Poissonian exponential functionals, q-series, q-integrals, and the moment problem for log-normal distributions

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Abstract. Moments formulae for the exponential functionals associated with a Poisson process provide a simple probabilistic access to the so-called q-calculus, as well as to some recent works about the moment problem for the log-normal distributions.

1. Introduction and main results

For an arbitrary random variable $X > 0$ with finite expectation, we denote by $\hat{X}$ a variable distributed according to the so-called length-biased law of X, viz.

$$\mathbb{E}\left(f(\hat{X})\right) = \frac{1}{\mathbb{E}(X)}\mathbb{E}\left(Xf(X)\right),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ stands for a generic Borel function. Several authors, including Chihara [10], Vardi *et al.* [25], Pakes and Khattree [20], Pakes [19], ... considered the situation when there is the identity in distribution

$$X \overset{(\mathrm{d})}{=} q\hat{X}, \tag{1.1}$$

for some fixed real number $q \in]0, 1[$. The main motivation for the aforementioned works stems from the easy fact that when (1.1) is fulfilled, then $\sqrt{q}X$ has the same entire moments as the log-normal variable $\exp(Y_{\sigma^2})$, where Y_{σ^2} denotes a centered Gaussian variable with variance $\sigma^2 = -\log q$, that is

$$\mathbb{E}\left((\sqrt{q}X)^n\right) = q^{-n^2/2} \qquad (n \in \mathbb{Z}). \tag{1.2}$$

Berg [3] explains how to go from Chihara's solutions (with countable support) to solutions with continuous densities. See also Stoyanov [22, 23] for a succinct discussion, and Christiansen [11, 12] and Gut [16] for other recent contributions to the indeterminate moment problems.

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The purpose of this note is to investigate an example for which (1.1) holds, that arises naturally in the study of exponential functionals of Poisson processes. Specifically, consider a standard Poisson process $(N_t, t \geq 0)$ and define its exponential functional by

$$I^{(q)} = \int_0^\infty dt\, q^{N_t} .$$

Note that we may also express $I^{(q)}$ in the form

$$I^{(q)} = \sum_{n=0}^{\infty} q^n \varepsilon_n , \tag{1.3}$$

where $\varepsilon_n = T_{n+1} - T_n$, $n = 0, 1, \ldots$ denote the waiting times between the successive jump times $T_n = \inf\{t : N_t = n\}$ of $(N_t, t \geq 0)$. In other words, $(\varepsilon_n, n \in \mathbb{N})$ is a sequence of i.i.d. exponential variables with parameter 1.

Next, define the random variable $L^{(q)}$ (or rather its distribution) in terms of that of $I^{(q)}$ by

$$\mathbb{E}\left(f(L^{(q)})\right) = \frac{1}{\mathbb{E}(1/I^{(q)})}\mathbb{E}\left(\frac{1}{I^{(q)}} f({I'}^{(q)}/I^{(q)})\right) \tag{1.4}$$

for every Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where ${I'}^{(q)}$ is an independent copy of $I^{(q)}$. We claim that $X = L^{(q)}$ satisfies (1.1). This is easily seen from the explicit calculation of the moments of $I^{(q)}$ as obtained in [6, 8]; details will be given in the next section. This observation incited us to investigate further the distributions of $I^{(q)}$ and $L^{(q)}$. In this direction, it is quite natural to use the so-called q-calculus (see, e.g. [15], [17], ...) which is associated with the basic hypergeometric series of Euler, Gauss, ... For the convenience of the reader, we have gathered in the Appendix the classical formulae attached to these series, by simply reproducing a selection from R. Askey's foreword to [15], which is exactly tailored to our needs. To state our main result, we introduce some standard notation from the q-calculus:

$$\begin{aligned} (a;q)_n &= \prod_{j=0}^{n-1}(1 - aq^j) , \qquad (1.5)\\ (a;q)_\infty &= \prod_{j=0}^{\infty}(1 - aq^j) ,\\ \Gamma_q(x) &= \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} . \end{aligned}$$

Theorem 1.1. (i) *The Laplace transform of $I^{(q)}$ is given by*

$$\mathbb{E}\left(\exp(\lambda I^{(q)})\right) = \frac{1}{(\lambda;q)_\infty} \qquad (\lambda < 1) , \tag{1.6}$$

its Mellin transform by

$$\mathbb{E}\left(\left(I^{(q)}\right)^s\right) = \frac{\Gamma(1+s)}{\Gamma_q(1+s)(1-q)^s} = \Gamma(1+s)\frac{(q^{1+s};q)_\infty}{(q;q)_\infty} , \tag{1.7}$$

and its density, which we denote as $(i^{(q)}(x), x \geq 0)$*, by*

$$i^{(q)}(x) = \sum_{n=0}^{\infty} \exp\left(-x/q^n\right) \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_\infty (q;q)_n}\,. \tag{1.8}$$

(ii) *The Mellin transform of* $L^{(q)}$ *is given by*

$$\mathbb{E}\left(\left(L^{(q)}\right)^s\right) = \frac{\Gamma(1+s)\Gamma(-s)}{(q;q)_\infty^2 \log(1/q)} (q^{1+s};q)_\infty (q^{-s};q)_\infty \qquad (-1 < Re\, s < 0) \tag{1.9}$$

and its density, which we denote as $(\lambda_q(x), x \geq 0)$*, by*

$$\lambda_q(x) = \frac{1}{(q;q)_\infty^3 \log(1/q)} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n\, q^{n(n+1)/2}}{q^n + x}\right) \tag{1.10}$$

$$= \frac{1}{x(-qx;q)_\infty (-1/x;q)_\infty (q;q)_\infty \log(1/q)} \tag{1.11}$$

Remarks. *(a) It is interesting to point out that expressions similar to (1.9) and (1.11) can be found in Pakes* [19]*; see in particular pages 834-5 there. Specifically, the Mellin transform* $M_0(t)$ *as given before Theorem 3.3 in* [19] *can be identified as* $\mathbb{E}((L^{(q)})^{-t})$*, i.e. the function* M_0 *coincides with the Mellin transform of* $1/L^{(q)}$*. In this direction, the expression (1.11) for the density of* $L^{(q)}$ *can be obtained from equation (3.11) in* [19] *specified for* $\gamma = 0$ *and Lemma 3.3 there. The identity between (1.10) and (1.11) can be found as a special case of an identity due to Bhargava and Adiga* [7]*; we shall also prove this below for the sake of completeness.*

(b) In [9]*, the authors obtain the density of* $\int_0^\infty h(N_t)dt$ *for a large class of functions* $h : \mathbb{N} \to \mathbb{R}_+$*, and in particular* i_q *when* $h(n) = q^n$*.*

As we were writing this paper, we became aware of the works of Lachal [18] who recognized that $I^{(q)}$ plays some role in a probabilistic model of DNA duplication introduced by Cowan and Chiu [13], while Dumas *et al.* [14] find the law of $I^{(q)}$ as an invariant measure related to a Transmission Control Protocol. In particular, the formula (1.8) was found independently from us by Dumas *et al.*, see Proposition 13 and its proof in [14].

Prior to this work and other cited references in the present paper, J. Pitman told us about another connection between the q-calculus and probability via Bernoulli trials; see e.g. Rawlings [21].

The rest of this work is organized as follows. In Section 2, we present a detailed proof of Theorem 1.1. In Section 3, we present further connections with the q-calculus based on the self-decomposability of $I^{(q)}$. Finally in the Appendix, we quote from Askey's foreword to [15] some key formulas of q-calculus (for an elementary approach, see Kac and Cheung [17]).

2. Proof of Theorem 1

We first develop some material on moments of the exponential functionals associated with a certain family of subordinators. Specifically, let $(\xi_t, t \geq 0)$ be a subordinator starting from 0 and assume that ξ has exponential moments of all orders. The Laplace transform

$$\mathbb{E}\left(\exp(\lambda \xi_t)\right) = \exp(t\Psi(\lambda)), \qquad (\lambda \in \mathbb{R}, t \geq 0)$$

is well defined and finite. For notational convenience, we write $\Phi(\lambda) = -\Psi(-\lambda)$ for $\lambda \in \mathbb{R}$, so

$$\mathbb{E}\left(\exp(-\lambda \xi_t)\right) = \exp(-t\Phi(\lambda)), \qquad (\lambda \in \mathbb{R}, t \geq 0),$$

i.e. Φ is the so-called Laplace exponent of ξ. We associate to ξ its exponential functional

$$I(\xi) = \int_0^\infty dt \exp(-\xi_t),$$

and lift from [6] and [8] some results about the moments of $I(\xi)$. First, $I(\xi)$ admits positive and negative moments of all orders, and there is the formula

$$\mathbb{E}\left(I(\xi)^s\right) = \frac{s}{\Phi(s)} \mathbb{E}\left(I(\xi)^{s-1}\right), \qquad (s \in \mathbb{R}, s \neq 0), \tag{2.1}$$

which extends to $s = 0$ as

$$\mathbb{E}(1/I(\xi)) = \mathbb{E}(\xi_1) = \Phi'(0).$$

Next, we introduce a variable $L(\xi)$ (or rather its distribution) via the following

$$\mathbb{E}\left(f(L(\xi))\right) = \frac{1}{\Phi'(0)} \mathbb{E}\left(\frac{1}{I(\xi)} f(I(\xi')/I(\xi))\right)$$

for every Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where ξ' is an independent copy of ξ. It follows that the Mellin transform of $L(\xi)$ satisfies the functional equation

$$E\left(L(\xi)^s\right) = \frac{\Psi(s)}{\Phi(s)} \mathbb{E}\left(L(\xi)^{s-1}\right), \tag{2.2}$$

where for $s = 0$, we agree that $\Psi(s)/\Phi(s) = 1$. We also note that, by the very construction of $L(\xi)$, we have $\mathbb{E}(L(\xi)^{-s}) = \mathbb{E}(L(\xi)^{s-1})$, so that (2.2) may also be written as

$$E\left(L(\xi)^s\right) = \frac{\Psi(s)}{\Phi(s)} \mathbb{E}\left(L(\xi)^{-s}\right). \tag{2.3}$$

The present paper is concerned with the Poisson case $\xi = (\log 1/q)N$, on which we focus from now on. Thus we write $I(\xi) = I^{(q)}$ and $L(\xi) = L^{(q)}$ in the sequel. We have

$$\Psi(s) = q^{-s} - 1 \quad , \quad \Phi(s) = 1 - q^s,$$

which yields the identity

$$\Psi(s)/\Phi(s) = q^{-s}.$$

In particular, we deduce from (2.1) that the entire moments of $I^{(q)}$ can be expressed using the notation (1.5) as

$$\mathbb{E}\left(\left(I^{(q)}\right)^j\right) = \frac{j!}{(q;q)_j}, \qquad j = 1,2,\dots \tag{2.4}$$

On the other hand, specifying (2.2) and (2.3) in this setting, we deduce by inversion of the Mellin transform the remarkable identities in distribution

$$q\hat{L}^{(q)} \overset{(\mathrm{d})}{=} L^{(q)} \overset{(\mathrm{d})}{=} \frac{1}{qL^{(q)}}. \tag{2.5}$$

In particular (1.1) holds for $X = L^{(q)}$.

We now turn our attention to the proof of Theorem 1.1, and in this direction, the following result, which is closely related to our paper [5] (see also [4]), provides the key to some calculations.

Lemma 2.1. *Let $(G_{q^n}; n = 1,2,\dots)$ be independent, geometrically distributed variables with respective parameters q^n, i.e.*

$$\mathbb{P}(G_a = k) = (1-a)a^k, \qquad k = 0,1,\dots,$$

and set

$$R^{(q)} \overset{(\mathrm{d})}{=} q^{\sum_{n=1}^{\infty} G_{q^n}}. \tag{2.6}$$

Then there is the identity in distribution

$$\varepsilon \overset{(\mathrm{d})}{=} I^{(q)} R^{(q)}, \tag{2.7}$$

where on the left-hand side, ε denotes a standard exponential law, and on the right-hand side, the variables $R^{(q)}$ and $I^{(q)}$ are supposed independent.

Remark. The identity (2.7) is closely connected to the construction of q-beta and q-gamma variables, which is done in [19], building upon e.g. Askey's papers [1, 2].

Proof. The Mellin transform of q^{G_a} is given by

$$\mathbb{E}\left(q^{sG_a}\right) = (1-a)\sum_{k=0}^{\infty} q^{sk}a^k = \frac{(1-a)}{(1-aq^s)}, \qquad s \geq 0,$$

and hence that of $R^{(q)}$ by

$$\mathbb{E}\left((R^{(q)})^s\right) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{s+n})} = \frac{(q;q)_\infty}{(q^{s+1};q)_\infty}. \tag{2.8}$$

In particular, taking $s = j$ integer, we find that the j-th moment of $R^{(q)}$ equals $(q;q)_j$, and we deduce from (2.4) that $I^{(q)}R^{(q)}$ has the same entire moments as ε. This proves our claim as the exponential law is determined by its entire moments. $\square$

We are now able to establish Theorem 1.1.

Proof. The identity (1.6) for the Laplace transform of $I^{(q)}$ derives immediately from the expression (1.3). The formula (1.7) for the Mellin transform of $I^{(q)}$ follows immediately from the fact that the Mellin transform of the standard exponential distribution is $\mathbb{E}(\varepsilon^s) = \Gamma(s+1)$, the identity (2.8) for the Mellin transform of $R^{(q)}$, and the factorization (2.7) of Lemma 2.1. The formula (1.9) for the Mellin transform of $L^{(q)}$ is then deduced from (1.7) and its definition (1.4) in terms of $I^{(q)}$.

We then turn our attention to the densities. First, we may rewrite Euler's formula (4.5) of the Appendix for $x = q^{1+s}$ as

$$(q^{1+s};q)_\infty \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} q^{n(1+s)}}{(q;q)_n}\,.$$

Plugging the identity

$$\Gamma(1+s) q^{n(s+1)} \;=\; \int_0^\infty dx\, x^s \exp(-x/q^n)\,,$$

this establishes (1.8) by inverting the Mellin transform (1.7).

In order to compute the density λ_q of $L^{(q)}$, we rewrite the Mellin transform (1.9) using the triple product identity (4.7). More precisely we get

$$\mathbb{E}\left(\left(L^{(q)}\right)^s\right) \;=\; \frac{\Gamma(1+s)\Gamma(-s)}{(q;q)_\infty^3 \log(1/q)} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} q^{ns}\right)\,, \qquad (-1 < Re\, s < 0)\,.$$

The identity

$$\Gamma(1+s)\Gamma(-s) \;=\; \int_0^\infty dv\, \frac{v^s}{1+v}\,,$$

now yields

$$\begin{aligned} \mathbb{E}\left(\left(L^{(q)}\right)^s\right) &= \frac{1}{(q;q)_\infty^3 \log(1/q)} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} \int_0^\infty dv\, \frac{(vq)^{ns}}{(1+v)}\right) \\ &= \frac{1}{(q;q)_\infty^3 \log(1/q)} \int_0^\infty dx\, x^s \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n\, q^{n(n+1)/2}}{q^n + x}\right)\,. \end{aligned}$$

This establishes the formula (1.10).

We finally turn our attention to the identity between (1.10) and (1.11), which amounts to check that

$$\frac{1}{x(-qx;q)_\infty(-1/x;q)_\infty} \;=\; \frac{1}{(q;q)_\infty^2} \left(\sum_{m=-\infty}^{\infty} \frac{(-1)^m\, q^{m(m+1)/2}}{q^m + x}\right)\,. \tag{2.9}$$

Note some similarity between this identity (2.9) and the triple product formula (4.7).

As a first step, we shall identify the residues on each side of the equality, for the pole $x_k := -q^{-k}$ with $k \geq 0$ (the calculations for $k < 0$ are similar); then we shall indicate how to modify this finding to complete the proof. So we first write the denominator on the left-hand side of (2.9) as

$$x(-qx;q)_\infty(-1/x;q)_\infty \;=\; A(x)(1+xq^k)B(x)C(x)$$

with

$$\begin{aligned} A(x) &= x(1+qx)(1+q^2x)\cdots(1+q^{k-1}x) \\ B(x) &= (1+xq^{k+1})(1+xq^{k+2})\cdots \\ C(x) &= (1+1/x)(1+q/x)(1+q^2/x)\cdots \end{aligned}$$

Thus, the residue in x_k is equal to $q^{-k}\left(A(x_k)B(x_k)C(x_k)\right)^{-1}$. We obtain

$$\begin{aligned} B(x_k) &= (1-q)(1-q^2)\cdots \;=\; (q;q)_\infty\,, \\ C(x_k) &= (1-q^k)(1-q^{k+1})\cdots \;=\; (q^k;q)_\infty\,. \end{aligned}$$

Finally,

$$\begin{aligned} A(x_k) &= -\left(\frac{1}{q^k}\right)\left(1-\frac{q}{q^k}\right)\left(1-\frac{q^2}{q^k}\right)\cdots\left(1-\frac{q^{k-1}}{q^k}\right) \\ &= \frac{(-1)^k}{q^{k^2}}(q-q^k)(q^2-q^k)\cdots(q^{k-1}-q^k) \\ &= \frac{(-1)^k}{q^{k^2}}q(1-q^{k-1})q^2(1-q^{k-2})\cdots q^{k-1}(1-q) \\ &= (-1)^k q^{-(k^2+k)/2}(1-q^{k-1})(1-q^{k-2})\cdots(1-q)\,. \end{aligned}$$

Putting the pieces together, we get that the sought residue at x_k is

$$\frac{(-1)^k q^{k(k-1)/2}}{(q;q)_\infty^2}\,.$$

But this is precisely the residue for $x_k = -q^{-k}$ as found on the right-hand side of (2.9) when taking $m = -k$.

To finish the proof, it suffices to write the fractional expansion for

$$\frac{1}{x(-qx;q)_n(-1/x;q)_n} \;=\; \frac{x^{n-1}}{(1+qx)\cdots(1+q^nx)(x+1)\cdots(x+q^{n-1})}$$

with the identification of the residues at the poles of this fraction (this is exactly the computation we have done, except that the infinite products are now replaced by finite ones), and to let n tend to ∞. This completes the proof of Theorem 1.1. $\square$

3. On the self-decomposability of $I^{(q)}$

Thanks to expression (1.3), the self-decomposability of the exponential law propagates to the law of $I^{(q)}$ (see formula (3.1) below), which allows to make more connections with the q-calculus (the basic formulae of which are recalled in the Appendix).

Proposition 3.1. (i) *For every $c \in]0,1[$, there is the decomposition*

$$I^{(q)} \overset{(\mathrm{d})}{=} cI^{(q)} + I_c^{(q)}\,, \tag{3.1}$$

where on the right-hand side, $I_c^{(q)}$ is independent of $I^{(q)}$, and satisfies

$$\mathbb{E}\left(\exp\left(\lambda I_c^{(q)}\right)\right) = \frac{(c\lambda;q)_\infty}{(\lambda;q)_\infty}\,, \qquad \lambda < 1\,, \tag{3.2}$$

$$\mathbb{E}\left(\left(I_c^{(q)}\right)^n\right) = \frac{(c;q)_n}{(q;q)_n}\,n!\,, \qquad n \in \mathbb{N}\,, \tag{3.3}$$

$$\mathbb{E}\left(\left(I_c^{(q)}\right)^s\right) = \Gamma(1+s)\frac{(q^{1+s};q)_\infty\,(c;q)_\infty}{(cq^s;q)_\infty(q;q)_\infty}\,, \qquad s \geq 0\,. \tag{3.4}$$

(ii) *Furthermore, for $c < q$, the variable $I_c^{(q)}$ is the exponential functional*

$$I\left(\xi_c^{(q)}\right) = \int_0^\infty dt \exp\left(-\xi_c^{(q)}(t)\right)$$

associated with $\xi_c^{(q)}$, a compound Poisson process whose Lévy measure is the probability

$$\nu_c^{(q)}(dx) = \sum_{m=1}^{\infty} (c/q)^{m-1}\,(1-c/q)\,\delta_{m\log(1/q)}(dx)\,.$$

Proof. (i) The expression for the Laplace transform (3.2) immediately derives from the self-decomposability (3.1) of $I^{(q)}$ and the formula (1.6). Then (3.3) is obtained from (3.2), using the series development of $\exp(\lambda x)$ and the q-binomial theorem, as expressed in formula (4.6) in the Appendix.

We shall now identify the Mellin transform of $I_c^{(q)}$ thanks to the following integral result due to Ramanujan; cf. formula (11) in Askey [2]: for every x such that $c < q^x$, we have

$$\int_0^\infty \frac{(-ct;q)_\infty}{(-t;q)_\infty} t^{x-1} dt = \frac{\pi}{\sin(\pi x)}\,\frac{(c;q)_\infty (q^{1-x};q)_\infty}{(cq^{-x};q)_\infty (q;q)_\infty}\,. \tag{3.5}$$

Let us replace on the left-hand side the ratio $(-ct;q)_\infty/(-t;q)_\infty$ using (3.2); we obtain

$$\int_0^\infty \mathbb{E}\left(\exp\left(-tI_c^{(q)}\right)\right) t^{x-1} dt = \Gamma(x)\mathbb{E}\left(\left(I_c^{(q)}\right)^{-x}\right)\,.$$

Since $\pi/\sin(\pi x) = \Gamma(x)\Gamma(1-x)$, this yields (3.4) by choosing $s = -x$.

(ii) One readily checks that the Laplace exponent Φ of the subordinator $\xi_c^{(q)}$ is given by

$$\Phi(n) \;=\; \frac{1-q^n}{1-cq^{n-1}} \qquad (n=1,2,\dots)\,.$$

The identity in distribution $I_c^{(q)} \overset{\text{(d)}}{=} I\left(\xi_c^{(q)}\right)$ follows by moment identification from (3.3) and the general formula

$$\mathbb{E}\left((I(\xi))^n\right) \;=\; \frac{n!}{\Phi(1)\cdots\Phi(n)}\,,$$

which stems from (2.1). □

This allows us, as a first application, to identify the law of $R_c^{(q)}$, an independent variable from $I_c^{(q)}$, which satisfies

$$\varepsilon \overset{\text{(d)}}{=} I_c^{(q)} R_c^{(q)}\,; \tag{3.6}$$

see [5]. Indeed, from (3.4) and the fact that $\mathbb{E}(\varepsilon^s) = \Gamma(1+s)$, we obtain

$$\mathbb{E}\left(\left(R_c^{(q)}\right)^s\right) \;=\; \frac{(q;q)_\infty (cq^s;q)_\infty}{(q^{1+s};q)_\infty (c;q)_\infty}\,. \tag{3.7}$$

It is interesting to note that in the above computation, we derived the Mellin transform of $I_c^{(q)}$ thanks to Ramanujan's identity (3.5), whereas for $c=0$, we used the representation of $R^{(q)}$ to obtain the Mellin transform of $I^{(q)}$.

Next, using (2.8), we rewrite (3.7) as

$$\mathbb{E}\left(\left(R^{(q)}\right)^s\right) \;=\; \mathbb{E}\left(\left(R_c^{(q)}\right)^s\right) \frac{(c;q)_\infty}{(cq^s;q)_\infty}\,; \tag{3.8}$$

which in turn, in the notation of Lemma 2.1, leads us to the identity

$$q^{\sum_{n=0}^\infty G_{q^n}} \overset{\text{(d)}}{=} R_c^{(q)} q^{\sum_{n=0}^\infty G_{cq^n}}\,. \tag{3.9}$$

Finally, we have obtained

Corollary 3.2. *The factorization (3.6) holds with*

$$R_c^{(q)} \overset{\text{(d)}}{=} q^{\sum_{n=0}^\infty X_{c,q^n}}\,,$$

with $X_{a,b}$ defined via, either:
(i) $X_{a,b} = \delta_{a,b}(1+G_b)$ *where $\delta_{a,b}$ is a Bernoulli variable with* $\mathbb{P}(\delta_{a,b}=0) = 1-\mathbb{P}(\delta_{a,b}=1) = (1-b)/(1-ab)$, *and G_b is independent from $\delta_{a,b}$;*
(ii) $G_b \overset{\text{(d)}}{=} G_{ab} + X_{a,b}$, *where $X_{a,b}$ and G_{ab} are independent.*

4. Appendix: Some basic q-formulae

(Extract from R. Askey's foreword to [15])

" Basic hypergeometric series are series $\sum_n c_n$, with c_{n+1}/c_n a rational function of q^n for a fixed parameter q, which is usually taken to satisfy $|q| < 1$, but at other times is a power of a prime. In this Foreword, $|q| < 1$ will be assumed.

Euler summed three basic hypergeometric series. The one which had the largest impact was

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q;q)_\infty , \tag{4.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) . \tag{4.2}$$

If

$$(a;q)_n = (a;q)_\infty / (aq^n;q)_\infty , \tag{4.3}$$

then Euler showed that

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} , \qquad |x| < 1 , \tag{4.4}$$

and

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} . \tag{4.5}$$

Eventually, all of these were contained in the q-binomial theorem

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n , \qquad |x| < 1 . \tag{4.6}$$

While (4.4) is clearly the special case (of (4.6)) $a = 0$, and (4.5) follows easily on replacing x by xa^{-1} and letting $a \to \infty$, it is not so clear how to obtain (4.1) from (4.6). The easiest way was discovered by Cauchy and many others. Take $a = q^{-2N}$, shift n by N, rescale and let $N \to \infty$. The result is called the triple product, and can be written as:

$$(x;q)_\infty (qx^{-1};q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n . \tag{4.7}$$

Then, $q \to q^3$ and $x = q$ gives Euler's formula (4.1)."

Askey then goes on describing the contributions of Gauss, Jacobi, Eisenstein, and finally Heine's introduction of a basic hypergeometric extension of ${}_2F_1(a, b, c; z)$, which we hope to deal with in a future paper...

References

[1] R. Askey (1978). The q-gamma and q-beta functions. *Appl. Anal.* **8**, 125-141.

[2] R. Askey (1980). Ramanujan's extensions of the gamma and beta functions. *Amer. Math. Monthly* **87**, 346-359.

[3] C. Berg (1998). From discrete to absolutely continuous solutions of indeterminate moment problems. *Arab J. Math. Sci.* **4**, 1-18.

[4] C. Berg and A. J. Duran (2002). A transformation from Hausdorff to Stieltjes moment sequences. Preprint.

[5] J. Bertoin and M. Yor (2001). On subordinators, self-similar Markov processes, and some factorizations of the exponential law. *Elect. Commun. Probab.* **6**, 95-106. Available at `http://www.math.washington.edu/ ejpecp/ecp6contents.html`.

[6] J. Bertoin and M. Yor (2002). On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse* **Série 6, vol. XI**, 33-45.

[7] S. Bhargava and C. Adiga (1994). A basic bilateral series summation formula and its applications. *Integral Transform. Spec. Funct.* **2**, 165-184.

[8] P. Carmona, F. Petit and M. Yor (1994). Sur les fonctionnelles exponentielles de certains processus de Lévy. *Stochastics and Stochastics Reports* **47**, 71-101.

[9] P. Carmona, F. Petit and M. Yor (1997). On the distribution and asymptotic results for exponential functionals of Levy processes. In: M. Yor (editor) *Exponential functionals and principal values related to Brownian motion.* Biblioteca de la Revista Matemática Iberoamericana.

[10] T. S. Chihara (1970). A characterization and a class of distribution functions for the Stieltjes-Wigert polynomials. *Canadian Math. Bull.* **13**, 529-532.

[11] J. S. Christiansen (2003). The moment problem associated with the q-Laguerre polynomials. *Constr. Approx.* **19**, 1-22.

[12] J. S. Christiansen (2003). The moment problem associated with the Stieltjes-Wigert polynomials. *J. Math. Anal. Appl.* **277**, 218-245.

[13] R. Cowan and S. N. Chiu (1994). A stochastic model of fragment formation when DNA replicates. *J. Appl. Probab.* **31**, 301-308

[14] V. Dumas, F. Guillemin and Ph. Robert (2002). A Markovian analysis of additive-increase, multiplicative-decrease (AIMD) algorithms. *Adv. in Appl. Probab.* **34**, 85-111.

[15] G. Gasper and M. Rahman (1990). *Basic Hypergeometric Series.* Cambridge University Press, Cambridge.

[16] A. Gut (2002). On the moment problem. *Bernoulli* **8**, 407-421.

[17] V. Kac and P. Cheung (2002). *Quantum Calculus.* Springer, Universitext, New York.

[18] A. Lachal (2003). Some probability distributions in modelling DNA replication. *Ann. Appl. Probab.* **13**, 1207-1230.

[19] A. G. Pakes (1996). Length biasing and laws equivalent to the log-normal. *J. Math. Anal. Appl.* **197**, 825-854.

[20] A. G. Pakes and R. Khattree (1992). Length-biasing, characterization of laws, and the moment problem. *Austral. J. Statist.* **34**, 307-322.

[21] D. Rawlings (1998). A probabilistic approach to some of Euler's number-theoretic identities. *Trans. Amer. Math. Soc.* **350**, 2939-2951.

[22] J. Stoyanov (1997). *Counterexamples in probability.* Wiley.

[23] J. Stoyanov (2000). Krein condition in probabilistic moment problems. *Bernoulli* **6**, 939-949.

[24] K. Urbanik (1992). Functionals of transient stochastic processes with independent increments. *Studia Math.* **103 (3)**, 299-315.

[25] Y. Vardi, L. A. Shepp and B. F. Logan (1981). Distribution functions invariant under residual-lifetime and length-biased sampling. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **56**, 415-426.

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