

Hecke group algebra of a Coxeter group at roots of
unity

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Analyse Algèbre et Géométrie

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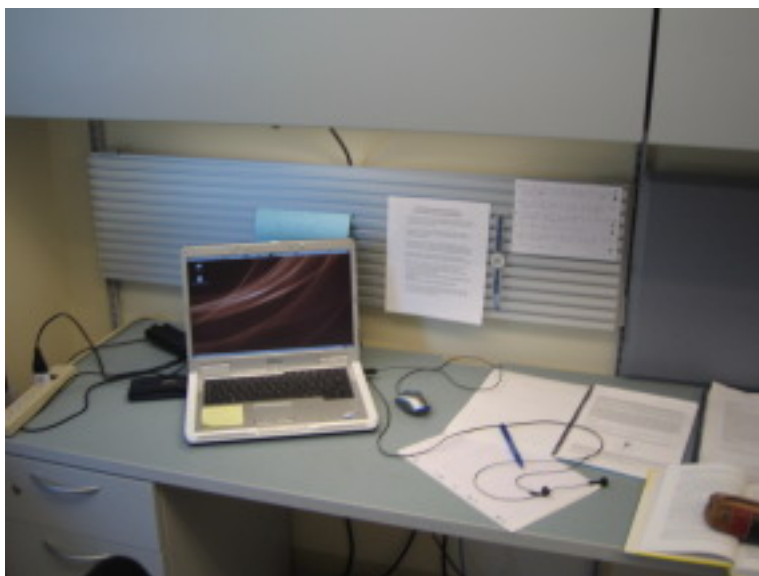
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1 Introduction

This work has been realised at the University of California Davis. I was there from the 17th of April to the 16th of July 2008. I worked in the Mathematical Science Building integrated in the Team of Anne Schilling. The aim of this travel was to take a real contact with the research and people who work in it. I had to work in team, to attempt several talks, to make my own work, to participate to the weekly seminar, to work on a software development. Another hard points were : discovering the combinatorics ways of thinking, practicing the English language, organizing myself with the time.



Let's see the beautifully Mathematical sciences building.



This was the part of the office that the M.S.B. (Mathematical Science Building)

gave to me. It was very nice from them. That allowed me to have the better conditions for working. I thank them and the team of Anne schilling for all they done for me. Especially, I thank mister Thiéry who made possible this adventure.

On a mathematical view, this study is about Hecke algebra and Hecke group algebra. This last algebraic structure was studied by Florent Hivert and Nicolas M. Thiéry [HT07]. The theory of usual Hecke algebra is not easy but there is already a lot of results about them and their comportment at roots of unity. The Hecke group algebra is a kind of globalising structure which contains all the classic Hecke algebras for a given Cartan type. The aim of this study is to explore what happen at roots of unity for Hecke group algebra. Nicolas M. Thiéry and Anne schilling have proven an hard result about that [HST08] ; they gave a limit for the roots of unity which make the representation degenerate. By the computer, I tried to see the more...

After explaining the context and the construction of the Hecke group algebra, I will say how I computed the representations and how I tried to improve my observation. I will give my success and my failures in optimizing the results.

For the curious reader, I leave in Annexe parts of implementation in SAGE [Ste08].

2 Context, Definitions and theory

2.1 Hecke algebra of a Coxeter group

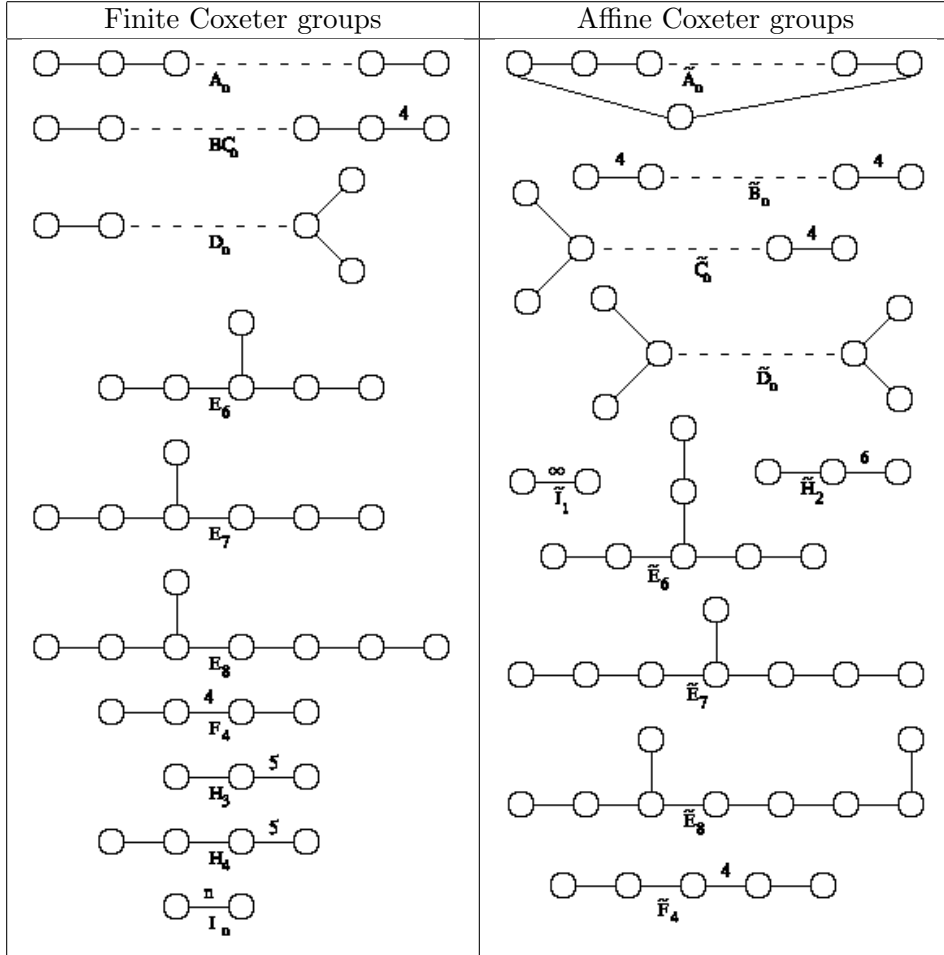
Definition 2.1 (Coxeter Group). [BB05] *A Coxeter system is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form : $(ss')^{m(s,s')} = 1$, where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S . In case no relation occurs for a pair s, s' , we make the convention that $m(s, s') = \infty$. Formally, W is the quotient F/N , where F is a free group on the set S and N is the normal subgroup generated by all elements $(ss')^{m(s,s')}$. When S is finite (this will be always the case in following studies), and $m(s, s') < \infty$ for all s, s' in S , we define $(m(s, s'))_{s,s' \in S}$ to be the Coxeter matrix M .*

Remark We call rank of W , $Rank(W) = Card(W)$. The rank of a Coxeter group is the smallest number of reflexions we need to span it.

The main example are permutation groups. S_n is generated by the transpositions $s_i = (i, i + 1)$ where $1 \leq i \leq n - 1$. So $S = \{s_1, s_2, \dots, s_{n-1}\}$ generate S_n and the Coxeter matrix of these groups are :

$$\begin{pmatrix} 1 & 3 & 2 & \dots & \dots & 2 \\ 3 & 1 & 3 & \ddots & & \vdots \\ 2 & 3 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 \\ \vdots & & \ddots & \ddots & \ddots & 3 \\ 2 & \dots & \dots & 2 & 3 & 1 \end{pmatrix}$$

The theory of reflexion group and Coxeter group is already describe in the book of James E.Humphrey's [Hum90]. These group are classified and recognize by their dynkin diagram. With the same notation, this graph has its vertices indexed by the set of generator S and we put an edge between the vertices s and s' if $m(s, s') \geq 2$. We label the edge by $m(s, s')$ if $m(s, s') \geq 3$.



Group Symbol	Rank	Order
A_n	n	$(n + 1)!$
$B_n = C_n$	n	$2^n n!$
D_n	n	$2^{n-1} n!$
$I_2(p)$	2	$2p$
H_3	3	120
F_4	4	1152
H_4	4	14400
E_6	6	51840
E_7	7	2903040
E_8	8	696729600

Definition 2.2 (Group algebra). *Let W be a finite group and K a field. The group algebra $K[W]$ is the vector space over K which admit a basis $T_w, w \in W$ indexed by element of W . The product in $K[W]$ is construct from the product*

in W and the mobility of the scalars of K .

For example, if W is the group $S_2 = \{Id, s\}$. $K[W]$ is a vector space of dimension 2. $\{Id, s\}$ is a base of this vector space and the rule of multiplication is given by this table :

\times	Id	s
Id	Id	s
s	s	Id

So $(5Id + 2s)(-4s) = -20s - 8Id$.

Definition 2.3 (Hecke algebra of a coxeter group). Suppose that (W, S) is a Coxeter system with the Coxeter matrix M . Fix a ground ring R (most commonly, R is the ring of the integers or an algebraically closed field, for us it will be \mathbb{C}). Let q be a formal indeterminate, and let $A = R[q, q^{-1}]$ be the ring of Laurent polynomials over R . Then the Hecke algebra $H(W)(q)$ defined by these data is the unital associative algebra over A with generators T_s for all $s \in S$ and the relations:

$$T_s T_t^{m(s,t)} = 1$$

for $s, t \in S, s \neq t$. These last are called the Braid relations. And for $s \in S$ the quadratic relation:

$$(T_s + q)(T_s - 1) = 0$$

This ring is also called the generic Hecke algebra, to distinguish it from the ring obtained from H by specializing the indeterminate q to an element of R (for example, a complex number if $R=\mathbb{C}$).

Remark Let (W, S) be a Coxeter system. Let $H(W)(q)$ be the generic Hecke algebra of the Coxeter group W . Let T_1, T_2, \dots, T_n be the Hecke generators indexed by S . A natural basis of $H(W)(q)$ is

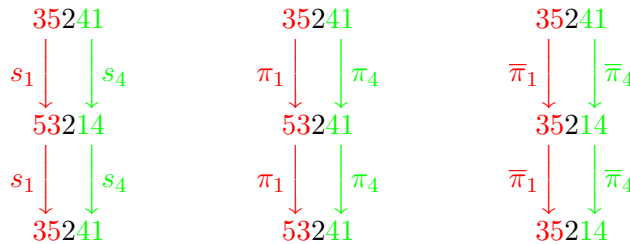
$$T_w = T_{i_1} T_{i_2} \dots T_{i_r} \text{ for } w = i_1 i_2 \dots i_r \text{ is a reduced word in } W$$

Definition 2.4 (0-(Iwahori)-Hecke algebra). Let W be a Coxeter group. the 0-Hecke algebra $H(W)(0)$ is the Hecke algebra of W for which q is specialized to 0. therefore if the generators are T_1, T_2, \dots, T_n , the quadratic relations becomes

$$T_i^2 = T_i, \text{ for } 1 \leq i \leq n$$

The braid relations stay similar.

For example, in type A_4 , this mean the symmetric group S_5 We can give an explicit description of the generators of $H(S_5)(0)$.



π_i act on the uplet $a_1a_2a_3a_4a_5$. if $a_{i+1} > a_i$, π_i sort them else π_i leave the uplet already sorted at the i^{th} position. For this Coxeter group, the 0-Hecke algebra is generated by $\pi_1, \pi_2, \pi_3, \pi_4$. The relations are :

$$\begin{aligned}\pi_i^2 &= \pi && \text{for all } 1 \leq i \leq 4, \\ \pi_i\pi_j &= \pi_j\pi_i && \text{for all } |i - j| > 1, \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} && \text{for all } 1 \leq i \leq 3.\end{aligned}$$

In this example, we can also see a realisation of the group algebra of S_5 which is $H(S_5)(1)$ too. $\mathbb{C}[S_5]$ is generated by s_1, s_2, s_3, s_4 and the relation are :

$$\begin{aligned}s_i^2 &= Id && \text{for all } 1 \leq i \leq 4, \\ s_i s_j &= s_j s_i && \text{for all } |i - j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } 1 \leq i \leq 3.\end{aligned}$$

Remark The Hecke generators T_i are invertible if and only if q is invertible. As we work over \mathbb{C} , T_i is invertible if and only if q is non zero. From the quadratic relation we conclude that

$$T_i^{-1} = q^{-1}T_s + (q^{-1} - 1)$$

Theorem 2.1. *Let W be a Coxeter group and $H(W)(q)$ the generic Hecke algebra of W over the field \mathbb{C} .*

$H(W)(1)$ is the group algebra $\mathbb{C}[W]$

$H(W)(0)$ is the 0-Hecke Algebra

For q non zero and not a root of unity, $H(W)(q)$ is isomorphic to $\mathbb{C}[W]$.

2.2 Natural representations, symmetries and projections

it's not my goal to expose all representations for any Weyl group, i will just present it for the main types. For further information, i invite the reader to open the excellent book of Humphrey's [Hum90].

The symmetric group S_n ($n \geq 2$) is denote by the type A_{n-1} . It can be thought of as a subgroup of the subgroup $O(n, \mathbb{R})$ of $n \times n$ orthogonal matrices in the following way. Make a permutation act on \mathbb{R}^n by permuting the standard basis vectors $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ (permute the subscripts). Observe that the transposition (i, j) acts as a reflexion, sending $\epsilon_i - \epsilon_j$ to its negative and fixing pointwise the orthogonal complement, which consists of all vectors in \mathbb{R}^n having equal i th and j th components. Since S_n is generated by transpositions, it is a reflexion group. Indeed, it is already generated by the transpositions $(i, i + 1)$ where $1 \leq i \leq n - 1$.

For the type B_n ($n \geq 2$). Again let $V = \mathbb{R}^n$, so S_n acts on V as above. Other reflections can be defined by sending an ϵ_i to its negative and fixing all other ϵ_j . These sign changes generate a group of order 2^n isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, which intersects S_n trivially and is normalized by S_n : conjugating the sign change $\epsilon_i \mapsto -\epsilon_i$ by a transposition yields another such sign change. Thus the semidirect product of S_n and the group of sign changes yields a reflection group W of order $2^n n!$.

For the type D_n ($n \geq 4$). We can get another reflection group acting on \mathbb{R}^n , a subgroup of index 2 in the group of type B_n just described : S_n clearly normalizes the subgroup consisting of sign changes which involve an *even* number of signs, generated by the reflections $\epsilon_i + \epsilon_j \mapsto -(\epsilon_i + \epsilon_j), i \neq j$. So the semidirect product is also a reflection group.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be the canonical basis of the vector space \mathbb{R}^n (\mathbb{R}^{n+1} and $\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}$ for the type A_n).

The simple roots are given by [HST08]

$$\text{Type } A_n : \alpha_i = \begin{cases} \epsilon_{n+1} - \epsilon_1 & \text{for } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i \leq n. \end{cases}$$

$$\text{Type } B_n : \alpha_i = \begin{cases} -\epsilon_1 - \epsilon_2 & \text{for } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i < n, \\ \epsilon_n & \text{for } i = n. \end{cases}$$

$$\text{Type } C_n : \alpha_i = \begin{cases} -2\epsilon_1 & \text{for } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i < n, \\ 2\epsilon_n & \text{for } i = n. \end{cases}$$

$$\text{Type } D_n : \alpha_i = \begin{cases} -\epsilon_1 - \epsilon_2 & \text{for } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i < n, \\ \epsilon_{n-1} + \epsilon_n & \text{for } i = n. \end{cases}$$

From this, we can construct the simple symmetries. they work on $x = (x_1, x_2, \dots, x_n)$ ($x = (x_1, x_2, \dots, x_{n+1})$ for the type A_n).

$$\text{Type } A_n : s_i(x) = \begin{cases} (x_{n+1}, x_2, \dots, x_n, x_1) & \text{for } i = 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_{n+1}) & \text{for } 1 \leq i \leq n. \end{cases}$$

$$\text{Type } B_n : s_i(x) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{for } i = 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{for } i = n. \end{cases}$$

$$\text{Type } C_n : s_i(x) = \begin{cases} (-x_1, x_2, \dots, x_n) & \text{for } i = 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{for } i = n. \end{cases}$$

$$\text{Type } D_n : s_i(x) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{for } i = 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n, \\ (x_1, \dots, x_{n-2}, -x_n, -x_{n-1}) & \text{for } i = n. \end{cases}$$

With this, the action of the projection on $x = (x_1, x_2, \dots, x_n)$ becomes ($x = (x_1, x_2, \dots, x_{n+1})$ for the type A_n)

$$\text{Type } A_n : \pi_i(x) = \begin{cases} (x_{n+1}, x_2, \dots, x_n, x_1) & \text{for } i = 0 \text{ and } x_{n+1} > x_1, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_{n+1}) & \text{for } 1 \leq i \leq n \text{ and } x_i > x_{i+1}, \\ x & \text{otherwise.} \end{cases}$$

$$TypeB_n : \pi_i(x) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{for } i = 0 \text{ and } x_1 + x_2 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{for } i = n \text{ and } x_n > 0, \\ x & \text{otherwise.} \end{cases}$$

$$TypeC_n : \pi_i(x) = \begin{cases} (-x_1, x_2, \dots, x_n) & \text{for } i = 0 \text{ and } x_1 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-1}, -x_n) & \text{for } i = n \text{ and } x_n > 0, \\ x & \text{otherwise.} \end{cases}$$

$$TypeD_n : \pi_i(x) = \begin{cases} (-x_2, -x_1, x_3, \dots, x_n) & \text{for } i = 0 \text{ and } x_1 + x_2 < 0, \\ (x_1, \dots, x_{i+1}, x_i, \dots, x_n) & \text{for } 1 \leq i < n \text{ and } x_i > x_{i+1}, \\ (x_1, \dots, x_{n-2}, -x_n, -x_{n-1}) & \text{for } i = n \text{ and } x_{n-1} + x_n > 0, \\ x & \text{otherwise.} \end{cases}$$

2.3 Regular representations and action in $\mathbb{C}W$

To say that W acts on itself by multiplication is tautological. If we consider this action as a permutation representation it is characterised as having a single orbit and stabilizer the identity subgroup $\{e\}$ of W . The regular representation of W , for a given field K , is the linear representation made by taking the permutation representation as a set of basis vectors of a vector space over K . The significance is that while the permutation representation doesn't decompose - it is transitive - the regular representation in general breaks up into smaller representations. For example if W is a finite group and K is the complex number field, the regular representation is a direct sum of irreducible representations, in number at least the number of conjugacy classes of W .

Let W be a finite Coxeter group and $\mathbb{C}W$ the vector space of dimension $|W|$ it spans. By its right regular representation, W act in $\mathbb{C}W$ and define matrices of permutation. these same matrices represent the generators of the group algebra $\mathbb{C}[W]$ as element of $End(\mathbb{C}W)$.

On the other side, we may realise the 0-Hecke algebra as follows. We saw before that the generating set S of W is describe in $\mathbb{R}^{Rank(W)}$ by the simple roots. We can also construct the simple reflexions by their action on $\mathbb{R}^{Rank(W)}$. For W a Coxeter group of type A, B, C or D , any element w of W is represented by a permutation of $\{1, 2, \dots, n\}$ or a signed permutation. So, we can associate to w a vector $((-1)^{\varepsilon_1} \epsilon_{\sigma(1)}, (-1)^{\varepsilon_2} \epsilon_{\sigma(2)}, \dots, (-1)^{\varepsilon_n} \epsilon_{\sigma(n)})$. The simple projections act and stabilise this set of permutation. Therefore, for any simple projection, we can construct a matrix in $End(\mathbb{C}W)$ filled with 0 and 1. As sorting operator, these matrices M_i can't be invertible but realise $M_i^2 = M_i$. These set of matrices generated the 0-Hecke algebra in $End(\mathbb{C}W)$.

Let W be a Coxeter group and z be a complex number, the Hecke algebra $H(W)(z)$ can be realised as acting on $\mathbb{C}W$ by interpolation, mapping T_i to

$(1 - z)\pi_i + zs_i$. Indeed, by identifying each $w \in \mathbb{C}W$ with T_w , one recovers the right regular representation of $H(W)(z)$, where

$$T_w T_i = \begin{cases} (1 - z)T_w + zT_{ws_i} & \text{for } i \text{ descent of } w, \\ T_{ws_i} & \text{otherwise.} \end{cases}$$

Through this mapping, $\bar{T}_i = (1 - z)\bar{\pi}_i + zs_i$.

2.4 Construction of the Hecke group algebra

Definition 2.5 (Hecke group algebra). *Let W be a finite Coxeter group, and $\mathbb{C}W$ the vector space of dimension $|W|$ it spans. As we have just seen, we may embed simultaneously the Hecke algebra $H(W)(0)$ and the group algebra $\mathbb{C}[W]$ in $\text{End}(\mathbb{C}W)$, via their right regular representation. The Hecke group algebra HW of W is the smallest subalgebra of $\text{End}(\mathbb{C}W)$ containing them both (see [HT07]).*

Remark The Hecke group algebra is therefore generated by $(\pi_i)_{i \in I}$ and $(s_i)_{i \in I}$. HW is generated by the simple projections and the simple reflexions. So by interpolation, it contains all $H(W)(z)$ for any z complex. A basis for HW is given by $\{w\pi_{w'} \mid D_R(w) \cap D_L(w') = \emptyset\}$. A more conceptual characterization is as follows; call a vector v in $\mathbb{C}W$ *i-left antisymmetric* if $s_i v = -v$; then, HW is the subalgebra of $\text{End}(\mathbb{C}W)$ of those operators which preserve all *i-left antisymmetries*.

Example If W is the Coxeter group of type A_1 . W is isomorphic to the symmetric group $S_2 = \{(1, 2), (2, 1)\}$. $\mathbb{C}W = \mathbb{C}Id + \mathbb{C}s$ the operators are :

$$id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \bar{\pi} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\bar{\pi}$ is the operator of antisort. this action on $\mathbb{C}W$ is describe :

	(1, 2)	(2, 1)
(1, 2)	1	1
(2, 1)	0	0

If the second number is lower than the first one, $\bar{\pi}$ permutes them. The Hecke group algebra of W is a vector space of dimension 3 over \mathbb{C} . As a vector space, $HW = \mathbb{C}Id + \mathbb{C}s + \mathbb{C}\pi$. Let specify q to a complex number z , then the generator of the Hecke algebra is (with the identity) :

$$T = (1 - z) \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ 1 & 1 - z \end{pmatrix}$$

3 Method, Computation and reading of the results

From a Cartan type of an affine Weyl group, we will first get the generators of the representation which are square matrices and after, for any valor of the parameter q , we will calculate the dimension as a vector space over \mathbb{C} of the built representation. We are specially interested of what happen at roots of unity.

3.1 Hard point before computation

Before to go on the code, we have to mind about the hardware. To get a dimension of a vector space, we will have to make some Gauss reduction on vectors over the field \mathbb{C} . It is clear that we can't make any operation over this field. We have to stay with the rational or with a finite extension of them. \mathbb{Q} or a finite extension of \mathbb{Q} are dealt the same way in a computer. With a formal mathematic software, calculus are exacts. Real and complex number are approximation in machine, the addition with this number are not associative. During my work, i tried to see what happen with the field \mathbb{C} just by curiosity, results are definitely different and wrong.

So, we will work with the rational but are we going to get the right result. We want the dimension over \mathbb{C} . An argument of tensorisation can answer the question.

$$\begin{array}{ccc} \langle T_0, T_1, \dots, T_n \rangle_{\mathbb{Q}} & \xrightarrow{\text{span}} & H_{\mathbb{Q}}(W)(q) \\ \otimes \mathbb{C} \downarrow & & \downarrow \otimes \mathbb{C} \\ \langle T_0, T_1, \dots, T_n \rangle_{\mathbb{C}} & \xrightarrow{\text{span}} & H_{\mathbb{C}}(W)(q) \end{array}$$

T_0, T_1, \dots, T_n are matrices with coefficients in \mathbb{Q} . $H_{\mathbb{Q}}(W)(q)$ admit a basis composed by products of the Hecke generators. This same basis view as element of $H_{\mathbb{C}}(W)(q)$ is still a basis but now over \mathbb{C} . So

$$\text{Dim}_{\mathbb{Q}}(\langle T_0, T_1, \dots, T_n \rangle_{\mathbb{Q}}) = \text{Dim}_{\mathbb{C}}(\langle T_0, T_1, \dots, T_n \rangle_{\mathbb{C}})$$

remark This same proof work any finite extension of \mathbb{Q} ; for example if we change \mathbb{Q} by a cyclotomic extension for z any root of the unity.

3.2 Construction of the generators

Let W be a finite weyl group and \overline{W} the affine weyl group corresponding.

We construct the generators as describe in the second part. This task can be a little bit difficult and long. I tried to make the construction easier and automatic but the work and the integration into SAGE isn't already done. So, for each Cartan type, I had to do these following steps.

First, i build the group W as a list of all these elements. for $W = S_3$, the list is $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$. thus I wrote a small program describing the action for each simple reflexion. By this action, I constructed the matrice of permutation corresponding. For the same example, the reflexion s_1 , permutation $(1, 2)$ give

	(1, 2, 3)	(1, 3, 2)	(2, 1, 3)	(2, 3, 1)	(3, 1, 2)	(3, 2, 1)
(1, 2, 3)	0	0	1	0	0	0
(1, 3, 2)	0	0	0	0	1	0
(2, 1, 3)	1	0	0	0	0	0
(2, 3, 1)	0	0	0	0	0	1
(3, 1, 2)	0	1	0	0	0	0
(3, 2, 1)	0	0	0	1	0	0

After, by adding the condition of action, I implemented the projection witch are projection. For the projection π_1 , I got

	(1, 2, 3)	(1, 3, 2)	(2, 1, 3)	(2, 3, 1)	(3, 1, 2)	(3, 2, 1)
(1, 2, 3)	0	0	0	0	0	0
(1, 3, 2)	0	0	0	0	0	0
(2, 1, 3)	1	0	1	0	0	0
(2, 3, 1)	0	0	0	0	0	0
(3, 1, 2)	0	1	0	0	1	0
(3, 2, 1)	0	0	0	1	0	1

With that, it is possible to construct the Hecke generators. We construct the good field adapt to the specialization (most of the time, it was \mathbb{Q} or any cyclotomic extension) and with the complex number chosen z . $T_1(z)$ look like

$$\begin{pmatrix} 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 1 & 0 & 1-z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1 & 0 & 1-z \end{pmatrix}$$

3.3 Description of the algorithm

Now, we have to deal with a problem of linear algebra. From a set of square matrices, we have to catch the full algebra generated by these matrices and check its dimension as a vector space over \mathbb{Q} . Clearly, the natural way to resolve the point is to start with the generators, make products and products with them and stop when we have a stability by multiplicative any element with any generator.

We have to construct a non commutative algebra so the better way to mind is to consider the set of generators as an alphabet and products of generators will be words over the alphabet.

This is the main algorithm...

```
MatrixAlgebra(Generators,Field):
    Current_space = Vector_space_generated_by(Generators)_over(field)
    Current_basis = basis_of_the_vector_space(Current_space)_over(Field)
    is_stable = False
    while (not is_stable)
        |
        | is_stable = True
        | List_products = list_of_products_of_any(Current_basis)_and_any(Generators)
        | for (Word) in (List_products)
        | |
        | | if (Word) is_not_in (Current_space)
        | | |
        | | | Current_space = Vector_space_generated_by(Current_basis+Word)_over(field)
        | | | Current_basis = basis_of_the_vector_space(Current_space)_over(Field)
        | | | is_stable = False
        | |
        |
```

| Print Dimension_of(Current_space)

return Current_space

Let consider T_1, T_2, \dots, T_n as an alphabet. We construct the list of words over this alphabet

$$L^* = \{Id, T_1, \dots, T_n, T_1T_1, \dots, T_1T_n, T_2T_1, \dots, T_nT_n, T_1T_1, T_1, \dots\}$$

We order this list by size of words and by lexicographic order. The algorithm construct, read this set and for each word, it looks if it is a linear combination of the preview. If it is, it forget about the word, if not, the word is necessary to span the representation. As this representation is finite (its dimension is lower than the size of the matrices which is $|W|^2$), the algorithm stop when it find the stability.

Let n be an integer and define $H_n(z)$ to be the subspace generated as vector space by words of size at most n . $(DimH_n(z))_{n \in \mathbb{N}}$ is strictly increasing suit from 0 to stability and stationary after stability. Therefore only the first numbers in this sequence are interesting. The valor for which the sequence stabilise is the dimension of the representation.

3.4 Reading of the results

I remained the reader that I call $H_n(q)$ (resp. $H_n(z)$) the subspace generated as vector space by words of size at most n over the alphabet $T_1(q), T_2(q), \dots, T_n(q)$ (resp. $T_1(z), T_2(z), \dots, T_n(z)$) I mean here that the generators are specialised to a complex number).

During its work, the algorithm print the dimension of $H_n(z)$. The algorithm stop at stability when make any products between any element of the generated space and any Hecke generator don't create a new element. These sequence of number the algorithm give is the information i can study. So, for each specialization, i recorded the sequence and I now present the results in tableau.

A_2	Dimension						
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5
$q = 2$	19	1	4	10	19	.	.
$q = 0$	19	1	4	10	19	.	.
$q = 1$	6	1	4	6	.	.	.
$q = -1$	18	1	4	10	16	18	.
$q = 1^{\frac{1}{3}}$	11	1	4	8	11	.	.
$q = 1^{\frac{1}{4}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{5}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{6}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{7}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{8}}$	19	1	4	10	19	.	.

The first column present the studied specialization. The second one, bold, show the dimension of representation. After, each column is labeled by Wn that mean we can read the dimension for $H_n(z)$. The last number before the dots is the dimension of the full algebra $H(z)$, the dots mean the sequence is now stationary. From these tableau, it is now time to search general results.

4 Questions and improve

The main part of my work was really focused on this linear algebra problem. Starting with a set of generators as algebra and search for the dimension as a vector space. There also were problems with specializations. What happen when we specialise ?

Another point was identify which roots of unity make the dimension of the representation decrease. I can read on the tableau for "small" Cartan type, but rapidly, it become impossible to identify them. To improve the observations and try to see more, we searched for a condition easily computable which would permit us to know before the end of the full calculus.

4.1 First results

There are a lot of way to dealt with the problem. The first one I felt is to relate this with scheme theory. Specialization is defined on a localisation. I didn't develop this way because I don't know enough about this theory and considering a scheme over an non commutative $\mathbb{C}(q)$ algebra appear to me impossible. So, I tried to stay on a lower level. There were observations to do just with linear algebra.

Lemma 4.1. *Let q a format parameter and z a complex number. Let $X_1(q), X_2(q), \dots, X_n(q)$ a family of a $\mathbb{C}(q)$ vector space which specialise in $X_1(z), X_2(z), \dots, X_n(z)$ by evaluation in z ; these lasts are element of a \mathbb{C} vector space.*

If the family $X_1(z), X_2(z), \dots, X_n(z)$ is free over \mathbb{C} thus the family $X_1(q), X_2(q), \dots, X_n(q)$ is free over $\mathbb{C}(q)$. In particular,

$$\text{Dim}(\langle X_1(z), X_2(z), \dots, X_n(z) \rangle_{\mathbb{C}}) \leq \text{Dim}(\langle X_1(q), X_2(q), \dots, X_n(q) \rangle_{\mathbb{C}(q)})$$

Proof. We proof this lemma by contraposition. A relation over $\mathbb{C}(q)$ induces a non trivial relation over \mathbb{C} . Assume that $X_1(q), X_2(q), \dots, X_n(q)$ is non free over $\mathbb{C}(q)$. That mean we have a relation such that :

$$\sum_{i=1}^k \frac{P_i(q)}{Q_i(q)} X_i(q) = 0$$

We want to specialise this relation but we have to be careful. let $v_{(q-z)}$ be the valuation of $\mathbb{C}[q]$ in $(q-z)$ that we extend on $\mathbb{C}(q)$ by :

$$v_{(q-z)}\left(\frac{P(q)}{Q(q)}\right) = v_{(q-z)}P(q) - v_{(q-z)}Q(q)$$

With this, we define m to be :

$$m = \min\{v_{(q-z)}\left(\frac{P_l(q)}{Q_l(q)}\right), 1 \leq l \leq k\}$$

Now, the relation

$$\sum_{i=1}^k (q-z)^{-m} \frac{P_i(q)}{Q_i(q)} X_i(q) = 0$$

still holds over $\mathbb{C}(q)$ but its specialization is now defined and non trivial over \mathbb{C} . □

I tried, in a second time, to push kernels. But it wasn't very easy. The big difficulty is that there is not natural morphism between objects. There are also hard tensorisation problem with this strategy. With what tensorize ? Are we going to have a commutative action on each object ? The specialization in z lie over $\mathbb{C}[q]_{(q-z)}$ and not over $\mathbb{C}(q)$. all my tentatives failed this way...

$$\begin{array}{ccc} H_i(q) & \xrightarrow{?} & H_{i+1}(q) \\ ? \downarrow & & \downarrow ? \\ H_i(z) & \xrightarrow{?} & H_{i+1}(z) \end{array}$$

Another hard point came from observation. For the type A_2 , we have : $\text{Dim}(H_3(-1)) = 16 = \text{Dim}(H_3(0)) - 3$ and $\text{Dim}(H_4(-1)) = 18 = \text{Dim}(H_4(0)) - 1$. On this example, a loss of dimension 3 reduces to one dimension at the end. So, the kernel can be push this step.

4.2 A lost conjecture

The following conjecture, which was suggested by the computer data allows for recognising "bad" q 's as soon as a dimension loss is observed. Any application would have been for type A_4 . The partial results for type A_4 would have been sufficient to determinate what roots are "bad".

Conjecture 4.1. Let q be a formal parameter. Let $T_1(q), T_2(q), \dots, T_n(q)$ be square matrices with coefficients in $\mathbb{C}[q]$. We define $A(q)$ to be the $\mathbb{C}(q)$ algebra generated by $T_1(q), T_2(q), \dots, T_n(q)$. Let $z \in \mathbb{C}$, we define $A(z)$ as the unital algebra generated by $T_1(z), T_2(z), \dots, T_n(z)$ where $T_i(z)$ is a matrix with each coefficient is the polynomial in q of $T_i(q)$ evaluated in z . For $i \in \mathbb{N}$, let $A_i(q)$ (resp. $A_i(z)$) be the subspace of $A(q)$ (resp. $A(z)$) generated by all products of at most i generators $T_1(q), T_2(q), \dots, T_n(q)$ (resp. $T_1(z), T_2(z), \dots, T_n(z)$).

Assume that $\dim_{\mathbb{C}}(A_j(z)) \leq \dim_{\mathbb{C}(q)}(A_j(q))$, for some j .

Then, $\dim_{\mathbb{C}}(A_{j'}(z)) \leq \dim_{\mathbb{C}(q)}(A_{j'}(q))$ for any $j' > j$.

And in particular

$$\dim_{\mathbb{C}}(A(z)) \leq \dim_{\mathbb{C}(q)}(A(q))$$

Proof. Let us first focus on the application of specialization : $q \leftarrow z$.

We define a partial section for specialization. That means for any element $X(z)$ in $A(z)$, we can lift $X(z)$ into an element $X(q)$ in $A(q)$ such that $X(z)$ specialises $X(q)$.

$$X(z) = \sum_{I \in J} \alpha_I T_{I_1}(z) T_{I_2}(z) \dots T_{I_k}(z)$$

where I is finite sequence of elements in $1, 2, \dots, n$, J is a finite set of such sequence and α_I is a complex number. Let's define a antecedent of the specialization :

$$X(q) = \sum_{I \in J} \alpha_I T_{I_1}(q) T_{I_2}(q) \dots T_{I_k}(q)$$

where I and J don't move, α_I is now a constant polynomial in q . So $T(z)$ is trivially the specialization of $T(q)$ when $q \leftarrow z$. By this way, we have define a kind a section for the specialization. That's proves that the specialization is surjective from $A_i(q)$ to $A_i(z)$ and from $A(q)$ to $A(z)$.

After, from the loss of dimension over words of size i , I tried to construct adapted basis in which I caught find a loss of dimension for word of size $i + 1$. But there was a mistake in my proof. I tried to ovoid the problem without success. After doubts, I ask the computer to search for a counterexample for this conjecture... \square

I found the following counterexample with the computer. (The way to found it can be seen in annexe.)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & q+1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} q-1 & q+1 & q-1 \\ 0 & q+1 & q-1 \\ 0 & q+1 & 0 \end{pmatrix}$$

If we specialised q to 1, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

with the first matrices, we generate the algebra over $\mathbb{C}(q)$, for the last ones, over \mathbb{C} . The results are

	$A_0(q)$	$A_1(q)$	$A_2(q)$	$A_3(q)$	$A_4(q)$	$A_5(q)$
dim	1	3	7	9	.	.
	$A_0(1)$	$A_1(1)$	$A_2(1)$	$A_3(1)$	$A_4(1)$	$A_5(1)$
dim	1	3	6	8	9	.

So we have a loss of dimension for the specialization but it is recovered at the end.

4.3 Conclusion

During this first contact with the research, I really learnt a lot about working methods. Manage owns time is an hard task. The use of the computer in algebraic combinatorics is also very important. There are times to observe, to conjecture and try to prove good results. Working in a team allowed you to have another look of your staff. I worked sometimes in a bad way but everytimes someone was there to bring my back in a good context. At the end, I didn't success to find the result I wanted to get. But this work is still open, not closed.

5 Annexe and results of computation

All my work has been implemented in Sage[Ste08].

5.1 Cartan type A

A_1	Dimension			
	Full Space	W 0	W 1	W 2
$q = 2$	3	1	3	.
$q = 0$	3	1	3	.
$q = 1$	2	1	2	.
$q = -1$	2	1	2	.
$q = 1^{\frac{1}{3}}$	3	1	3	.
$q = 1^{\frac{1}{4}}$	3	1	3	.
$q = 1^{\frac{1}{5}}$	3	1	3	.
$q = 1^{\frac{1}{6}}$	3	1	3	.
$q = 1^{\frac{1}{7}}$	3	1	3	.
$q = 1^{\frac{1}{8}}$	3	1	3	.

A_2	Dimension						
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5
$q = 2$	19	1	4	10	19	.	.
$q = 0$	19	1	4	10	19	.	.
$q = 1$	6	1	4	6	.	.	.
$q = -1$	18	1	4	10	16	18	.
$q = 1^{\frac{1}{3}}$	11	1	4	8	11	.	.
$q = 1^{\frac{1}{4}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{5}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{6}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{7}}$	19	1	4	10	19	.	.
$q = 1^{\frac{1}{8}}$	19	1	4	10	19	.	.

A_3	Dimension										
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7	W 8	W 9
$q = 2$	211	1	5	15	35	69	121	181	207	211	.
$q = 0$	211	1	5	15	35	69	121	181	207	211	.
$q = 1$	24	1	5	15	23	24
$q = -1$	125	1	5	15	33	59	89	115	125	.	.
$q = 1^{\frac{1}{3}}$	152	1	5	15	35	68	112	139	149	152	.
$q = 1^{\frac{1}{4}}$	112	1	5	15	33	58	86	108	112	.	.
$q = 1^{\frac{1}{5}}$	211	1	5	15	35	69	121	181	207	211	.
$q = 1^{\frac{1}{6}}$	211	1	5	15	35	69	121	181	207	211	.
$q = 1^{\frac{1}{7}}$	211	1	5	15	35	69	121	181	207	211	.
$q = 1^{\frac{1}{8}}$	211	1	5	15	35	69	121	181	207	211	.

5.2 Cartan type B

B_2	Dimension								
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7
$q = 2$	33	1	4	9	17	26	31	33	.
$q = 0$	33	1	4	9	17	26	31	33	.
$q = 1$	8	1	4	7	8
$q = -1$	16	1	4	9	14	16	.	.	.
$q = 1\frac{1}{3}$	26	1	4	9	17	25	26	.	.
$q = 1\frac{1}{4}$	22	1	4	9	16	21	22	.	.
$q = 1\frac{1}{5}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{6}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{7}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{8}$	33	1	4	9	17	26	31	33	.

B_3	Dimension														
	repr.	W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7	W 8	W 9	W 10	W 11	W 12	W 13
$q = 0$	819	1	5	14	31	59	101	161	242	346	468	585	678	736	771

5.3 Cartan type C

As type B and type C created isomorphic structures, I just try to computed the type B_2 to realise it.

C_2	Dimension								
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7
$q = 2$	33	1	4	9	17	26	31	33	.
$q = 0$	33	1	4	9	17	26	31	33	.
$q = 1$	8	1	4	7	8
$q = -1$	16	1	4	9	14	16	.	.	.
$q = 1\frac{1}{3}$	26	1	4	9	17	25	26	.	.
$q = 1\frac{1}{4}$	22	1	4	9	16	21	22	.	.
$q = 1\frac{1}{5}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{6}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{7}$	33	1	4	9	17	26	31	33	.
$q = 1\frac{1}{8}$	33	1	4	9	17	26	31	33	.

5.4 Cartan type D

I didn't get any result for type D . Not only a lack of time, the first group start at D_4 . Thus, this group contain $2^3 4! = 192$ elements. Generators would have contain 36864 coefficients. This calculi is probably not adapted to classical machine.

5.5 The main algorithm

The main algorithm ask in arguments a list of generators and a field. From this, it constructs the full algebra over the field generated by the entries.

```
def MatrixAlgebra(GEN=[[1]],K=QQ):
    n = len(GEN)                /*number of generators*/
    m = len(trans_matrix(GEN[0])) /*Size of the generators*/
    V = VectorSpace(K,m)        /*Ambient space*/
    Dim = n                      /*starting dimension*/
    Current_Basis = map(V, map(trans_matrix, GEN))
    Current_Space = V.subspace(Current_Basis)
    Current_Basis = Current_Space.basis()
```

```

Dim = Current_Space.dimension()
print Dim;
not_ended = true;                /*by default, no stability*/
while not_ended:
    new = []                      /*Intialising list of new words*/
    is_stable = true              /*We suppose all is generated*/
    for i in range(n-1):
        for j in range(len(Current_Basis)):
            op_left = trans_matrix(GEN[i+1]*recompose_matrix(Current_Basis[j]))
            if ((V(op_left) not in Current_Space)):
                new.append(V(op_left))
                is_stable = false /*A new word, no stability*/
    if is_stable:                  /*If we don't find any new word...*/
        not_ended = false;        /*...the job is done*/
    else:
        Current_Space = V.subspace(Current_Basis + new)
        Current_Basis = Current_Space.basis()
        Dim = Current_Space.dimension()
        print Dim                 /*Subspace generated by word of lenght . */
Return Current_Space;

```

5.6 Searching a conterexample of the conjecture

To search a conter example to the conjecture i followed during my work, i rapidly realised that i would not be easy possible to find it in a human way. This time, the computer was clearly needed. I was searching it with square matrices of size 3, so the ambient space was of dimension 9. I was needed, with the identity, two random matrices, which don't commute most of the time. Calculus over the field $\mathbb{C}(q)$ with q formal was very heavy even the matrices are small.

My first intuition was a good one this time. I constructed a program which give randomly one of this 4 events : $0, 1, q+1, q-1$. I just choice that by equiprobability with the random function of SAGE [Ste08]. The specialization was in 1 this choice must be in correlation with the choices $q+1, q-1$. After, i just programmed a test by generate the generic Hecke algebra over $\mathbb{C}(q)$ and generate a specialised Hecke algebra with $q = 1$. therefore, i had just to wait and hope to see a loose a dimension with a recovery for last step...

Let's see a log file of the searching algorithm.

```

How to read this --->
[Id, Generator_1, Generator_2]
[suit of dimensions over C(q)]
[suit of dimensions for q=1 over C]
Match if conter example, No match else.
-----
[[1 0 0]   [ 0 q + 1 q - 1]   [q + 1   0 q + 1]
[0 1 0]   [q + 1   0   1]   [q - 1 q - 1 q - 1]
[0 0 1],  [ 1   1   0],   [ 1   1 q - 1]]
[1, 3, 7, 9]
[1, 3, 7, 9]
No Match
-----
[[1 0 0]   [q - 1 q + 1   1]   [q + 1 q - 1   0]
[0 1 0]   [q + 1   0   1]   [ 1   0 q + 1]
[0 0 1],  [q + 1   0 q - 1],   [q + 1   1   0]]
[1, 3, 7, 9]
[1, 3, 7, 9]
No Match
-----
[[1 0 0]   [q - 1 q - 1   0]   [q - 1   1   0]

```

$[0\ 1\ 0]$ $[\quad 0 \quad 0\ q - 1]$ $[q - 1\ q - 1\ q - 1]$
 $[0\ 0\ 1],$ $[q + 1\ q - 1\ q - 1],$ $[q + 1 \quad 1 \quad 0]$
 $[1, 3, 7, 9]$
 $[1, 3, 4]$

No Match

$[1\ 0\ 0]$ $[\quad 1 \quad 0 \quad 0]$ $[\quad 0\ q - 1 \quad 1]$
 $[0\ 1\ 0]$ $[q + 1 \quad 0 \quad 1]$ $[\quad 0 \quad 1 \quad 1]$
 $[0\ 0\ 1],$ $[q + 1\ q - 1\ q + 1],$ $[\quad 0 \quad 0 \quad 1]$
 $[1, 3, 7, 9]$
 $[1, 3, 7]$

No Match

$[1\ 0\ 0]$ $[q + 1\ q - 1\ q + 1]$ $[\quad 1\ q - 1 \quad 0]$
 $[0\ 1\ 0]$ $[q - 1 \quad 1\ q - 1]$ $[\quad 1\ q + 1\ q + 1]$
 $[0\ 0\ 1],$ $[\quad 0 \quad 0 \quad 1],$ $[q + 1\ q - 1 \quad 1]$
 $[1, 3, 7, 9]$
 $[1, 3, 6, 7]$

No Match

$[1\ 0\ 0]$ $[q + 1 \quad 0\ q + 1]$ $[q - 1 \quad 1\ q + 1]$
 $[0\ 1\ 0]$ $[q - 1\ q - 1 \quad 0]$ $[\quad 0\ q - 1 \quad 0]$
 $[0\ 0\ 1],$ $[q + 1\ q - 1 \quad 0],$ $[q - 1\ q + 1 \quad 1]$
 $[1, 3, 7, 9]$
 $[1, 3, 7]$

No Match

$[1\ 0\ 0]$ $[q - 1\ q + 1 \quad 1]$ $[\quad 0\ q + 1\ q + 1]$
 $[0\ 1\ 0]$ $[q + 1 \quad 0\ q + 1]$ $[\quad 0\ q - 1\ q - 1]$
 $[0\ 0\ 1],$ $[\quad 1\ q + 1 \quad 1],$ $[q - 1\ q + 1\ q - 1]$
 $[1, 3, 7, 9]$
 $[1, 3, 7, 9]$

No Match

$[1\ 0\ 0]$ $[\quad 0 \quad 1 \quad 0]$ $[q - 1\ q + 1\ q - 1]$
 $[0\ 1\ 0]$ $[\quad 1\ q + 1 \quad 1]$ $[\quad 0\ q + 1\ q - 1]$
 $[0\ 0\ 1],$ $[\quad 0 \quad 0 \quad 1],$ $[\quad 0\ q + 1 \quad 0]$
 $[1, 3, 7, 9]$
 $[1, 3, 6, 8, 9]$

Match

For the last try, we see a loose dimension for words of size 2 which is recovered with words of size 4. So, the set of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & q+1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q-1 & q+1 & q-1 \\ 0 & q+1 & q-1 \\ 0 & q+1 & 0 \end{pmatrix}$$

give a counterexample to the conjecture by specialise q to 1.

A_3	Dimension								
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7
$q = 2$	24	1	4	9	15	20	23	24	.
$q = 0$	24	1	4	9	15	20	23	24	.
$q = 1$	24	1	4	9	15	20	23	24	.
$q = -1$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{3}}$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{4}}$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{5}}$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{6}}$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{7}}$	24	1	4	9	15	20	23	24	.
$q = 1^{\frac{1}{8}}$	24	1	4	9	15	20	23	24	.

A_2	Dimension					
	Full Space	W 0	W 1	W 2	W 3	W 4
$q = 2$	6	1	3	5	6	.
$q = 0$	6	1	3	5	6	.
$q = 1$	6	1	3	5	6	.
$q = -1$	6	1	3	5	6	.
$q = 1^{\frac{1}{3}}$	6	1	3	5	6	.
$q = 1^{\frac{1}{4}}$	6	1	3	5	6	.
$q = 1^{\frac{1}{5}}$	6	1	3	5	6	.
$q = 1^{\frac{1}{6}}$	6	1	3	5	6	.
$q = 1^{\frac{1}{7}}$	6	1	3	5	6	.
$q = 1^{\frac{1}{8}}$	6	1	3	5	6	.

B_2	Dimension						
	Full Space	W 0	W 1	W 2	W 3	W 4	W 5
$q = 2$	8	1	4	5	7	8	.
$q = 0$	8	1	4	5	7	8	.
$q = 1$	8	1	4	5	7	8	.
$q = -1$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{3}}$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{4}}$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{5}}$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{6}}$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{7}}$	8	1	4	5	7	8	.
$q = 1^{\frac{1}{8}}$	8	1	4	5	7	8	.

B_3	repr.	Dimension										
		W 0	W 1	W 2	W 3	W 4	W 5	W 6	W 7	W 8	W 9	W 10
$q = 2$?	1	5	14	31	59	101	161	242	346	468	585
$q = 0$?	1	5	14	31	59	101	161	242	346	468	
$q = 1$	48	1	5	14	31	45	48	
$q = -1$?	1	5	14	31	59	100					
$q = 1^{\frac{1}{3}}$?	1	5	14	31	59	100					
$q = 1^{\frac{1}{4}}$?	1	5									
$q = 1^{\frac{1}{5}}$?	1	5									
$q = 1^{\frac{1}{6}}$?	1	5									
$q = 1^{\frac{1}{7}}$?	1	5									
$q = 1^{\frac{1}{8}}$?	1	5									

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