Renormalization of Multitangent Functions and Applications.

Combinatorics of Mathematical Renormalization: a special day.

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Plan

1. Reminders on Multizeta Values.
2. Introduction to Multitangent Functions.
3. Multitangent Functions Renormalization.
4. Relations Between Multizeta Values Obtained via the Study of Multitangent Functions, for Small Weights.
5. Some Conjectures.

Let : $\mathcal{S}_d^\star = \{\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)\ ;\ s_1 \geq 2\}$.

Multizeta Values Definition.

Let $\underline{\mathbf{s}} \in \mathcal{S}_d^\star$.

We put: $\mathcal{Z}e^{\underline{\mathbf{s}}} = \mathcal{Z}e^{s_1, \cdots, s_r} = \displaystyle\sum_{1 \leq n_r < \cdots < n_1 < +\infty} \frac{1}{{n_1}^{s_1} \cdots {n_r}^{s_r}}$.

Today, multizetas values have deep connections with many other mathematical topics:

1. Number theory.
2. Quantum groups, knot theory or mathematical physics.
3. Resurgence theory and holomorphics invariants.
4. Feynman diagrams.
5. $\mathbb{P}^1 - \{0\,;1\,;\infty\}$ (with Grothendieck-Ihara's program) .
6. Absolute Galois group of $\mathbb{Q}$.

Reminders on Multizeta Values: Integral Representation.

Multizeta values can be represented by two different ways:

by a **sum** or by an **iterated integral**.

Example: $\forall n \in \mathbb{N}^*$, $\frac{1}{n^2} = \int_0^1 \left(\int_0^u v^{n-1} \, dv \right) \frac{du}{u}$.

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \int_0^1 \left(\int_0^u \sum_{n=1}^{+\infty} v^{n-1} \, dv \right) \frac{du}{u} = \int_0^1 \left(\int_0^u \frac{dv}{1-v} \right) \frac{du}{u} .$$

More generally, we let: $\mathcal{W}a^{\alpha_1, \cdots, \alpha_r} = \int_{0<t_1<\cdots<t_r<1} \omega_{\alpha_1} \cdots \omega_{\alpha_r}$,

where $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{1-z}$.

Then: $\mathcal{Z}e^{s_1, \cdots, s_r} = \mathcal{W}a^{1,0^{[s_r-1]}, \cdots, 1, 0^{[s_1-1]}}$.

Reminders on Multizeta Values: the Two Multiplication Tables 1/2 .

Shuffle product or symmetrality.

Consider the alphabet $X = \{x_0 \,;\, x_1\}$.

Fact : $\mathcal{S}_d^\star \simeq x_0 X^\star x_1$:
$$\underline{\mathbf{s}} \longmapsto x_{\underline{\mathbf{s}}} = x_0^{s_1 - 1} x_1 \cdots x_0^{s_r - 1} x_1 \ .$$

We define $\zeta : x_0 X^\star x_1 \longrightarrow \mathbb{C}$ by $\zeta(x_{\underline{\mathbf{s}}}) = \mathcal{Z}e^{\underline{\mathbf{s}}}$, and we extend it by linearity to $\mathbb{Q}\varepsilon \oplus x_0 \mathbb{Q}<X> x_1$.

Shuffle product:

We define $ш$ recursively by:

$$\begin{cases} \varepsilon \,ш\, \omega = \omega \,ш\, \varepsilon = \omega \ . \\ (x_i\omega_1) \,ш\, (x_j\omega_2) = x_i\big(\omega_1 \,ш\, (x_j\omega_2)\big) + \\ \qquad\qquad\qquad\qquad\quad x_j\big((x_i\omega_1) \,ш\, \omega_2\big) \ . \end{cases}$$

Fact : For $(\omega_1 \,;\, \omega_2) \in \big(x_0\mathbb{Q}<X> x_1\big)^2$,
$$\zeta(\omega_1)\zeta(\omega_2) = \zeta(\omega_1 \,ш\, \omega_2) \ .$$

Quasi-shuffle product or symmetrelity.

Consider the alphabet $Y = \{y_i \,;\, i \in \mathbb{N}\}$.

Fact : $\mathcal{S}_d^\star \simeq \{\omega = y_i\widetilde{\omega} \in Y^\star \,;\, i \geq 2\}$:
$$\underline{\mathbf{s}} \longmapsto y_{\underline{\mathbf{s}}} = y_{s_1} \cdots y_{s_r} \ .$$

We define $\zeta : x_0 X^\star x_1 \longrightarrow \mathbb{C}$ by $\zeta(y_{\underline{\mathbf{s}}}) = \mathcal{Z}e^{\underline{\mathbf{s}}}$, and we extend it by linearity to $\mathbb{Q}\varepsilon \oplus \{y_i \,;\, i \geq 2\}\mathbb{Q}<Y>$.

Stuffle product:

We define $\star$ recursively by:

$$\begin{cases} \varepsilon \star \omega = \omega \star \varepsilon = \omega \ . \\ (y_i\omega_1) \star (y_j\omega_2) = y_i\big(\omega_1 \star (y_j\omega_2)\big) + \\ \qquad\qquad\qquad\qquad\quad y_j\big((y_i\omega_1) \star \omega_2\big) + \\ \qquad\qquad\qquad\qquad\quad y_{i+j}(\omega_1 \star \omega_2) \ . \end{cases}$$

Fact : For $(\omega_1 \,;\, \omega_2) \in \big(\{y_i \,;\, i \geq 2\}\mathbb{Q}<Y>\big)^2$
$$\zeta(\omega_1)\zeta(\omega_2) = \zeta(\omega_1 \star \omega_2) \ .$$

Shuffle product or symmetrality.

Example:

$$\mathcal{Z}e^{2}\mathcal{Z}e^{3} = \mathcal{Z}e^{2,3} + \mathcal{Z}e^{3,2} + \mathcal{Z}e^{5} \ .$$

Quasi-shuffle product or symmetrelity.

Example:

$$\mathcal{Z}e^{2}\mathcal{Z}e^{3} = \mathcal{Z}e^{2,3} + 3\mathcal{Z}e^{3,2} + 6\mathcal{Z}e^{4,1} \ .$$

So:

$$\mathcal{Z}e^{5} = 2\mathcal{Z}e^{3,2} + 6\mathcal{Z}e^{4,1} \ .$$

Lemma :

For all $\theta \in \mathbb{C}$, there exists a unique symmetr<u>e</u>l extension of $\mathcal{Z}e^{\bullet}$ to $\text{seq}(\mathbb{N}^*)$, such that $\mathcal{Z}e^{1} = \theta$.

Remark : We can choose $\theta = 0$ for simplicity, but there exists some simple passage formulas if we chose $\theta \neq 0$.

Sketch of proof:

• By symmetr<u>**é**</u>lity, for sequences $\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)$ beginning with 1s, we can recursively **remove the 1s to the right**.

For example: $\mathcal{Z}e^{1,2} = \mathcal{Z}e^{2}\mathcal{Z}e^{1} - \mathcal{Z}e^{2,1} - \mathcal{Z}e^{3}$.

• So, if $\mathcal{M}ZV_{\text{CV}} = \text{Vect}_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}_d^\star}$, we have:

$$\forall \underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ , \ \mathcal{Z}e^{\underline{\mathbf{s}}} \in \mathcal{M}ZV_{\text{CV}}\left[\mathcal{Z}e^{1}\right] \ .$$

□

Reminders on Multizeta Values: Third Type of Relations.

Lemma :

Let $y_i = {x_0}^{i-1} x_1$, for $i \in \mathbb{N}^*$.
Then, $y_1 \sqcup\!\sqcup\, y_{\underline{s}} - y_1 \star y_{\underline{s}}$ is a linear combination of words of $x_0 X^\star x_1$ and is in the kernel of ζ .

Example : $x_1 \sqcup\!\sqcup\, x_0 x_1 - x_1 \star x_0 x_1 = x_0 x_1^2 - x_0^2 x_1$, so $\mathcal{Z}e^{2,1} = \mathcal{Z}e^3$.

Diophantine Conjecture:

The kernel of ζ is exactly described by the three types of relations that we have seen:

- shuffle relations $\longleftrightarrow$ relations of symmetr**a**lity.
- stuffle relations $\longleftrightarrow$ relations of symmetr**e**lity.
- regularization relations

Henceforth, we shall be interested only in these three types of relations between multizeta values.

Introduction to Multitangent Functions: Definition.

Definition:

Let $\mathcal{S}^{\star}_{df} = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ ; \ s_1 \geq 2 \text{ et } s_r \geq 2\}$.
For all sequence $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{df}$, we consider:

$$\mathcal{T}e^{s_1, \cdots, s_r} : \begin{array}{ccc} \mathbb{C} - \mathbb{Z} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \displaystyle\sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{1}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \end{array} .$$

Remarks :

1. Multitangent functions are a generalization of Eisenstein series $(r = 1)$.

2. Multitangent functions appear naturally in problems of holomorphic dynamics.

Introduction to Multitangent Functions: First Properties.

Property :

1. **Differential property.**

   Let $\underline{\mathbf{s}} = (s_1; \cdots s_r) \in \mathcal{S}^\star_{df}$.

   The function $\mathcal{T}e^{\underline{s}}$ is holomorphic on $\mathbb{C} - \mathbb{Z}$; it is a uniformly convergent series on any compact subset of $\mathbb{C} - \mathbb{Z}$ and satisfies:

$$\frac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z} = -\sum_{i=1}^{r} s_i \mathcal{T}e^{s_1, \cdots, s_{i-1}, s_i+1, s_{i+1}, \cdots, s_r} \ .$$

2. **Parity property.**

$$\forall z \in \mathbb{C} - \mathbb{Z} \ , \ \forall \underline{\mathbf{s}} \in \mathcal{S}^\star_{df} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||} \mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z) \ .$$

3. **Symmetrelity.**

   $\mathcal{T}e^{\bullet}$ is symm<u>e</u>trel.

Introduction to Multitangent Functions: Reduction into Monotangent Functions, First Version.

Remark : A monotangent function is a multitangent function with length 1 .

Let: $\mathcal{MZV} = \text{Vect}_{\mathbb{Q}} \left(\mathcal{Z}e^{\underline{\mathbf{s}}} \right)_{\underline{\mathbf{s}} \in \mathcal{S}_d^\star}$.

$$m(\underline{\mathbf{s}}) = \max(s_1 \, ; \cdots ; s_r), \text{ for all } \underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ .$$

Property: Reduction of Convergent Multitangent Functions into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathcal{S}_{df}^\star \ , \ \exists (z_2 \, ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{MZV}^{m(\underline{\mathbf{s}})-1} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = \sum_{\substack{\cancel{k=1} \\ k=2}}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k \ .$$

Sketch of proof:

1. Partial fraction expansion of $\dfrac{1}{(n_1 + X)^{s_1} \cdots (n_r + X)^{s_r}}$.
2. Using an analytic argument:

$$\forall z \in \mathbb{C} - \mathbb{R} \ , \ |\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq 4r \left(\frac{2}{|\Im m \ z|} \right)^{s_1 + \cdots + s_r - r - 1} \frac{e^{-\pi |\Im m \ z|}}{1 - e^{-\pi |\Im m \ z|}} \ .$$

Introduction to Multitangent Functions: Examples of Reduction into Monotangent Functions.

Weight 4

$$\mathcal{T}e^{2,2} = 2\zeta(2)\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,3} = -3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,2} = 3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,2} = 0 .$$

Weight 5

Weight 6

$$\mathcal{T}e^{2,4} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} - 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

$$\mathcal{T}e^{3,3} = -\frac{12}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{4,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} + 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

$$\mathcal{T}e^{2,2,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,1,3} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} + \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,1,2} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} - \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,1,2} = \frac{4}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

Multitangent Functions Renormalization: the Property.

Property: Multitangent Functions Renormalization.

There exists an extension of multitangent functions to $\mathrm{seq}(\mathbb{N}^*)$ such that:

1. $\mathcal{T}e^{\bullet}$ is always symmetr**e**l.
2. $\forall z \in \mathbb{C} - \mathbb{Z}$, $\mathcal{T}e^{1}(z) = \dfrac{\pi}{\tan(\pi z)}$.

This extension automatically satisfies: the differential property.
the parity property.

The removal to the right algorithm does not apply:

$$\mathcal{T}e^{1,2}(z) = \underbrace{\mathcal{T}e^{1}(z)}_{\text{known by hypothesis}} \times \underbrace{\mathcal{T}e^{2}(z)}_{\text{convergent multitangent function}} - \underbrace{\mathcal{T}e^{2,1}(z)}_{\text{problematics =unknown}} - \underbrace{\mathcal{T}e^{3}(z)}_{\text{convergent multitangent function}} .$$

Multitangent Functions Renormalization: Notations.

Let $\mathcal{S}_d^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)\ ;\ s_1 \geq 2\}$.

$\mathcal{S}_f^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)\ ;\ s_r \geq 2\}$.

$\mathcal{S}_{df}^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)\ ;\ s_1 \geq 2 \text{ et } s_r \geq 2\}$.

Let us consider the symmetre<u>l</u> moulds $\mathcal{H}e_+^\bullet$, $\mathcal{H}e_-^\bullet$ and $\mathcal{C}e^\bullet$, defined by:

$$\forall \underline{\mathbf{s}} \in \mathcal{S}_d^\star\ ,\ \mathcal{H}e_+^{\underline{\mathbf{s}}}(z) = \sum_{0<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \text{ and } \mathcal{H}e_+^{\emptyset}(z) = 1\ .$$

$$\forall \underline{\mathbf{s}} \in \mathcal{S}_f^\star\ ,\ \mathcal{H}e_-^{\underline{\mathbf{s}}}(z) = \sum_{-\infty<n_r<\cdots<n_1<0} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \text{ and } \mathcal{H}e_-^{\emptyset}(z) = 1\ .$$

$$\forall \underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)\ ,\ \mathcal{C}e^{\underline{\mathbf{s}}}(z) = \begin{cases} 1 & \text{si} \quad \underline{\mathbf{s}} = \emptyset\ . \\ \dfrac{1}{z^s} & \text{si} \quad l(\underline{\mathbf{s}}) = 1\ . \\ 0 & \text{si} \quad l(\underline{\mathbf{s}}) > 1\ . \end{cases}$$

Multitangent Functions Renormalization: Trifactorisation 1 / 2 .

Property:

$$\forall \underline{\mathbf{s}} \in \mathcal{S}^{\star}_{df} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = \sum_{\substack{(\underline{\mathbf{s}}^1 ; \underline{\mathbf{s}}^2 ; \underline{\mathbf{s}}^3) \in \mathcal{S}^{\star}_{d} \times \mathrm{seq}(\mathbb{N}^*) \times \mathcal{S}^{\star}_{f} \\ \underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 \cdot \underline{\mathbf{s}}^3 = \underline{\mathbf{s}}}} \mathcal{H}e_{+}^{\underline{\mathbf{s}}^1} \ \mathcal{C}e^{\underline{\mathbf{s}}^2} \ \mathcal{H}e_{-}^{\underline{\mathbf{s}}^3} \quad .$$

We write this in a simpler way: $\mathcal{T}e^{\bullet} = \mathcal{H}e_{+}^{\bullet} \times \mathcal{C}e^{\bullet} \times \mathcal{H}e_{-}^{\bullet} \quad .$

Notation:

In a general manner, we write $(A^{\bullet} \times B^{\bullet})^{\underline{\mathbf{s}}}$ when we consider the sum:

$$\sum_{\underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 = \underline{\mathbf{s}}} A^{\underline{\mathbf{s}}^1} \ B^{\underline{\mathbf{s}}^2} \quad .$$

Sketch of proof:

Recall that $\mathcal{T}e^{\underline{\mathbf{s}}}(z) = \displaystyle\sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$.

We deal with the position of 0 in the decreasing sequence $n_r < \cdots < n_1$:

$$n_r < \cdots < n_{i+1} < 0 < n_i < \cdots < n_1 \ .$$

or

$$n_r < \cdots < n_{i+1} < n_i = 0 < n_{i-1} < \cdots < n_1 \ .$$

$\rightsquigarrow$ The terms to the left of 0 give $\mathcal{H}e_+^{s_r,\cdots,s_{i+1}}(z)$.

$\rightsquigarrow$ The term $n_i = 0$ gives $\mathcal{C}e^{s_i}(z)$.

$\rightsquigarrow$ The terms to the right of 0 give $\mathcal{H}e_-^{s_i,\cdots,s_1}(z)$ or $\mathcal{H}e_-^{s_{i-1},\cdots,s_1}(z)$.

We obtain all the terms $\mathcal{H}e_+^{\underline{\mathbf{s}}^1}\ \mathcal{C}e^{\underline{\mathbf{s}}^2}\ \mathcal{H}e_-^{\underline{\mathbf{s}}^3}$ where $\underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 \cdot \underline{\mathbf{s}}^3 = \underline{\mathbf{s}}$.

□

Multitangent Functions Renormalization: Consequences of trifactorisation 1 /3 .

Lemma:

Let $(\Phi_+ \,;\, \Phi_-) \in \mathcal{H}(\mathbb{C} - \mathbb{Z})^2$.
$\mathcal{H}e_+^{\bullet}$ et $\mathcal{H}e_-^{\bullet}$ have a unique symmetr$\underline{\text{e}}$l extension to $\text{seq}(\mathbb{N}^*)$ such that

$$\begin{cases} \mathcal{H}e_+^{1} = \Phi_+ \ . \\ \\ \mathcal{H}e_-^{1} = \Phi_- \ . \end{cases}$$

Sketch of proof:

Identical to the renormalisation property of the multizeta values, by the removal to the right of the 1s algorithm.

□

Multitangent Functions Renormalization: Consequences of trifactorisation 2 /3 .

Example:

$$\begin{aligned}
\mathcal{T}e^{2,1}(z) &= \mathcal{H}e_+^{2,1}(z) + \mathcal{H}e_+^{2}(z)\left(\frac{1}{z} + \underbrace{\mathcal{H}e_-^{1}(z)}_{\substack{\text{divergent}\\ \text{term}}}\right) + \frac{1}{z^2}\underbrace{\mathcal{H}e_-^{1}(z)}_{\substack{\text{divergent}\\ \text{term}}} + \underbrace{\mathcal{H}e_-^{2,1}(z)}_{\substack{\text{divergent}\\ \text{term}}} \\
&= \mathcal{H}e_+^{2,1}(z) + \mathcal{H}e_+^{2}(z)\left(\frac{1}{z} + \underbrace{\mathcal{H}e_-^{1}(z)}_{\substack{\text{divergent}\\ \text{term}}}\right) + \frac{1}{z^2}\underbrace{\mathcal{H}e_-^{1}(z)}_{\substack{\text{divergent}\\ \text{term}}} \\
&\quad + \mathcal{H}e_-^{2}(z)\underbrace{\mathcal{H}e_-^{1}(z)}_{\substack{\text{term}\\ \text{divergent}}} - \underbrace{\mathcal{H}e_-^{1,2}(z)}_{\substack{\text{convergent}\\ \text{term}}} - \mathcal{H}e_-^{3}(z)\ .
\end{aligned}$$

Multitangent Functions Renormalization: Consequences of trifactorisation 3 /3 .

Property: Multitangent Functions Renormalization.

There exists an extension of multitangent functions to $\mathrm{seq}(\mathbb{N}^*)$ such that:

1. $\mathcal{T}e^\bullet$ is always symmetr<u>e</u>l.
2. $\forall z \in \mathbb{C} - \mathbb{Z}\ ,\ \mathcal{T}e^1(z) = \dfrac{\pi}{\tan(\pi z)}$.

This extension automatically satisfies: the differential property.
the parity property.

<u>Sketch of proof:</u>

The extension is given by:

$$\begin{cases} \mathcal{H}e^1_+(z) = \displaystyle\sum_{n\geq 1}\left(\frac{1}{n+z} - \frac{1}{n}\right) . \\ \mathcal{H}e^1_-(z) = \displaystyle\sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{n-z}\right) . \\ \forall \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)\ ,\ \mathcal{T}e^{\underline{\mathbf{s}}} = (\mathcal{H}e^\bullet_+ \times \mathcal{C}e^\bullet \times \mathcal{H}e^\bullet_-)^{\underline{\mathbf{s}}} . \end{cases}$$

□

Multitangent Functions Renormalization: Generating Functions 1/5.

Problem: Divergent multitangent functions are not written in an internal way...

We would like to express divergent multitangent functions with convergent multitangent functions.

Let us consider colored multizeta values and colored multitangent functions:

For $\binom{\underline{\varepsilon}}{\underline{s}} \in \text{seq}\left(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*\right)$ and $e_k = e^{-2i\pi\varepsilon_k}$, for $k \in [\![1 \,;\, n]\!]$, we denote:

$$\mathcal{Z}e^{\left(\begin{smallmatrix}\varepsilon_1 & ,\cdots, & \varepsilon_r\\ s_1 & ,\cdots, & s_r\end{smallmatrix}\right)} = \sum_{1 \leq n_r < \cdots < n_1} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{{n_1}^{s_1} \cdots {n_r}^{s_r}} \; .$$

$$\mathcal{H}e_+^{\left(\begin{smallmatrix}\varepsilon_1 & ,\cdots, & \varepsilon_r\\ s_1 & ,\cdots, & s_r\end{smallmatrix}\right)}(z) = \sum_{0 < n_r < \cdots < n_1 < +\infty} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \; .$$

$$\mathcal{T}e^{\left(\begin{smallmatrix}\varepsilon_1, & ,\cdots, & \varepsilon_r\\ s_1, & ,\cdots, & s_r\end{smallmatrix}\right)}(z) = \sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \; .$$

For a symmetre$\underline{\mathbf{e}}$l mould $Me^{\bullet}$, we can consider its generating functions, which we denote $Mig^{\bullet}$:

$$\begin{cases} Mig^{\emptyset} = 1 \ . \\ \\ Mig^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}} = \displaystyle\sum_{s_1, \cdots, s_r \geq 1} M\underline{\mathbf{e}}^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r}} \ {v_1}^{s_1 - 1} \cdots {v_r}^{s_r - 1} \ . \end{cases}$$

$\rightsquigarrow \mathcal{Z}ig^{\bullet}$, $\mathcal{Z}ig_{-}^{\bullet}$, $\mathcal{H}ig^{\bullet}$, $\mathcal{H}ig_{-}^{\bullet}$ and $\mathcal{T}ig^{\bullet}$.

Let $\mu^{n_1,\cdots,n_r} = \dfrac{1}{r_1!\cdots r_n!}$ where the sequence $\underline{\mathbf{n}} = (n_1;\cdots;n_r) \in \mathrm{seq}(\mathbb{N}^*)$ attains r_1 times its highest value, r_2 times its second highest value, and so on.

Lemma: The Generating Function $\mathcal{Z}ig^\bullet$ (J. Ecalle) .

For all $k \in \mathbb{N}^*$, let $do\mathcal{Z}ig_k^\bullet$ and $co\mathcal{Z}ig_k^\bullet$ be defined by:

$$do\mathcal{Z}ig_k^{\binom{\varepsilon_1,\cdots,\varepsilon_r}{v_1,\cdots,v_r}} = \begin{cases} \displaystyle\sum_{1\le n_r<\cdots<n_1<k} \frac{e_1{}^{n_1}\cdots e_r{}^{n_r}}{(n_1-v_1)\cdots(n_r-v_r)} & \text{, if } r\neq 0\,.\\ 1 & \text{, if } r=0\,.\end{cases}$$

$$co\mathcal{Z}ig_k^{\binom{\varepsilon_1,\cdots,\varepsilon_r}{v_1,\cdots,v_r}} = \begin{cases} \displaystyle(-1)^r \sum_{1\le n_r\le\cdots\le n_1<k} \frac{\mu^{n_1,\cdots,n_r}}{n_1\cdots n_r} & \text{, if } \underline{\boldsymbol{\varepsilon}}\neq\underline{\mathbf{0}}\,.\\ 0 & \text{, if } \underline{\mathbf{u}}=\underline{\mathbf{0}}\,.\end{cases}$$

Then, the mould $\mathcal{Z}ig^\bullet$ admits a mould factorization:

$$\mathcal{Z}ig^\bullet = \lim_{k\longrightarrow+\infty}\left(co\mathcal{Z}ig_k^\bullet \times do\mathcal{Z}ig_k^\bullet\right)\,.$$

Multitangent Functions Renormalization: Generating Functions 4/5.

Theorem: The Generating Functions $\mathcal{T}ig^{\bullet}$.

Let:

$$\begin{cases} Qig^{\emptyset}(z) = 0 \ . \\ Qig^{\binom{\varepsilon_1}{v_1}}(z) = -Te^{\binom{\varepsilon_1}{1}}(v_1 - z) \ . \\ Qig^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}}(z) = 0 \text{ , si } r \geq 2 \ . \end{cases}$$

$$\begin{cases} \delta^{\emptyset} = 0 \ . \\ \delta^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}} = \begin{cases} \dfrac{(i\pi)^r}{r!} \mathbb{1}_{\{0\}}(\varepsilon_1) \cdots \mathbb{1}_{\{0\}}(\varepsilon_r) & \text{, if } r \text{ is even.} \\ 0 & \text{, if } r \text{ is odd.} \end{cases} \end{cases}$$

Then:

$$\mathcal{T}ig^{\bullet}(z) = \delta^{\bullet} + \mathcal{Z}ig^{\bullet \rfloor} \times \mathcal{Q}ig^{\lceil \bullet \rceil}(z) \times \mathcal{Z}ig_{-}^{\lfloor \bullet} \ .$$

Sketch of proof:

1. $\mathcal{T}ig^{\bullet}(z) = \underbrace{\mathcal{H}ig_{+}^{\bullet}(z)}_{=\mathcal{Z}ig^{\bullet}(z)} \times Cig^{\bullet}(z) \times \underbrace{\mathcal{H}ig_{-}^{\bullet}(z)}_{=\mathcal{Z}ig_{-}^{\bullet}(z)}$.

2. We use the expression of the generating function $\mathcal{Z}ig^{\bullet}$:

$$\mathcal{T}ig^{\bullet}(z) = \lim_{N \longrightarrow +\infty} \left(co\mathcal{Z}ig_{N}^{\bullet} \times Tig_{N}^{\bullet}(z) \times co\mathcal{Z}ig_{-,N}^{\bullet} \right) ,$$

where: $Tig_{N}^{\binom{\varepsilon_1, \cdots, u_r}{v_1, \cdots, v_r}}(z) = \left(do\mathcal{Z}ig_{N}^{\bullet}(z) \times Cig^{\bullet}(z) \times do\mathcal{Z}ig_{-,N}^{\bullet}(z) \right)^{\binom{\varepsilon_1, \cdots, u_r}{v_1, \cdots, v_r}}$

$$= \sum_{-N < n_r < \cdots < n_1 < N} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1 - V_1 + z) \cdots (n_r - V_r + z)} .$$

3. We use a partial fraction expansion of $Tig_{N}^{\bullet}$.

4. We compute $co\mathcal{Z}ig_{N}^{\bullet} \times Tig_{N}^{\bullet}(z) \times co\mathcal{Z}ig_{-,N}^{\bullet}$ to conclude.

□

Multitangent Functions Renormalization: Reduction into Monotangent Functions, second version.

Property: Reduction into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) \ , \ \exists (z_1 \, ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{M}ZV^{m(\underline{\mathbf{s}})} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = \delta^{\underline{\mathbf{s}}} + \sum_{k=1}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k \ ,$$

where $\delta^{\underline{\mathbf{s}}} = \begin{cases} \dfrac{(i\pi)^r}{r!} & \text{, if } \underline{\mathbf{s}} = 1^{[r]} \text{ and if } r \text{ is even.} \\ 0 & \text{, else.} \end{cases}$

Important remark: $z_1 = 0 \iff \underline{\mathbf{s}} \neq 1^{[r]}$ or $\begin{cases} \underline{\mathbf{s}} = 1^{[r]} \ . \\ r \text{ is even} \ . \end{cases}$

Multitangent Functions Renormalization: Examples of Reduction into Monotangent Functions.

Weight 2

$$\mathcal{T}e^{1,1} = -3\,\zeta(2)\ .$$

$$\mathcal{T}e^{1,2} = 0\ .$$

$$\mathcal{T}e^{2,1} = 0\ .$$

$$\mathcal{T}e^{1,1,1} = -\zeta(2)\mathcal{T}e^{1}\ .$$

Weight 3

Weight 4

$$\mathcal{T}e^{1,3} = -\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{3,1} = -\zeta(2)\mathcal{T}e^{2}\ .$$

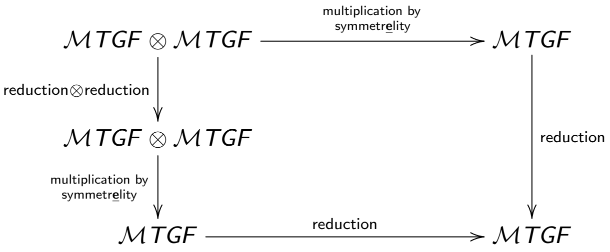
$$\mathcal{T}e^{1,1,2} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,2,1} = 0\ .$$

$$\mathcal{T}e^{2,1,1} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,1,1,1} = \frac{3}{2}\zeta(2)^{2}\ .$$

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: a Commutative Diagram.



Property:

The symmetrelity of $\mathcal{T}e^{\bullet}$ and the precedent commutative diagram show the symmetrelity of $\mathcal{Z}e^{\bullet}$.

Remark: We obtain other relations between multizeta values, for example some of regularization.

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: No $\mathcal{T}^1$ in relation of reduction.

Property:

There is no composant $\mathcal{T}e^1$ in the reduction of $\mathcal{T}e^{\underline{\mathbf{s}}}$ into monotangent if:

$$\underline{\mathbf{s}} \neq 1^{[r]} \text{ or } \begin{cases} \underline{\mathbf{s}} = 1^{[r]} . \\ r \text{ is even} . \end{cases}$$

Remark: We obtain some symmetr**a**lity relations between multizeta values and some regularization relations.

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: For weight 3 et 4.

Multitangent functions give us the following relations between multizetas values:

- By the commutative diagram:

$$6\mathcal{Z}e^{2,2} + 8\mathcal{Z}e^{4} = 5\left(\mathcal{Z}e^{2}\right)^{2} .$$

$$2\mathcal{Z}e^{2,2} + \mathcal{Z}e^{4} = \left(\mathcal{Z}e^{2}\right)^{2} .$$

- By the absence of composant $\mathcal{T}e^{1}$:

$$\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3} .$$

$$2\mathcal{Z}e^{2,2} + 4\mathcal{Z}e^{3,1} = \left(\mathcal{Z}e^{2}\right)^{2} .$$

Consequences:

- For the weight 3, we find all the regularization relations.
- For the weight 4, we find all the symmetr**e**lity and symmetr**a**lity relations. We find sufficiently many relations to deduce those of regularization.

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: For weight 5.

Convergent multitangent functions give us the following relations between multizeta values:

- By the commutative diagram:

$$\begin{aligned} \mathcal{Z}e^{2,3} + \mathcal{Z}e^{3,2} + \mathcal{Z}e^{5} &= \mathcal{Z}e^{2}\mathcal{Z}e^{3} . \\ \mathcal{Z}e^{2,1,2} + 2\mathcal{Z}e^{2,2,1} + \mathcal{Z}e^{2,3} + \mathcal{Z}e^{4,1} &= \mathcal{Z}e^{2}\mathcal{Z}e^{2,1} . \\ \mathcal{Z}e^{2,1,2} + 2\mathcal{Z}e^{2,2,1} + 2\mathcal{Z}e^{2,3} + 3\mathcal{Z}e^{3,2} + 7\mathcal{Z}e^{4,1} &= \mathcal{Z}e^{2}\left(\mathcal{Z}e^{3} + \mathcal{Z}e^{2,1}\right) . \\ 4\mathcal{Z}e^{2,3} + 6\mathcal{Z}e^{3,2} + 15\mathcal{Z}e^{5} &= 10\mathcal{Z}e^{2}\mathcal{Z}e^{3} . \end{aligned}$$

- By the absence of composant $\mathcal{T}e^{1}$:

$$\mathcal{Z}e^{2,3} + 3\mathcal{Z}e^{3,2} + 6\mathcal{Z}e^{4,1} = \mathcal{Z}e^{2}\mathcal{Z}e^{3} .$$

Consequences:
- The last one is not independant of the others.
- For the weight 5, we obtain all the symmetr**e**lity relations, but the following symmetr**a**lity relation is missing:

$$3\mathcal{Z}e^{2,2,1} + 6\mathcal{Z}e^{3,1,1} + \mathcal{Z}e^{2,1,2} = \mathcal{Z}e^{2,1}\mathcal{Z}e^{2} .$$

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: General Statement.

Conjecture:

- With convergent multitangent functions, we find :
  1. All the symmetr<u>**e**</u>lity relations.
  2. A few of symmetr<u>**a**</u>lity relations.
  3. A few of regularization relations.

- Using the divergent multitangent functions, we find:
  1. All the symmetr<u>**e**</u>lity relations.
  2. A few more of symmetr<u>**a**</u>lity relations.
  3. A few more of regularization relations.

Let : $\mathcal{M}ZV = Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}_d^\star}$ and $\mathcal{M}ZV_2 = Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}_2)}$.

$\mathcal{M}TGF = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}_{df}^\star}$. and $\mathcal{M}TGF_2 = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}_2)}$.

Here, $\mathbb{N}_2 = \mathbb{N} - \{0\,;1\}$.

Conjecture: Projection on Multitangent Function Space.

$$\forall (\underline{\mathbf{s}}^1\,;\underline{\mathbf{s}}^2) \in \mathcal{S}_d^\star \times \mathcal{S}_{df}^\star \ , \ \mathcal{Z}e^{\underline{\mathbf{s}}^1}\mathcal{T}e^{\underline{\mathbf{s}}^2} \in \mathcal{M}TGF_2 \ .$$

Remark: It is sufficient to show that: $\forall \underline{\mathbf{s}} \in \mathcal{S}_{df}^\star, \ \mathcal{Z}e^{\underline{\mathbf{s}}}\mathcal{T}e^2 \in \mathcal{M}TGF_2$.

This has been verified for all the sequences $\underline{\mathbf{s}} \in \mathcal{S}_d^\star$ of weight less than 12 .

Some Conjectures: Examples of Projections on Multitangent Function Space.

$\|\|\underline{\mathbf{s}}\|\| = 4$	$\mathcal{Z}e^{2}\mathcal{T}e^{2} = \frac{1}{2}\mathcal{T}e^{2,2}$.
$\|\|\underline{\mathbf{s}}\|\| = 5$	$\mathcal{Z}e^{3}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{3,2} - \frac{1}{6}\mathcal{T}e^{2,3}$. $\mathcal{Z}e^{2,1}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{3,2} - \frac{1}{6}\mathcal{T}e^{2,3}$.
$\|\|\underline{\mathbf{s}}\|\| = 6$	$\mathcal{Z}e^{4}\mathcal{T}e^{2} = -\frac{1}{6}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{3,1}\mathcal{T}e^{2} = -\frac{1}{24}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{2,2}\mathcal{T}e^{2} = -\frac{1}{8}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{2,1,1}\mathcal{T}e^{2} = -\frac{1}{6}\mathcal{T}e^{3,3}$.
$\|\|\underline{\mathbf{s}}\|\| = 7$	$\mathcal{Z}e^{5}\mathcal{T}e^{2} = -\frac{1}{30}\mathcal{T}e^{5,2} - \frac{1}{15}\mathcal{T}e^{4,3} + \frac{1}{15}\mathcal{T}e^{3,4} + \frac{1}{30}\mathcal{T}e^{2,5}$. $\vdots$

Some Conjectures: Multizeta values and Multitangent Functions Cleansing 1/3 .

For multizeta values :

Theorem: (conjectured by J. Blumlein, proved by par J. Ecalle.)

Let $\mathcal{M}ZV = \text{Vect}_{\mathbb{Q}}\left(\mathcal{Z}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\mathcal{S}_d^\star}$.
$\mathcal{M}ZV_2 = \text{Vect}_{\mathbb{Q}}\left(\mathcal{Z}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\text{seq}(\mathbb{N}_2)}$.

Then, for all sequence $\underline{\mathbf{s}} \in \mathcal{S}_d^\star$, $\mathcal{Z}e^{\underline{\mathbf{s}}}$ can be expressed (explicitly) in terms of $\mathcal{M}ZV_2$.

In other words:

$$\mathcal{M}ZV = \mathcal{M}ZV_2 \ .$$

Some Conjectures: Multizeta values and Multitangent Functions Cleansing 2/3 .

For multitangent functions:

Conjecture:

Let $\mathcal{M}TGF = \text{Vect}_{\mathbb{Q}}\left(\mathcal{T}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{df}}$.
$\mathcal{M}TGF_2 = \text{Vect}_{\mathbb{Q}}\left(\mathcal{T}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}_2)}$.

Then, for all sequence $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{df}$, $\mathcal{T}e^{\underline{\mathbf{s}}}$ can be expressed (explicitly) in terms of $\mathcal{M}TGF_2$.

In other words:

$$\mathcal{M}TGF = \mathcal{M}TGF_2 \ .$$

Some Conjectures: Multizeta values and Multitangent Functions Cleansing 3/3 .

$p=6$	$\mathcal{T}e^{3,1,2} = \frac{1}{6}\mathcal{T}e^{3,3} + \frac{1}{4}\mathcal{T}e^{2,4} - \frac{1}{4}\mathcal{T}e^{4,2}$. $\mathcal{T}e^{2,1,3} = \frac{1}{6}\mathcal{T}e^{3,3} - \frac{1}{4}\mathcal{T}e^{2,4} + \frac{1}{4}\mathcal{T}e^{4,2}$. $\mathcal{T}e^{2,1,1,2} = -\frac{1}{3}\mathcal{T}e^{3,3}$.
$p=7$	$\mathcal{T}e^{4,1,2} = \frac{1}{6}\mathcal{T}e^{2,2,3} - \frac{1}{6}\mathcal{T}e^{3,2,2} - \frac{1}{3}\mathcal{T}e^{5,2} + \frac{7}{48}\mathcal{T}e^{4,3} + \frac{23}{48}\mathcal{T}e^{3,4} + \frac{1}{3}\mathcal{T}e^{2,5}$ $\mathcal{T}e^{3,1,3} = \frac{1}{5}\mathcal{T}e^{2,3,2}$. $\mathcal{T}e^{2,1,4} = \frac{1}{3}\mathcal{T}e^{3,2,2} + \frac{1}{3}\mathcal{T}e^{5,2} + \frac{13}{24}\mathcal{T}e^{4,3} + \frac{5}{24}\mathcal{T}e^{3,4} - \frac{1}{3}\mathcal{T}e^{2,5}$.

Some Conjectures: Absence of Relations between Multizeta Values and Multitangent Functions of Different Weights 1/2 .

For multizeta values:

Conjecture:

There is no relation between multizeta values of different weights:

$$\sum_{k\in\mathbb{N}^*} \mathcal{Z}_k = \bigoplus_{k\in\mathbb{N}^*} \mathcal{Z}_k \text{ , où } \mathcal{Z}_k = \begin{cases} \mathbb{Q} & \text{, if } k = 0 \text{ .} \\ \{0\} & \text{, if } k = 1 \text{ .} \\ Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\substack{\underline{\mathbf{s}}\in\mathcal{S}_d^\star \\ ||\underline{\mathbf{s}}||=k}} & \text{, if } k \geq 2 \text{ .} \end{cases}$$

Some Conjectures: Absence of Relations between Multizeta Values and Multitangent Functions of Different Weights 2/2 .

For multitangent functions:

Conjecture:

There is no relation between multitangent functions of different weights:

$$\sum_{k\in\mathbb{N}^*} \mathcal{T}_k = \bigoplus_{k\in\mathbb{N}^*} \mathcal{T}_k \ , \text{ où } \mathcal{T}_k = \begin{cases} \mathbb{Q} & , \text{ si } k = 0 \ . \\ \{0\} & , \text{ si } k = 1 \ . \\ Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\substack{\underline{\mathbf{s}}\in\mathcal{S}^\star_{df} \\ ||\underline{\mathbf{s}}||=k}} & , \text{ si } k \geq 2 \ . \end{cases}$$

Some Conjectures: What are the Implications Between the Various Conjectures?

- $\left\{ \begin{array}{c} \text{Projection on multitangent} \\ \text{function space} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{Multitangent Function} \\ \text{cleansing} \end{array} \right\}$

  If the conjecture concerning the projection on multitangent functions space is true, then:

  $$\left\{ \begin{array}{c} \text{multizeta values} \\ \text{cleansing} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{multitangent functions} \\ \text{cleansing} \end{array} \right\}$$

- $\displaystyle\sum_{k \in \mathbb{N}^*} \mathcal{Z}_k = \bigoplus_{k \in \mathbb{N}^*} \mathcal{Z}_k \Longrightarrow \sum_{k \in \mathbb{N}^*} \mathcal{T}_k = \bigoplus_{k \in \mathbb{N}^*} \mathcal{T}_k$.

  $$\left\{ \begin{array}{l} \displaystyle\sum_{k \in \mathbb{N}^*} \mathcal{T}_k = \bigoplus_{k \in \mathbb{N}^*} \mathcal{T}_k \\ \text{projection on } \mathcal{M}\text{TGF} \end{array} \right. \Longrightarrow \sum_{k \in \mathbb{N}^*} \mathcal{Z}_k = \bigoplus_{k \in \mathbb{N}^*} \mathcal{Z}_k \ .$$

Conclusion:

1. Multitangent functions seem to be an interesting functional model for the study of multizetas values.

2. There is a deep link between multizeta values and multitangent functions, **the reduction into monotangent functions** ; another important link, but a conjectural one, is **projection on multitangent function space**.

3. Multitangent functions don't allow us to find the full dimorphy of multizeta values: we find only $1 + \frac{1}{2}$ symmetries.

4. The missing relations seem to be retrievable, but in a more complicated way...