Free cumulants and representations of large symmetric groups

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Abstract. We show how free cumulants, which have appeared in free probability theory, can be used to evaluate characters of symmetric groups. They provide the key for understanding the asymptotic behaviour of the representation theory of symmetric groups of large order.

1. Introduction

Although the representation theory of finite symmetric groups is well developped, it is a difficult problem to understand the asymptotic properties of this theory when the symmetric groups become large. Some results, in connection with representation theory of infinite symmetric groups, have been obtained by S. Kerov and A. Vershik [**KV2**] who considered representations associated with Young diagrams with n boxes having largest row and column of the order of n. Considering the Plancherel measure on Young diagrams, or other natural measures (see Section 4 below), one sees that most Young diagrams have their largest row and column of the order of $\sqrt{n}$, therefore some new tools are needed in order to understand the asymptotics of representation theory in this regime. Indeed the problem resembles that of statistical mechanics where one has to find the relevant macroscopic parameters of a system whose microscopic description is known. Here the "microscopic" description correspond to the usual algorithms like the Littlewood-Richardson rule or the Murnagham-Nakayama formula, whereas we want to argue that the right "macroscopic" parameters for describing the representation theory are provided by the free cumulants. We shall define free cumulants, which originate from free probability theory, in Section 1 and 2, and explain in Section 3 how they describe the asymptotic of representations of large symmetric groups. Finally we mention in the last section that free cumulants also yield explicit universal formulas for characters.

2. Free cumulants

Let $(m_n)_{n\geq 1}$ be a complex sequence and form the generating series

$$G(z)=\frac{1}{z}+\sum_{n=1}^{\infty}m_nz^{-n-1}$$

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This series has an inverse, for composition of power series, in the form

$$K(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n z^{n-1}$$

The coefficients R_n form the free cumulant sequence associated with the sequence m_n, they can be expressed as polynomials in the m_n, for example the first few terms are

$$R_1 = m_1; \quad R_2 = m_2 - m_1^2; \quad R_3 = m_3 - 3m_2m_1 + 2m_1^3$$

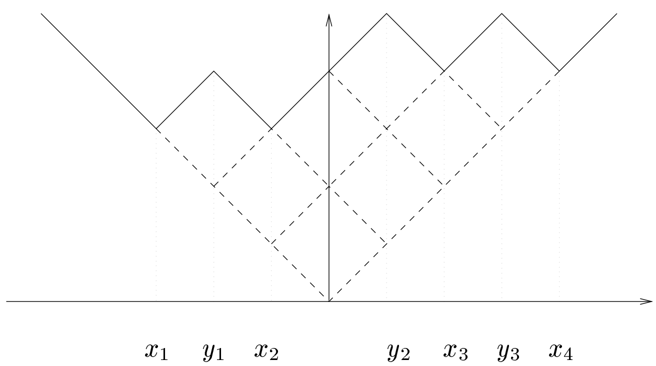
These relations can be inverted and yield polynomial expressions of the m_n in terms of the R_n, e.g.

$$m_1 = R_1; \quad m_2 = R_2 + R_1^2; \quad m_3 = R_3 + 3R_2R_1 + R_1^3$$

Free cumulants have appeared in the theory of free probability, in the case where the m_n form the sequence of moments of a probability measure on the real line. Then the free cumulants linearize the free convolution of measures, in the same way that ordinary cumulants linearize the classical convolution of measures, see e.g. [**VDN**]. The relation between moments and free cumulants can be described combinatorially using non-crossing partitions, as discovered by R. Speicher [**S**].

3. Young diagrams

It is well known that irreducible representations of the symmetric group on n letters are parametrized by Young diagrams with n boxes. We shall use a non-standard description of such Young diagrams, which has been extensively used by S. Kerov and A. Vershik, namely we shall represent a Young diagram by a function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(x) = |x|$ for $|x|$ large enough, and ω is a piecewise affine function, with slopes ± 1, see the following picture which shows the Young diagram corresponding to the partition $8 = 3 + 2 + 2 + 1$.



Alternatively we can encode the Young diagram using the local minima and local maxima of the function ω, denoted by $x_1, \ldots, x_k$ and $y_1, \ldots, y_{k-1}$ respectively, which form two interlacing sequences of integers. These are (-3,-1,2,4) and (-2,1,3) respectively in the above picture. Associated with the Young diagram there is a unique probability measure m_ω on the real line, such that

$$\int_{\mathbb{R}} \frac{1}{z-x} m_\omega(dx) = \frac{\prod_{i=1}^{k-1}(z-y_i)}{\prod_{i=1}^{k}(z-x_i)} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R} \tag{3.1}$$

This probability measure is supported by the set $\{x_1, \dots, x_k\}$ and is called the transition measure of the diagram, see [**K1**]. One can give the following interpretation to this probability measure. Consider the representation λ of S_n associated with the Young diagram, and build the following $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & [\lambda](1\,2) & [\lambda](1\,3) & \dots & [\lambda](1\,n-1) & [\lambda](1\,n) \\ 1 & [\lambda](1\,2) & 0 & [\lambda](2\,3) & \dots & [\lambda](2\,n-1) & [\lambda](2\,n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & [\lambda](1\,n) & [\lambda](2\,n) & [\lambda](3\,n) & \dots & [\lambda](n-1\,n) & 0 \end{pmatrix}$$

whose entries are in $End(V)$, where V is the vector space of the representation λ. Here $[\lambda](i\,j)$ denotes the image of the transposition $(i\,j)$ in the representation λ. It turns out that this operator, considered as a $dim(V) \times (n+1)$-dimensional matrix, has real spectrum $\{x_1, \dots, x_k\}$ and the relative multiplicity (i.e. the multiplicity divided by the total dimension of the space) of the eigenvalue x_i is equal to $m_\omega(\{x_i\})$, see [**B1**].

The set of Young diagrams can be embedded in the set of functions $\omega : \mathbb{R} \to \mathbb{R}$ such that

1. $\omega(x) = |x|$ for $|x|$ large enough.
2. $\omega(x) - \omega(y)| \leq |x - y|$ for all x, y.

For such functions a probability measure m_ω can be defined uniquely by the identity

$$\frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{x-z} d\alpha(x) = \int_{\mathbb{R}} \frac{1}{z-x} m_\omega(dx) \tag{3.2}$$

which extends (3.1). Here $\alpha(u) = (\omega(u) - |u|)/2$ is Lipschitz, hence differentiable almost everywhere, with bounded derivative. Such functions are called *generalized diagrams*, and a sequence $R_n(\omega)$ of free cumulants can also be associated to them.

4. Asymptotics of representation theory

4.1. Asymptotic of characters. As we shall now see the free cumulants provide the asymptotic value of characters for large symmetric groups. Let σ denote the conjugacy class in S_n of a permutation with k_2 cycles of length 2, k_3 of length 3, etc.. We shall keep $k_2, k_3, \ldots$ fixed while we let $n \to \infty$. Suppose that a sequence of Young diagrams with n boxes is chosen such that the corresponding rescaled functions $n^{-\frac{1}{2}} \omega_n(n^{\frac{1}{2}} x)$ converge uniformly to some generalized diagram ω as $n \to \infty$. Denote by χ_n the normalized character associated with the Young diagram ω_n, then one has the following asymptotic result from [**B1**]

$$\chi_n(\sigma) = n^{-(k_2+2k_3+3k_4+\dots)/2} \left(\prod_j R_{j+1}^{k_j}(\omega) + O(n^{-1}) \right)$$

Using this asymptotic evaluation of characters, one can describe the behaviour of representation theory of large symmetric groups.

4.2. Asymptotic of restriction. Let ω be a generalized diagram and $t \in [0,1]$, then there exists a unique generalized diagram ω_t whose free cumulants satisfy $R_n(\omega_t) = t^{n-1} R_n(\omega)$. Suppose, as in the preceding subsection, that we choose a sequence ω_n, such that the rescaled Young diagrams converge to ω, and let p_n be such that $p_n/n \to t$, then as $n \to \infty$ almost all diagrams (with respect to the natural measure on diagrams), occuring in the restriction of the representation to S_{p_n}, suitably rescaled, become close to the generalized diagram ω_t.

4.3. Asymptotic of induction. For generalized diagrams ω, ω', define a new generalized diagram by $R_n(\omega \boxplus \omega') = R_n(\omega) + R_n(\omega')$. This operation, which corresponds to the operation of free convolution on the associated measures, becomes relevant in the asymptotic description of the Littlewood-Richardson rule. Choose two sequences p_n and q_n of integers growing like $\sqrt{n}$, and sequences of Young diagrams with p_n and q_n boxes such that ω_n and ω'_n rescaled by $\sqrt{n}$, converge to the diagrams ω and ω'. Consider the natural induced representation of $S_{p_n+q_n}$ built from ω_n and ω'_n, then as $n \to \infty$ almost all Young diagrams occuring in the decomposition of this induced representation, rescaled by $\sqrt{n}$, are close to the generalized diagram $\omega \boxplus \omega'$.

4.4. Asymptotic of Kronecker tensor products. In order to describe the asymptotic of tensor products, one needs to introduce the generalized diagrams whose free cumulants all vanish except the second. Let

$$\Omega(x) = \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4-x^2}) \quad \text{for } |x| \leq 2$$
$$\Omega(x) = |x| \quad \text{for } |x| > 2$$

then $R_2(\Omega) = 1$ and $R_n(\Omega) = 0$ for $n \neq 2$. The corresponding measure is Wigner's semi-circle distribution. The diagram Ω appears as the limit shape for Young diagrams under Plancherel measure [**KV1**], [**LS**]. Let ω_n, ω'_n be sequences of diagrams with n boxes and with $O(\sqrt{n})$ rows and columns, then as $n \to \infty$, almost all diagrams occuring in the decomposition of the tensor product representation of S_n, when rescaled by $\sqrt{n}$, become close to Ω.

4.5. Asymptotic factorization of positive definite functions. It follows from the asymptotic result of Subsection 3.1, that normalized irreducible characters satisfy an asymptotic factorization property, namely for large n one has

$$\chi(\sigma_1\sigma_2) \sim \chi(\sigma_1)\chi(\sigma_2)$$

for permutations σ_1, σ_2 with bounded disjoint supports. There is a converse result for normalized positive definite functions [**B2**], namely consider a normalized positive definite function ψ on S_n, (i.e. $\psi(e) = 1$), then it decomposes as a convex combination of irreducible characters

$$\psi = \sum_\lambda p_\lambda \chi_\lambda$$

Now if n is large and ψ satisfies an approximate factorization property,

$$\psi(\sigma_1\sigma_2) \sim \psi(\sigma_1)\psi(\sigma_2)$$

then it follows that the measure p_λ is concentrated on a family of Young diagrams whose shape are close to a certain given shape, determined by the sequence of cumulants $R_k = n^{(k-1)/2}\psi(c_k)$, where c_k is a cycle of order k. Using this result one can extend the classical results of Kerov-Vershik [**KV1**] and Logan-Shepp [**LS**] on the Plancherel measure, to more general measures [**B2**]. Consider the natural representation of S_n on $(\mathbb{C}^N)^{\otimes n}$. The multiplicity of a Young diagram in this representation can be obtained in terms of Schur functions, by a classical formula of Frobenius (see e.g. Mac Donald [**M**], formula 7.8). We can describe the asymptotic shape of a typical Young diagram arising in this representation when $n \to \infty$. There is a critical regime which is $n/\sqrt{N} \to c \in [0, \infty[$. The case $c = 0$ gives the same result as in the Plancherel measure case, i.e. the typical Young diagram rescaled by $\sqrt{n}$ has the limit shape Ω where Ω is as in subsection 3.3. For other values of c we obtain limit shapes described by the following formulas. Define

$$h(c,u) = \frac{2}{\pi}\left(u \arcsin\left[\frac{u+c}{2\sqrt{1+uc}}\right] + \frac{1}{c}\arccos\left[\frac{2+uc-c^2}{2\sqrt{1+uc}}\right] + \frac{1}{2}\sqrt{4-(u-c)^2}\right)$$

for $0 < c < \infty$, and $u \in [c-2, c+2]$. Then the diagrams are given by P_c where

$$P_c(u) = \begin{cases} h(c,u) & \text{if } u \in [c-2, c+2] \\ |u| & \text{if } u \notin [c-2, c+2] \end{cases}$$

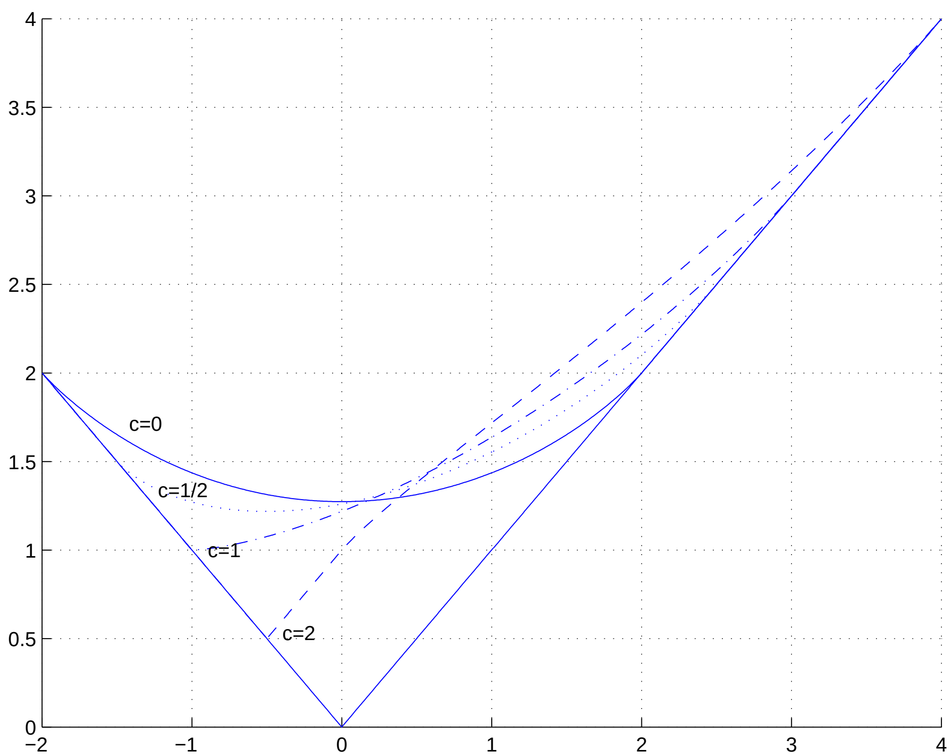
for $0 < c < 1$

$$P_1(u) = \begin{cases} \frac{u+1}{2} + \frac{1}{\pi}\left((u-1)\arcsin(\frac{u-1}{2}) + \sqrt{4-(u-1)^2}\right) & \text{if } u \in [-1, 3] \\ |u| & \text{if } u \notin [-1, 3] \end{cases}$$

$$P_c(u) = \begin{cases} u + \frac{2}{c} & \text{if } u \in [-\frac{1}{c}, c-2] \\ h(c,u) & \text{if } u \in [c-2, c+2] \\ |u| & \text{if } u \notin [-\frac{1}{c}, c+2] \end{cases}$$

for $c > 1$.

The diagrams are depicted below for various values of c.



There are also more general results exhibiting connections with freely infinitely divisible distributions, see [**B2**].

5. Exact formulas for characters via free cumulants

As remarked by S. Kerov [**K2**], free cumulants can also be used to get universal, exact formulas for character values. More precisely consider the following quantities

$$\Sigma_k(\omega) = n(n-1)\dots(n-k+1)\chi_\omega(c_k)$$

for $k \geq 1$ where χ_ω is a normalized character of S_n associated with a Young diagram ω with n boxes, and c_k is a cycle of order k (with $c_1 = e$).

THEOREM 5.1 (Kerov's formula for characters). *There exist universal polynomials $K_1, K_2, \dots, K_m, \dots,$ with integer coefficients, such that the following identities hold for any Young diagram ω with n boxes*

$$\Sigma_k(\omega) = K_k(R_2(\omega), R_3(\omega), \dots, R_{k+1}(\omega)).$$

We list the few first such polynomials

$$\begin{aligned}
\Sigma_1 &= R_2 \\
\Sigma_2 &= R_3 \\
\Sigma_3 &= R_4 + R_2 \\
\Sigma_4 &= R_5 + 5R_3 \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned}$$

Assign a degree k to the variable R_k, then the highest degree term appearing in Σ_k is R_{k+1}, as follows from Subsection 4.1. Apart from this little is known on the coefficients of the polynomials K_k, although they seem to have interesting combinatorial significance. In particular Kerov conjectured that all non zero coefficients are positive integers, but at present this remains unproved.

Note that the above formula can also been considered as an asymptotic expansion for the characters evaluated on cycles.

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