1, i-1$ is a subword of the reading of u'. Remark that i and $i+1$ are adjacent for some inscription of C if and only if the same holds for C'. It follows that for any two paths C_1 and C_2 joining t to u, we have $i \preccurlyeq_{C_1} i+1 \Leftrightarrow i \preccurlyeq_{C_2} i+1$. Now let s be an inscription in W such that there exists a path C_1 joining t to s and let r be an inscription representing T^c which is on the north-east of both s and u and has no common cell with them. There exists certainly a path C_2 joining s to r and a path C_3 joining u to r and such that the elements of C_3 are obtained from u by a sequence of translations of rows and columns. Remark that we have $\preccurlyeq_{C_3} = \preccurlyeq_u$. Now, we have $i \preccurlyeq_{C_1 \cup C_2} i+1 \Leftrightarrow i \preccurlyeq_{C \cup C_3} i+1 \Leftrightarrow i \preccurlyeq_C i+1$, whence Lemma 7. ▮

We use the notation $i \not\preccurlyeq_C j$ *if* $i \preccurlyeq_C j$ does not hold.

LEMMA 8. *Let t, u, C be as above. If $i \preccurlyeq_C j$, then there exists an admissible permutation σ for W with a reduced expression $\sigma = s_p \cdots s_1$ such that for some k $(i \leqslant k \leqslant j-1)$ we have*

$$\sigma(i) = k, \qquad \sigma(j) = k+1, \qquad \sigma(l) = l \quad \text{if } l < i \text{ or } l > j.$$

Proof. We will reason by induction on $j - i$. Suppose that $i \not\preccurlyeq_C j$ with $j - i > 1$. If $i \not\preccurlyeq_C i+1$, then by Lemma 7, $i \not\preccurlyeq_W i+1$ and $s = (i, i+1)$ is admissible for W. Since $s(j) - s(i) = j - i - 1$, the proposition holds true by induction. We suppose now that $i \preccurlyeq_C i+1$, which implies that $i+1 \not\preccurlyeq_C j$. Let σ' be a product promised by the lemma for the pair $(i+1, j)$. If $\sigma'(j) < j$, we can conclude by induction, since $\sigma'(i) = i$. If not then $\sigma'(j) = j$, so $\sigma'(i+1) = j-1$. We have $j-1 \not\preccurlyeq_{\sigma' C} j$, so by Lemma 7, $j-1 \not\preccurlyeq_{\sigma' W} j$ and the product $\sigma = (j-1, j)\sigma'$ is admissible for W. We have $\sigma(i) = i$ and $\sigma(j) = j-1$, so induction applies in this case also. ∎

We can now conclude the proof of Theorem 5: supposing that $i \not\preccurlyeq_C j$, let $C, t, u, \sigma = s_p \cdots s_1$ and k be as above. Since the product σ is admissible for W, σC is a path joining σt to σu. We have $k \not\preccurlyeq_{\sigma C} k+1$, so by Lemma 7, $k \not\preccurlyeq_{\sigma W} k+1$; it follows that $i \not\preccurlyeq_W j$. ∎

If a and b are letters written on the plane with coordinates x_a, y_a and x_b, y_b, we put $a \downarrow b$ if $x_a < x_b$ and $y_a > y_b$. From now on we identify T with the inscription representing T such that the letter 1 lies on the point $(1, 1)$. If $a_1 > \cdots > a_p$ is the shape of T, we identify T^c with the inscription representing T^c such that the letter n lies on the point (a_1, p). Let C be a path joining T to T^c. By Theorem 5, we have $\preccurlyeq_W = \preccurlyeq_C$. We can write $C = (T_0, T_1, ..., T_k)$ with $T_0 = T$, $T_{i+1} = T_i \gamma_{x_i}$ $(0 \leqslant i \leqslant k-1)$, $T_k = T^c$. We denote by $\mathscr{T}_{0,i} = T_i$, $\mathscr{T}_{1,i}$, ..., $\mathscr{T}_{n_i,i} = T_{i+1}$ the consecutive "tableaux with a blank" which allow us to pass from T_i to T_{i+1} through a sequence of elementary switches. The blank of T_i is by convention the starting point of T_i. We denote by C_i the word $\mathscr{T}_{0,i}, \mathscr{T}_{1,i}, ..., \mathscr{T}_{n_i-1,i}$ and by C' the word $C_0, C_1, ..., C_{k-1}, T^C$; C' will be called the *detailed path* of C. For example, if $T = {}^{3}_{1}\,{}^{\cdot}_{2}\,{}^{\cdot}_{4}$, then

$$C = \begin{pmatrix} 3 & \cdot & \cdot & & 1 & 3 & \cdot & & 1 & 3 & 4 \\ 1 & 2 & 4\text{'} & & \cdot & 2 & 4\text{'} & & \cdot & \cdot & 2 \end{pmatrix}$$

$$C' = \begin{pmatrix} 3 & \bullet & \cdot & & \bullet & 3 & \cdot & & 1 & 3 & \bullet & & 1 & 3 & 4 & & 1 & 3 & 4 \\ 1 & 2 & 4\text{'} & & 1 & 2 & 4\text{'} & & \cdot & 2 & 4\text{'} & & \cdot & 2 & \bullet\text{'} & & \cdot & \cdot & 2 \end{pmatrix},$$

where the sign $\bullet$ stands for blanks. For sake of simplicity let us relabel the elements of C': $C' = C'_1, ..., C'_q$. We put $D_l = \{C'_1, ..., C'_l\}$ for $1 \leqslant l \leqslant q$. We have $\preccurlyeq_{C'} = \preccurlyeq_{D_q} = \preccurlyeq_W$, so $\mathscr{G}_{D_q} = \mathscr{G}(T)$. We will use the notation $\mathscr{G}_l$ for $\mathscr{G}_{D_l}$.

THEOREM 6. *It is possible to draw the graph $\mathscr{G}(T)$ on the plane in such a way that*:

(1) *$\mathscr{G}(T)$ is planar.*

(2) *The vertices of $\mathscr{G}(T)$ are drawn on $N \times N$.*

(3) *If $i<j$ are consecutive with respect to $\preccurlyeq_W$, then the edge(i, j) is either vertical or horizontal and ij or (respectively) ji is a subword of some element of W.*

(4) *$i \nearrow j$ on $\mathscr{G}(T)$ if and only if $i \preccurlyeq_W j$.*

(5) *If $i \downarrow j$ on $\mathscr{G}(T)$, then ij is a subword of any element of W.*

Proof. Parts (3) and (5) are equivalent to (3′) and (5′):

• (3′) If $i<j$ are consecutive with respect to $\preccurlyeq_W$, then the edge (i, j) is either horizontal or vertical and ij or (respectively) ji is a subline or (respectively) a subcolumn of some element of $C' = D_q$.

• (5′) If $i \downarrow j$ on $\mathscr{G}(T)$, then $i \downarrow j$ for any element of $C' = D_q$.

The following proof by induction yields an algorithm in q steps to construct $\mathscr{G}(T)$.

We suppose that (1), (2), (3′), (4), (5′), are verified when replacing $\mathscr{G}(T)$ by $\mathscr{G}_l$ and W and D_q by D_l $(1 \leqslant l \leqslant q-1)$. If $\preccurlyeq_{D_{l+1}} \neq \preccurlyeq_{D_l}$, then the blank of C'_l has an immediate left neighbour (say j) and an immediate bottom neighbour (say i). Suppose first that $j > i$. The lth switch moves j on top of i, and we have $i \not\preccurlyeq_{D_l} j$ an $i \preccurlyeq_{D_{l+1}} j$,

$$\begin{matrix} j & \bullet \\ \cdot & i \end{matrix} \;\to\; \begin{matrix} \bullet & j \\ \cdot & i \end{matrix}$$

$$C'_l \to C'_{l+1}$$

(α) Let $\preccurlyeq_{i,j}$ be the order defined by the single relation $i \preccurlyeq_{i,j} j$. The order $\preccurlyeq_{D_{l+1}}$ is the transitive closure of the union $\preccurlyeq_{D_l} \cup \preccurlyeq_{i,j}$. It follows that i and j are consecutive with respect to $\preccurlyeq_{D_{l+1}}$.

(β) If for some x we have $j \downarrow x \downarrow i$ on $\mathscr{G}_l$, then by (5′) $j \downarrow x \downarrow i$ for any element of D_l; in particular $j \downarrow x \downarrow i$ on C'_l, a contradiction.

(γ) If $\mathscr{G}_l$ has a vertical edge (x, j) with $x<j$, then by (3′), x and j are consecutive with respect to $\preccurlyeq_{D_l}$ and there is an element C'_r of D_l for which jx is a subcolumn. For $r \leqslant v \leqslant l$, either C'_v contains the subcolumn jx or $j \downarrow x$ since if not we would have for some w, $r \leqslant w \leqslant l-1$,

$$\begin{matrix} j & \bullet \\ x & y \end{matrix} \;\to\; \begin{matrix} \bullet & j \\ x & y \end{matrix}$$

$$C'_w \to C'_{w+1}$$

and $x \preccurlyeq_{D_l} y \preccurlyeq_{D_l} j$, a contradiction. Keeping track of the respective locations of i, j, x along D_l, we conclude that we have necessarily $x \preccurlyeq_{D_l} i$, and finally $x \nearrow i$ on $\mathscr{G}_l$ (by 4).

(δ) Let us consider in $\mathscr{G}_l$ the rectangle with horizontal and vertical sides and ji as a diagonal:

$$\begin{array}{ccccc} j & \square & \cdots & \square & \eta) \\ \square & \diamond & \cdots & \diamond & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \diamond & \cdots & \diamond & \square \\ \zeta) & \square & \cdots & \square & i \end{array}$$

By (β), the region marked $\diamond$ on the above figure is empty and by (γ) the left side of the rectangle is empty, except for j and possibly for the cell ζ. In a similar way it is easy to check that the other sides are also empty, except for i and possibly for the cell η.

Now let us erase the edges of $\mathscr{G}_l$, and define the set X_l as the set of vertices x of $\mathscr{G}_l$ such that ($j \nearrow x$) or ($i \nearrow x$ and $i \neq x$) or $i \downarrow x$. By translating X_l to the right until j lies above i, we obtain a set of points Y_l which by (δ) and (4) has the following property: $x \preccurlyeq_{D_{l+1}} y$ if and only if $x \nearrow y$ on Y_l. We state that if we create an edge between each pair of points in Y_l lying on the same row or on the same column, we obtain the graph of $\preccurlyeq_{D_{l+1}}$ and that this graph verifies all the required properties. This can be checked by reasoning by cases. In case we have $z \nearrow j$ on $\mathscr{G}_l$ with z on the north-west of the cell ζ, then the last edge of every path joining z to j on $\mathscr{G}_l$ is horizontal, so moving j does not create any difficulty. The other cases are easy. Finally, if $j < i$ a similar construction applies, and the case $l = 1$ is trivial, which concludes the proof. ▮

The mechanism of the proof allows us to derive a fast algorithm for constructing $\mathscr{G}(T)$ (see Fig. 1 for an example of computation). Remark that it follows from the theorem that $\preccurlyeq_W$ is a lattice order, and that $(i, i+1)$ is admissible for W if and only if i and $i+1$ are not comparable with respect to $\nearrow$ on $\mathscr{G}(T)$.

We give now an alternative proof of Theorem 4 which yields an algorithm for the computation of plaques. We denote by $\mathscr{G}(\mathscr{P})$ the graph obtained from $\mathscr{G}(T)$ by erasing the labels $1, 2, ..., n$ on its vertices. This notation is valid because by Proposition 6 the set $\mathscr{P}^* = \{\mathscr{G}(R), R \in \mathscr{P}\}$ is the set of those labellings L of $\mathscr{G}(P)$ sucht that we have ($i \nearrow j$ on $L \Rightarrow i < j$).

We call *column graph* (respectively *row graph*) of $\mathscr{G}(\mathscr{P})$ and we denote by $\mathscr{G}_1$ (respectively $\mathscr{G}_2$) the labelling in $\mathscr{P}^*$ such that if $i \downarrow j$ on $\mathscr{G}_1$ (respectively

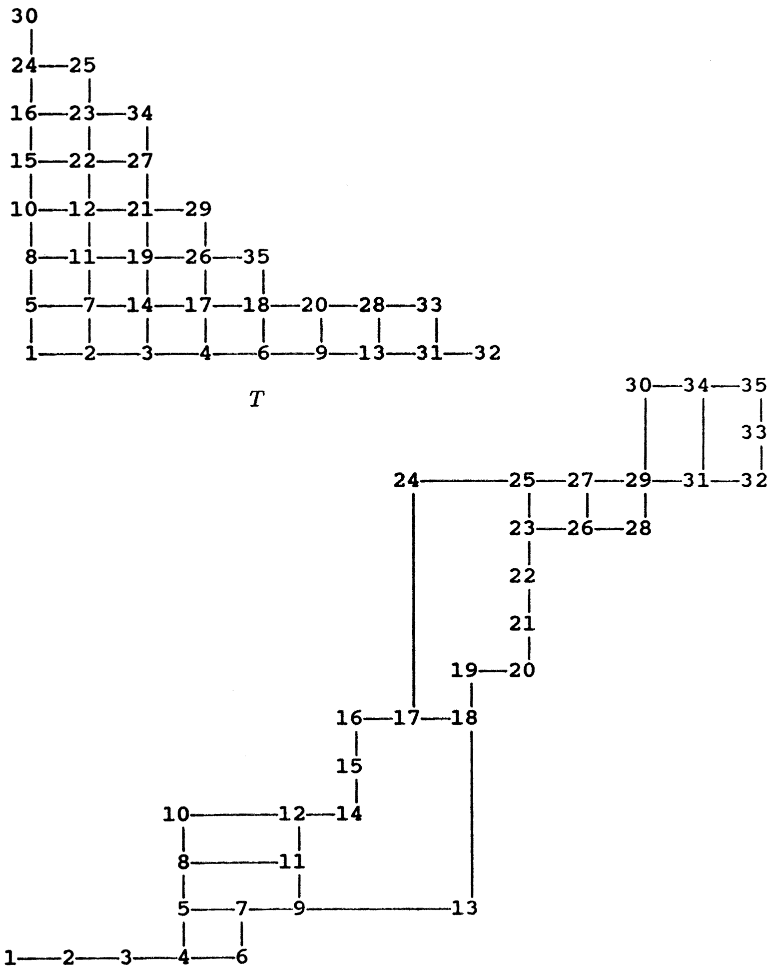
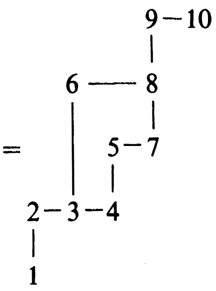


FIG. 1. The graph $\mathscr{G}(T)$.

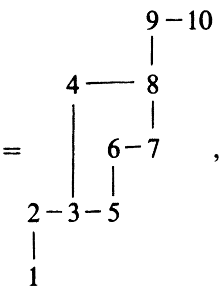
$\mathscr{G}_2$), then $i<j$ (respectively $j<i$); $\mathscr{G}_1$ and $\mathscr{G}_2$ are uniquely determined by these conditions because the inversions of their reading permutations are completely defined, so these permutations are themselves uniquely determined. Let g_1, g_T, g_2 be the readings of $\mathscr{G}_1, \mathscr{G}(T), \mathscr{G}_2$. Since $\mathscr{I}(g_1) \subseteq \mathscr{I}(g_T) \subseteq \mathscr{I}(g_2)$, we have $g_1 \leqslant_B g_T \leqslant_B g_2$. Let σ_1 and σ_2 be the permutations such that $g_1 = \sigma_1 g_T$ and $g_2 = \sigma_2 g_T$; σ_1 and σ_2 are admissible for W and we have $\sigma_1 T \leqslant_B T \leqslant_B \sigma_2 T$, by induction. Now suppose that $w \in \mathscr{S}_n$ is such that $\sigma_1 T \leqslant_B w \leqslant_B \sigma_2 T$. By Lemma 5, w is a tableau of same shape as T, and by Lemma 4, w is in $\mathscr{P}$, which concludes the proof. ∎

As claimed, this proof provides an algorithm for computing the plaque of a given Young tableau. For example, if

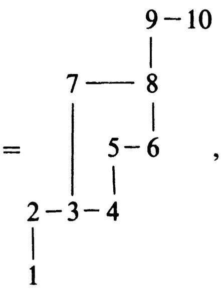
$$T = \begin{array}{llll} 9 & & & \\ 6 & 8 & & \\ 2 & 5 & 10 & \\ 1 & 3 & 4 & 7 \end{array} \qquad \text{then } \mathscr{G}(T) =$$



$\mathscr{G}_1 =$



, $\mathscr{G}_2 =$



,

and $\sigma_1 = (4, 5)(5, 6)$, $\sigma_2 = (6, 7)$, so

$$\sigma_1 T = \begin{array}{llll} 9 & & & \\ 4 & 8 & & \\ 2 & 6 & 10 & \\ 1 & 3 & 5 & 7 \end{array} \qquad \text{and} \qquad \sigma_2 T = \begin{array}{llll} 9 & & & \\ 7 & 8 & & \\ 2 & 5 & 10 & \\ 1 & 3 & 4 & 6 \end{array},$$

and the plaque of T is the interval $\{(9, 4, 8, 2, 6, 10, 1, 3, 5, 7), (9, 7, 8, 2, 5, 10, 1, 3, 4, 6)\}$. Figure 2 contains the plaques corresponding to the shape 3, 3, 1.

If P is a Young tableau then $P\amalg = I$ depends on the shape of P only. In particular $I\Pi = I\Pi$, so I is an involution [6]; I is in fact the only Young tableau of same shape as P which is an involution, and it can be obtained by an easy algorithm.

PROPOSITION 7. *Let P, Q, and I be Young tableaux of same shape, with I as above. Then if P or Q is in the plaque of I, we have*

$$(P \cdot I \cdot Q^{-1})\Pi = P \qquad \textit{and} \qquad (P \cdot I \cdot Q^{-1})\amalg = Q.$$

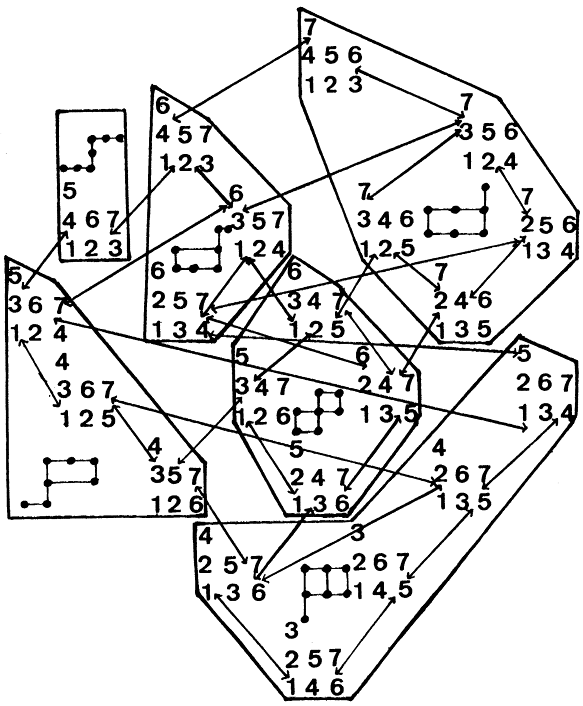


FIG. 2. The plaques of shape 3, 3, 1.

Proof. Suppose for instance that P is in the plaque of I, and let σ be the permutation such that $\sigma I = P$. We have $Q^{-1}\Pi = I$, $Q^{-1}\amalg = Q$ [6], and Theorem 3 allows us to conclude. ▮

In the case the shape of P is rectangular the proposition has a startling corollary:

THEOREM 7. *If P and Q are rectangular tableaux of same shape, then*

$$(P \cdot I \cdot Q^{-1})\Pi = P \qquad \textit{and} \qquad (P \cdot I \cdot Q^{-1})\amalg = Q.$$

Proof. It suffices to remark that $P = P^c$, so by Theorem 5, P, I, and Q are necessarily in the same plaque. ▮

Remark that in the rectangular case, I has a particularly simple form, since it is the top element of the set of tableaux of that shape, or *hyperstandard* tableau. For example, if $P = \begin{smallmatrix} 2 & 5 & 6 \\ 1 & 3 & 4 \end{smallmatrix}$ and $Q = \begin{smallmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{smallmatrix}$, then $I = \begin{smallmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{smallmatrix}$ and $P \cdot I \cdot Q^{-1} = (256134)(456123)(451263) = (251364)$.

PROPOSITION 8. *The set of all tableaux of a given shape form a plaque if and only if that shape is rectangular.*

Proof. It remains to verify that the tableaux of a non-rectangular shape decompose into more than one plaque. The domain corresponding to such a shape contains necesseraly two cells $a-1$, b and $a, b-1$ with a, b a starting point for the (dual) jeu de taquin. It is then easy to construct a tableau of that same shape containing two consecutive letters i and $i+1$ in those two cells. Now for that particular tableau the transposition $(i, i+1)$ is not admissible. ▮

Another extremal situation is met when the shape is a hook (see the Introduction), in which case every tableau makes up a plaque of its own.

PROPOSITION 9. *For a given shape, every plaque is reduced to one element if and only if the shape is a hook.*

Proof. If the shape of a tableau is a hook, then no transposition $(i, i+1)$ may be admissible, by an easy induction. Inversely, given a shape that is not a hook, there exists a tableau of that particular shape containing the subword 342, and by Lemma 2(2) and Theorem 5, if a tableau contains the subword *bca*, then *ab* is admissible (with the notations of Lemma 3). ▮

EXAMPLES. (1) *The graph* $\mathscr{G}(T)$. Figure 1 contains an example of the graph $\mathscr{G}(T)$.

(2) *Plaques and permutohedron.* The graph of the weak Bruhat order is sometimes called *permutohedron.* Figure 2 represents the subinterval of the permutohedron of $\mathscr{S}_7$ consisting of the Young tableaux of shape 3, 3, 1. Within every plaque $\mathscr{P}$ the graph $\mathscr{G}(\mathscr{P})$ is represented.

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