Gauss-Laguerre and related quadratures - Explicit formula of the error term

Olivier Bouillot

December 31, 2023

Abstract

We investigate a new quadrature, namely the Gauss-Laguerre-like quadrature, close to the Gauss-Laguerre quadrature, those whose weight function is modified from $z \longmapsto e^{-z}$ to $z \longmapsto (1+e^{-z})^{-1}$.

Firstly, we prove an explicit error formula for the Gauss-Laguerre and the Gauss-Laguerre-like quadratures, which apply to a subclass of holomorphic functions with isolated singularities outside of $]0;+\infty[$.

Then, we explore the asymptotic of the respective error terms on two simple examples but sufficiently general to be meaningful for the reader. In order to control explicitly their relative error and to compute the related integrals, we also give explicit upper bounds of the error in these two examples.

Contents

1 Introduction

In 1905, Lindelöf (see [6], chapter 3) has shown, under precise hypothesis on functions, called *the 1D-Lindelöf hypothesis* in [2], the following:

Theorem 1.1. *(Lindelöf,* 1905*)*
Let $m_0 > \dfrac{1}{2}$ *be a real number,* m *a positive integer such that* $m \geqslant m_0$ *and* $f : \Omega \longrightarrow \mathbb{C}$ *be an holomorphic function over* $\Omega = \left\{z \in \mathbb{C}\ ,\ Re\ z \geqslant m_0 - \frac{1}{2}\right\}$ *satisfying:*

- *the 1D-Lindelöf hypothesis ;*
- $\displaystyle\sum_{\nu \geqslant m} f(\nu)$ *is a convergent series.*

Then, $f \in \mathcal{L}^1([m_0 - \frac{1}{2}; +\infty[)$ *and*

$$\sum_{\nu \geqslant m} f(\nu) = \int_{m-\frac{1}{2}}^{+\infty} f(t)\ dt - i \int_0^{+\infty} \frac{f(m - \frac{1}{2} + it) - f(m - \frac{1}{2} - it)}{e^{2\pi t} + 1}\ dt\ . \quad (1)$$

In many cases, the first integral can be explicitly computed while the second can not.

Thus, to numerically compute sums of functions evaluated at positive integers by Equation (1), we focus on computing integrals $\displaystyle\int_0^{+\infty} f(t) \frac{dt}{e^t + 1}$. To this end, we introduce and tabulate a new quadrature close to the Gauss-Laguerre one. Its weight function moves from $t \longmapsto e^{-t}$ as in the case of the Gauss-Laguerre quadrature, to $t \longmapsto \dfrac{1}{1 + e^t}$.

The integral expressions seems to be similar. But, the main change concerns the apparition of singularities on the weight function: we have modified an entire function for a function with infinite simple poles in $\mathbb{C}$ . . . This change has a significant impact on the error term of the quadrature: automatically, the error term will contain an infinite number of terms.

In Section 2, we will first review the Gauss-Laguerre quadrature. In particular, we will focus on the state of the art on its error term e_N^{GL}. Then, we define the Gauss-Laguerre-like quadrature.

In particular, in this Section, we will introduce two main examples that will be continued later. The first example concerns the meromorphic function $z \longmapsto \dfrac{e^{-z}}{1+z^2}$ integrated numerically by the Gauss-Laguerre quadrature. The second one deals with the meromorphic function $z \longrightarrow \dfrac{1}{1+e^z}$, integrated this time by the Gauss-Laguerre-like quadrature.

These examples are quite simple but turn out to be sufficiently general to be meaningful on other examples of applications of these quadratures when applied to meromorphic functions: each newly added term is of the form of the error term of the first example, *i.e.* easily depending on the quantity

$$\mathcal{U}_n(z) = \int_0^{+\infty} \frac{t^n}{(t-z)^{n+1}} \; dt \; , \tag{2}$$

for a complex number z, according to the presence of a new singularity on the integrand function in comparison with the two main examples.

The main goal of the paper is then to *explore in detail the error term of this new quadrature, in particular, to provide explicit formula and upper bounds for this error term.* For simplicity reasons, we will restrict ourselves to holomorphic functions with isolated singularities outside of $]0;+\infty[$. Another restriction is the growth rate of the integrand function: it must not be bigger than a polynomial on an infinite family of circles centered in 0 whose radius are growing to $+\infty$.

To this end, we will first prove the following result on the error of the Gauss-Laguerre quadrature before extending it to the Gauss-Laguerre-like quadrature.

Theorem 1.2. *Let N be a positive integer.*
Let also $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on $\mathbb{C} - \mathcal{S}(f)$ such that:

- *$\mathcal{S}(f)$ is a closed subset of $\mathbb{C}$, called the set of the isolated singular points of f;*
- *f has only isolated singularities on $\mathbb{C}$ and no singularities in $]0;+\infty[$;*
- *the integral $\displaystyle\int_0^{+\infty} f(t)\; e^{-t} \; dt$ is well-defined.*

Let us also consider an increasing sequence $(R_n)_{n\in\mathbb{N}}$ of positive real numbers growing to $+\infty$ such that:

- *no singular point of f and no zero of L_n are located on the circles $\mathcal{C}(0,R_k)$, $k = 0, 1, \cdots$;*
- *f does not grow faster than a polynomial on all the circles $\mathcal{C}(0,R_k)$, $k = 0, 1, \cdots$:*

$$\exists n_0 \in \mathbb{N} \; , \; \exists C > 0 \; , \; \forall k \in \mathbb{N} \; , \; \forall z \in \mathbb{C} \; , \; |z| = R_k \Longrightarrow |f(z)| \leqslant C|z|^{n_0} \; . \tag{3}$$

If $N > n_0$, the error $e_N^{GL}(f)$ of the N-point Gauss-Laguerre quadrature is then given by:

$$e_N^{GL}(f) = \int_0^{+\infty} \sum_{s \in \mathcal{S}(f)} Res \left(z \longmapsto \frac{f(z)}{(t-z)^{N+1} L_N(z)}, s \right) \cdot t^N \ e^{-t} \ dt \ . \qquad (4)$$

In Section 3, we will first prove Theorem 1.2 and extends it to the Gauss-Laguerre-like quadrature (Theorem 3.3). Then, we apply these two theorems to our two main examples to find out the explicit expressions of the error term.

Let us precise here that the extension to the Gauss-Laguerre-like quadrature requires a precise study of a family of functions, postponed in Annex 6 to keep the focus on the main goal of the article.

The error obtained in the first example is essentially a term $\mathcal{U}_n(i\pi)$, divided by an evaluation of a Laguerre polynomial. The second example is more complicated: the quadrature error expresses as the integral of a series. If we could term-by-term integrate it (which will be proved later, in Section 5), therefore, we obtain a series of terms similar to this of the first example.

To understand the asymptotics of an error term we now have to elucidate the behaviour, when n goes to infinity, of $\mathcal{U}_n(z)$ for all complex numbers z, which is obtained in Section 4, using the Laplace's method.

With Perron's formula giving the asymptotics of Laguerre's polynomials when $n \longrightarrow +\infty$ (see Equation (125)), we also need to understand the behaviour on $i\mathbb{R}$ of these polynomials. Therefore, in Section 5, we give explicit upper bounds of all the terms involved in the error terms of our two main examples and prove their asymptotics. Therefore, we can predict numerically how these errors go to 0, and then compute the corresponding integral with a predefined number of exact digits. Moreover, we give Conjectures on sharp upper bounds concerning the error terms of the two main examples.

2 The Gauss-Laguerre-like quadrature

In this preamble Section, we review here known results on the Gauss-Laguerre quadrature. We also introduce the Gauss-Laguerre-like quadrature, as well as the two main example that will be continued during the whole article.

2.1 The Gauss-Laguerre quadrature

The N-points Gauss-Laguerre quadrature is defined for continuous functions $f : \mathbb{R}^+ \longrightarrow \mathbb{C}$ such that $t \longmapsto f(t)e^{-t} \in \mathcal{L}^1(\mathbb{R}^+)$ by (see [3], §3.6):

$$\int_0^{+\infty} f(t)e^{-t} \ dt = \sum_{k=1}^{N} w_{k,N}^{GL} f(t_{k,N}^{GL}) + e_N^{GL}(f) \ , \qquad (5)$$

Nodes					*Weights*				
0.(1)	87649	41047	89278	40360	0.	20615	17149	57800	99433
0.	46269	63289	15080	83188	0.	33105	78549	50884	16599
1.	14105	77748	31226	85688	0.	26579	57776	44214	15260
2.	12928	36450	98380	61633	0.	13629	69342	96377	53998
3.	43708	66338	93206	64523	0.(1)	47328	92869	41252	18978
5.	07801	86145	49767	91292	0.(1)	11299	90008	03394	53231
7.	07033	85350	48234	13040	0.(2)	18490	70943	52631	08643
9.	43831	43363	91938	78395	0.(3)	20427	19153	08278	46013
12.	21422	33688	66158	73694	0.(4)	14844	58687	39812	98771
15.	44152	73687	81617	07676	0.(6)	68283	19330	87119	95644
19.	18015	68567	53134	85466	0.(7)	18810	24841	07967	32139
23.	51590	56939	91908	53182	0.(9)	28623	50242	97388	16196
28.	57872	97428	82140	36752	0.(11)	21270	79033	22410	29674
34.	58339	87022	86625	81453	0.(14)	62979	67002	51786	77872
41.	94045	26476	88332	63547	0.(17)	50504	73700	03551	28204
51.	70116	03395	43318	36434	0.(21)	41614	62370	37285	51904

Table 1: The 16-points Gauss-Laguerre quadrature.

where $e_N^{GL}(f)$ denotes the error of the quadrature, $(x_{k,N}^{GL})_{k\in[\![1;N]\!]}$ are the nodes of the quadrature, *i.e.* the zeros of the N-th Laguerre polynomial $L_N(x)$, and finally $w_{k,N}^{GL}$ are the associated weights defined by:

$$w_{k,N}^{GL} = \frac{1}{L_n'(x_{k,N}^{GL})} \int_0^{+\infty} \frac{L_N(x)e^{-x}}{x - x_{k,N}^{GL}}\, dx = \frac{x_{k,N}^{GL}}{(N+1)^2 \Big(L_{N+1}(x_{k,N}^{GL})\Big)^2} \tag{6}$$

(see [15], Formula (3.4.3) and [1], Formula 25.4.45).

In Table 1, we have tabulated the nodes and the weights of the 16-points Gauss-Laguerre quadrature, up to 20 significant digits. The numbers in the parentheses stand for the number of zeros between the decimal points and the first significant digit.

Example 1. *Let us consider the function* a *defined over* $\mathbb{R}^+$ *by* $a(x) = \dfrac{1}{1+x^2}$. *If we denote respectively by* Ci *and* Si *the integral cosine and integral sine defined by*

$$Ci(x) = \int_x^{+\infty} \frac{\cos x}{x}\, dx \qquad , \qquad Si(x) = \int_x^{+\infty} \frac{\sin x}{x}\, dx \quad , \tag{7}$$

then, it is not difficult to see that:

$$\begin{aligned} \int_0^{+\infty} \frac{e^{-x}}{1+x^2}\, dx &= \sin(1)Ci(1) + \Big(\frac{\pi}{2} - Si(1)\Big)\cos(1) \\ &\approx 0.621449624235813... \end{aligned} \tag{8}$$

We can use Gauss-Laguerre quadratures to experimentally compute this value:

N	*Approximation given by the $N-$points Gauss-Laguerre Quadrature*	*Error* $e_n^{GL}(a)$
1	0. 5	0. 121449624235813
2	0. 64705 88235 29412	−0.(1) 256091992935984
4	0. 63642 69950 05227	−0.(1) 149773707694137
8	0. 62007 49908 74038	0.(2) 137463336177555
16	0. 62150 65102 75135	−0.(4) 568860393211221
32	0. 62144 92814 62043	0.(6) 342773770163874
64	0. 62144 96240 04992	0.(9) 230821032938805
128	0. 62144 96242 35839	−0.(13) 257593489624824
256	0. 62144 96242 35813	0.(19) 418491851920082

Let us mention that, for greater powers of two, we need to use a library for arbitrary-precision floating-point arithmetic to obtain more precise results.

We remark here that when N is quadrupled, the number of zeros before the first significative digits essentially doubles. So, we could conjecture that there exists $C \in \mathbb{R}$ such that $\ln\left|e_n^{GL}(a)\right| \underset{N\longrightarrow+\infty}{\sim} C\sqrt{N}$. More precisely, we can also conjecture that $C = -2\sqrt{2}$ (see Conjecture 1).

2.2 A Gauss-Laguerre-like quadrature.

Let us now define a quadrature associated with the weight function $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by: $\omega(t) = \dfrac{1}{1+e^t}$ for all $t \in \mathbb{R}^+$.

Therefore, we can successively write:

$$\begin{aligned}\int_0^{+\infty} \frac{f(t)}{1+e^t}\,dt &= \int_0^{+\infty} \frac{f(t)}{1+e^{-t}} e^{-t}\,dt \\ &= \sum_{k=1}^{N} \frac{w_{k,N}^{GL}}{1+e^{-t_{k,N}^{GL}}} f(t_{k,N}^{GL}) + e_N^{GL}\left(t \longmapsto \frac{f(t)}{1+e^{-t}}\right) \\ &= \sum_{k=1}^{N} w_{k,N} f(t_{k,N}) + E_N(f)\ , \end{aligned} \tag{9}$$

where:

- the nodes $t_{k,N}$ are defined by the Gauss-Laguerre's nodes: $t_{k,N} = t_{k,N}^{GL}$;
- the weights $w_{k,N}$ are defined by $w_{k,N} = \dfrac{w_{k,N}^{GL}}{1+e^{-t_{k,N}^{GL}}}$;

Nodes					*Weights*				
0.(1)	87649	41047	89278	40360	0.	10759	02368	07348	58471
0.	46269	63289	15080	83188	0.	20315	48527	96083	18217
1.	141057	77483	12268	56877	0.	20143	96546	44327	58248
2.	12928	36450	98380	61632	0.	12181	08849	32714	38727
3.	43708	66338	93206	64524	0.(1)	45854	33411	77478	66314
5.	07801	86145	49767	91292	0.(1)	11229	91251	12025	92098
7.	07033	85350	48234	13040	0.(2)	18475	00668	97130	75948
9.	43831	43363	91938	78395	0.(3)	20425	56535	96797	11668
12.	21422	33688	66158	73694	0.(4)	14844	51325	39238	55128
15.	44152	73687	81617	07676	0.(6)	68283	17987	66214	98887
19.	18015	68567	53134	85466	0.(7)	18810	24832	27814	17876
23.	51590	56939	91908	53182	0.(9)	28623	50242	79853	60917
28.	57872	97428	82140	36752	0.(11)	21270	79033	22327	84516
34.	58339	87022	86625	81453	0.(14)	62979	67002	51786	17641
41.	94045	26476	88332	63547	0.(17)	50504	73700	03551	28173
51.	70116	03395	43318	36434	0.(21)	41614	62370	37285	51904

Table 2: The 16-points Gauss-Laguerre-like quadrature.

- the error $E_N(f)$ is defined by:

$$E_N(f) = e_N^{GL}\left(t \longmapsto \frac{f(t)}{1+e^{-t}}\right) \ . \tag{10}$$

In Table 2, we have tabulated the nodes and the weights of the 16-points Gauss-Laguerre-like quadrature, up to 20 significant digits. One more time, the numbers in the parentheses stand for the number of zeros between the decimal points and the first significant digit.

Example 2. It is not difficult to compute exactly $\int_0^{+\infty} \frac{dt}{1+e^t}$*:*

$$\int_0^{+\infty} \frac{dt}{1+e^t} = \sum_{k>0} (-1)^k \int_0^{+\infty} e^{-kt}\ dt = \sum_{k>0} \frac{(-1)^k}{k} = \ln 2 \ . \tag{11}$$

Consequently, let us test this quadrature to the constant function $f \equiv 1$*:*

N	*Approximation given by the* $N-$ *points Gauss-Laguerre--like Quadrature*				*Error*	
1	0.	73105	85786	30005	0.(1)	379113980700596
2	0.	69010	24700	33026	−0.(2)	304471052691971
4	0.	69312	37952	35122	−0.(4)	233853248232775
8	0.	69314	27730	86691	−0.(5)	440747325425636
16	0.	69314	71785	75662	−0.(8)	198428359047497
32	0.	69314	71805	54994	−0.(11)	495161506186579
64	0.	69314	71805	59946	−0.(16)	387982796897492

Here again, a library for arbitrary-precision floating-point arithmetic is needed to obtain more precise results for greater powers of two.

This table suggests that the logarithm of the error $\ln|E_n(1)|$ *has a simple equivalent: there exists* $C \in \mathbb{R}$, $\ln|E_n(1)| \underset{n \longrightarrow +\infty}{\sim} C\sqrt{n}$. *More precisely, we conjecture that* $C = -2\sqrt{2\pi}$ *(see Conjecture 2).*

2.3 Error formula and convergence properties.

The most well-known formula concerning the error $e_n(f)$ of the Gauss-Laguerre quadrature, but useless in a practical context, is due to Markov (see [3], Equation 3.6.3 or [5], Equation 1.18):

$$\forall n \in \mathbb{N}^* \ , \ \exists \xi \in \mathbb{R}_+^* \ , \ e_n(f) = \frac{n!^2}{(2n)!} f^{(2n)}(\xi) \ , \tag{12}$$

if the function f is $\mathcal{C}^{2n}(\mathbb{R}^+)$.

However, this error formula is mostly unusable... and can not be used to prove that the remainder $e_n(f)$ or $E_n(f)$ goes to 0 when n goes to $+\infty$. On one hand, it can be inextricable to compute explicitly the $(2n)^{\text{th}}$-derivative, depending on the function f; on the other hand, this derivative can take arbitrary large values even if the error is small.

Let us remind the reader that the Gauss-Laguerre quadrature is unfortunately known to have poor convergence properties, even to be unstable when used to numerically integrate functions over $[0; +\infty[$ by the very bad following rule:

$$\int_0^{+\infty} f(t) \ dt = \int_0^{+\infty} \left(f(t)e^t\right) e^{-t} \ dt \approx \sum_{k=1}^{N} w_{k,N}^{GL} e^{t_{k,N}^{GL}} f(t_{k,N}^{GL}) \ . \tag{13}$$

Sometimes, this is explained by the difficulty of managing the infinite upper integration limit, in opposition with the Stieljes result (see [13]) saying that for all continuous function $f : [-1; 1] \longrightarrow \mathbb{C}$ the N^{th}-point Gauss-Legendre quadrature scheme converge, when N goes to infinity, to the value of the integral of f over $[-1; 1]$.

Actually, this is due to the growth rate of $\widetilde{f} : t \longmapsto f(t)e^t$, or saying it differently, $\widetilde{f}$ can not be well-approximate by polynomials over $[0; +\infty[$. More precisely, in 1928, Uspensky (see [12]) has shown convergence properties for the Gauss-Laguerre quadrature:

Theorem 2.1. ***(Uspensky,*** 1928***)***
If the function f *satisfies the following property:*

$$\exists \rho > 0 \ , \ \exists x_0 > 0 \ , \ \exists C > 0 \ , \ \forall x \geqslant x_0 \ , \ |f(x)| \leqslant \frac{Ce^x}{x^{1+\rho}} \ , \tag{14}$$

then $e_N^{GL}(f) \underset{N \longrightarrow +\infty}{\longrightarrow} 0$.

Example 3. Let us come back to Example 1 and 2.
Of course, Uspensky's theorem can be applied to the function a, as well as to the function $b : t \longrightarrow \dfrac{1}{1+e^{-x}}$. *Therefore, we have proven that*

$$e_N^{GL}(a) \underset{N\longrightarrow+\infty}{\longrightarrow} 0 \ . \tag{15}$$

$$E_N(1) = e_N^{GL}(b) \underset{N\longrightarrow+\infty}{\longrightarrow} 0 \ . \tag{16}$$

This is the first step in the direction of proving that $\ln|e_N^{GL}(a)| \underset{N\longrightarrow+\infty}{\sim} -2\sqrt{2N}$ *and* $\ln|E_N(1)| \underset{N\longrightarrow+\infty}{\sim} -C\sqrt{N}$, *but there is still a long way to go.*

Uspensky's theorem is a theoretical theorem which is nowadays a bit frustrating for the numerical scientist working with high-power computation tools. Our main goal is to provide an explicit formula for the error $e_n^{GL}(f)$ for large class of functions f, as well as find out a way to describe the rate of convergence of $\big(e_N^{GL}(f)\big)_{N\in\mathbb{N}}$ or $\big(E_N(f)\big)_{N\in\mathbb{N}}$ to 0.

In this direction, Mastroianni & Monegato's result (see [7]) is important:

Theorem 2.2. ***(Mastroianni & Monegato,*** **1995*)***
Let p and q be two non-negative integers such that $0 \leqslant p \leqslant q$. *Let us denote* $\mathcal{C}_p^q[0;\infty)$ *the subset of* $\mathcal{C}^p([0;+\infty[) \cap \mathcal{C}^q(]0;+\infty[)$ *defined by:*

$$\begin{aligned} \mathcal{C}_p^q[0;\infty) \quad &= \quad \{f \in \mathcal{C}^p([0;+\infty[) \cap \mathcal{C}^q(]0;+\infty[) \ ; \\ & \qquad x \longmapsto x^i f^{(p+i)}(x) \in \mathcal{C}^0([0;+\infty[), \ i = 1, \cdots, q-p\} \ . \end{aligned} \tag{17}$$

With $f \in \mathcal{C}_p^q[0;\infty)$, *we associate the auxiliary function* $\Phi \in \mathcal{C}^q([0;+\infty[)$ *defined by* $\Phi(x) = x^{q-p} f(x)$, *so that* $e_N^{GL}(f)$ *satisfies:*

$$|e_N^{GL}(f)| = \begin{cases} \mathcal{O}(N^{-\frac{q}{2}}) \ E_{N-p-1}\big(\Phi^{(q)}, e^{-\frac{x}{2}}\big) & , \textit{ if } q \leqslant 2p+1 \ , \\ \mathcal{O}(N^{-(p+1)} \ln n) \ E_{N-p-1}\big(\Phi^{(q)}, e^{-\frac{x}{2}}\big) & , \textit{ if } q \leqslant 2p+2 \ , \\ \mathcal{O}(N^{-(p+1)}) \ E_{N-p-1}\big(\Phi^{(q)}, e^{-\frac{x}{2}}\big) & , \textit{ if } q \geqslant 2p+3 \ , \end{cases} \tag{18}$$

where $E_n(f,\omega) = \underset{P_n\in\mathbb{C}_{n-1}[X]}{inf} ||\omega(f-P_n)||_{\infty,[0;+\infty[}$ *is the error of the best polynomial approximation of* f.

Moreover, following the notations introduced in [8], (at the beginning of Section 3), Mastroianni and Szabados have proved estimation of the quantity $E_n(f, e^x)$ for a special class of functions:

Lemma 2.3. ***(Mastroianni & Szabados, 2007)*** *For all functions* $f \in W_1^\infty(e^{-x})$, *there exists* $C > 0$ *and a sequence of numbers* $(a_n)_{n\in\mathbb{N}}$ *satisfying* $a_n \underset{n\longrightarrow+\infty}{\sim} n$ *such that we have:*

$$E_n\big(f, e^{-x}\big) \leqslant \frac{C\sqrt{a_n}}{n} ||f'(x)\sqrt{x}e^{-x}||_\infty \ . \tag{19}$$

Example 4. *Let us apply Theorem 2.2 to Example 1.*

The function a is an element of $\mathcal{C}_p^q[0;+\infty)$ for all integers p and q such that $0 \leqslant p \leqslant q$. In particular, when $p \geqslant 0$ and $q = 2p$, we have:

$$e_N^{GL}(a) \underset{N \longrightarrow +\infty}{=} \mathcal{O}(N^{-p}) \cdot E_{N-p-1}\big(\Phi^{(2p)}, e^{-\frac{x}{2}}\big) \ , \tag{20}$$

where $\Phi(x) = x^p f(x)$.

According to $\Phi^{(p)} \in W_1^{\infty}(e^{-x})$ and Lemma 2.3, we have:

$$E_{N-p-1}\big(\Phi^{(q)}, e^{-\frac{x}{2}}\big) \underset{N \longrightarrow +\infty}{=} \mathcal{O}(N^{-\frac{1}{2}}) \ . \tag{21}$$

Therefore, we deduce that:

Lemma 2.4. *For all non-negative integers p, we have:*

$$e_N^{GL}(a) \underset{N \longrightarrow +\infty}{=} \mathcal{O}(N^{-p}) \ . \tag{22}$$

Example 5. *Following the same scheme of proof, as in Example 4, we prove the following result:*

Lemma 2.5. *For all non-negative integers p, we have:*

$$E_N(1) \underset{N \longrightarrow +\infty}{=} \mathcal{O}(N^{-p}) \ . \tag{23}$$

Even if it will not be possible to apply it in our context of holomorphic functions over $\mathbb{C} - \mathcal{S}(f)$, where $\mathcal{S}(f)$ is the closed subset of $\mathbb{C}$ of the singularities of f, let us remind, for completeness, the Lubinsky's result (see [10])

Theorem 2.6. ***(Lubinsky,*** *1983)*
Let f be an entire function defined by its series expansion $f(z) = \sum_{n \geqslant 0} c_n z^n$ for all $z \in \mathbb{C}$.
Let us consider the real number $A = \limsup_{n \longrightarrow +\infty} \left(\dfrac{n|c_n|^{\frac{1}{n}}}{2} \right)$.
If $A < 1$, we have for sufficiently large n:

$$\left| e_n^{GL}(f) \right| \leqslant A^{2n} \ . \tag{24}$$

3 Error formula of the quadrature, by contour integration

Without forgetting methods based on Peano error estimates (see [14] for example), it is well known that, when applied to holomorphic functions, there exist three main methods to obtain estimations of the remainder of a quadrature (see [5], §4). Among the oldest are the estimates based on contour integration ; there are also these based on the Hilbert space norm estimates, as well as the estimates obtained via approximation theory (see [17] for a recent result).

Here, we will be interested in the contour integration method, applied to meromorphic functions over $\mathbb{C}$.

3.1 Proof of Theorem 1.2 concerning the error term of Gauss-Laguerre quadrature

Let us fix a positive integer n. Let also $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an holomorphic function on $\mathbb{C} - \mathcal{S}(f)$ such that:

- $\mathcal{S}(f)$ is a closed subset of $\mathbb{C}$ called the set of the isolated singular points of f;
- f has only isolated singularities on C and no singularities in $]0;+\infty[$;
- the integral $\displaystyle\int_0^{+\infty} f(t)\ e^{-t}\ dt$ is well-defined.

According to the general method described in [3] (see §4.6, p. 303) or in [11] (see §3.2), let us consider a closed circle $\mathcal{C}(0,R)$, centered in 0, with radius R, enclosing all the zeros $x_{1,n}^{GL}, \cdots, x_{n,n}^{GL}$ of the n-th Laguerre ploynomial L_n. such that no pole of f are on $\mathcal{C}(0,R)$.

Thus, according to Cauchy's residue theorem, we have for all positive real numbers t such that $t \leqslant R$ and $t \neq x_{k,n}^{GL}$, $k \in [\![1;n]\!]$:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathcal{C}_R}\frac{f(z)\ dz}{(z-t)L_n(z)}\\ =\ &\frac{f(t)}{L_n(t)}+\sum_{k=1}^{n}\frac{f(x_{k,n}^{GL})}{(x_{k,n}^{GL}-t)L_n'(x_{k,n}^{GL})}+\sum_{\substack{s\in\mathcal{S}(f)\\|s|<R}}Res\left(z\longmapsto\frac{f(z)}{(z-t)L_n(z)},s\right)\end{aligned} \tag{25}$$

Let us moreover assume that there exists an increasing sequence $(R_n)_{n\in\mathbb{N}}$ of positive real numbers growing to $+\infty$ such that

- no singular point of f is located on the circles $\mathcal{C}(0,R_k)$, $k=0,\ 1,\ \cdots$;
- no zero of L_n is located on the circles $\mathcal{C}(0,R_k)$, $k=0,\ 1,\ \cdots$;
- f does not grow faster than a polynomial on all the circles $\mathcal{C}(0,R_k)$, $k=0$, $1,\ \cdots$:

$$\exists n_0 \in \mathbb{N}\ ,\ \exists C>0\ ,\ \forall k\in\mathbb{N}\ ,\ \forall z\in\mathbb{C}\ ,\ |z|=R_k \Longrightarrow |f(z)|\leqslant C|z|^{n_0}\ . \tag{26}$$

Therefore, we have the following:

Lemma 3.1. *If such a sequence $(R_k)_{k\in\mathbb{N}}$ is available for the function $f : \mathbb{C} \longrightarrow \mathbb{C}$, holomorphic on $\mathbb{C} - \mathcal{S}(f)$, for all $n > n_0$ and all $t \in \mathbb{R}^+$, we have:*

$$\frac{1}{2\pi i}\int_{\mathcal{C}(0,R_k)} \frac{f(z)\ dz}{(z-t)L_n(z)} \underset{k\longrightarrow+\infty}{\longrightarrow} 0\ . \tag{27}$$

Proof. The dominant term of the polynomial $L_n(X)$ is $\dfrac{(-X)^n}{n!}$. Therefore, there exist an integer $k_1 \geqslant 0$ such that for all integers $k \geqslant k_1$ and all $z \in \mathcal{C}(0, R_k)$, we have:

$$\frac{|z|^n}{2\cdot n!} \leqslant |L_n(z)| \leqslant \frac{2|z|^n}{n!} \tag{28}$$

Let us now fix $t \geqslant 0$. So, there exists an integer $k_2 \geqslant k_1$ such that for all $k \geqslant k_2$, we have for all $z \in \mathcal{C}(0, R_k)$:

$$\left\{\begin{array}{l} R_k > t\ . \\ \left|\dfrac{f(z)}{(z-t)L_n(z)}\right| \leqslant \dfrac{CR_k^{n_0}}{(R_k-t)\cdot\dfrac{R_k^n}{2n!}} \end{array}\right. . \tag{29}$$

Finally, this gives us:

$$\left|\frac{1}{2\pi i}\int_{\mathcal{C}(0,R_k)} \frac{f(z)\ dz}{(z-t)L_n(z)}\right| \leqslant \frac{2Cn!R_k^{n_0+1}}{(R_k-t)R_k^n} \tag{30}$$

and proves for all non-negative real numbers t the convergence to 0 when $k \longrightarrow +\infty$ of the sequences of integrals since $n > n_0$. □

As a corollary, from Equation (25), we deduce that $\displaystyle\sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(z-t)L_n(z)}, s\right)$ is well-defined if $n > n_0$ and $t \in \mathbb{R}^+ - \left\{x_{k,n}^{GL}\ ;\ 1 \leqslant k \leqslant n\right\}$.

Now, it is possible to send $R \longrightarrow +\infty$ in Equation (25) and then re-organising it. This gives the following Equation, valid a priori for all $t \in \mathbb{R}^+ - \left\{x_{k,n}^{GL}\ ;\ 1 \leqslant k \leqslant n\right\}$, but which can be extended to all $t \in \mathbb{R}^+$ using the continuity of each term:

$$\begin{aligned} f(t)\ e^{-t} \quad &= \quad \sum_{k=1}^{n} \frac{f(x_{k,n}^{GL})}{L_n'(x_{k,n}^{GL})} \frac{L_n(t)\ e^{-t}}{t - x_{k,n}^{GL}} \\ &\quad + \sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right) \cdot L_n(t)\ e^{-t}\ . \end{aligned} \tag{31}$$

Moreover, for all $k \in [\![1; n]\!]$, $t \longmapsto \dfrac{L_n(t)\ e^{-t}}{t - x_{k,n}^{GL}}$ defines a continuous and integrable functions over $[0; +\infty[$. From the existence of $\displaystyle\int_0^{+\infty} f(t)e^{-t}\ dt$, we deduce that $\displaystyle\int_0^{+\infty} \sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right) \cdot L_n(t)\ e^{-t}\ dt$ is also well-defined. Therefore, we have:

$$\begin{aligned}\int_0^{+\infty} & f(t)\ e^{-t}\ dt \\ & = \sum_{k=1}^{n} \frac{f(x_{k,n}^{GL})}{L_n'(x_{k,n}^{GL})} \int_0^{+\infty} \frac{L_n(t)\ e^{-t}}{t - x_{k,n}^{GL}}\ dt \\ & \quad + \int_0^{+\infty} \sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right) \cdot L_n(t)\ e^{-t}\ dt\ . \end{aligned} \tag{32}$$

According to Equations (5) and (6), this finally gives us a nice expression of the error term of the Gauss-Laguerre quadrature for such functions f:

$$e_n^{GL}(f) = \int_0^{+\infty} \sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right) \cdot L_n(t)\ e^{-t}\ dt\ . \tag{33}$$

Therefore, using Rodrigue's formula $L_n(x) = \dfrac{e^x}{n!}\left(x^n e^{-x}\right)^{(n)}$ and successive integrations by parts, we have:

$$\begin{aligned} e_N^{GL}(f) &= \frac{1}{n!}\int_0^{+\infty} \sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right) \cdot \left(t^n\ e^{-t}\right)^{(n)}\ dt \\ &= \frac{(-1)^n}{n!}\int_0^{+\infty}\left(\sum_{s\in\mathcal{S}(f)} Res\left(z \longmapsto \frac{f(z)}{(t-z)L_n(z)}, s\right)\right)^{(n)} \cdot t^n\ e^{-t} dt \end{aligned} \tag{34}$$

Using Laurent's expansion, it is easy to see that, if $z \longmapsto a(z)$ is a meromorphic function near $s \in C$, then we have for all integers n:

$$\frac{d^n}{dt^n} Res\left(z \longmapsto \frac{a(z)}{t-z}, s\right) = (-1)^n n!\ Res\left(z \longmapsto \frac{a(z)}{(t-z)^{n+1}}, s\right)\ . \tag{35}$$

Therefore, plugging Equation (35) to Equation (34) concludes the proof of Theorem 1.2.

Remark 1. *Let us remark that if the function f is entire and satisfies the hypothesis of Theorem 1.2, then, necessarily, f is a polynomial with degree d satisfying $d \leqslant n_0$. Then, by definition, the N-point Gauss-Laguerre quadrature applied to f is exact if $n_0 \leqslant 2N+1$. This is, of course, stronger than the result of Theorem 1.2 in this particular case.*

Consequently, this theorem is useful only for functions which have singularities and explain why Lubinski's result on geometric convergence to 0 *of the error can not be applied in our context.*

Example 6. *Let us come back to Example 1 defining the meromorphic function a over $\mathbb{C}$ by $a(z) = \dfrac{1}{1+z^2}$ for all $z \in \mathbb{C} - \{i; -i\}$.*

Let us also fix a positive integer N and consider an increasing sequence $(R_k)_{k\in\mathbb{N}}$ growing to $+\infty$ such that $R_0 \geqslant 2x^{GL}_{N+1,N+1}$.

The function a has two simple poles, i and $-i$. Therefore, Theorem 1.2 applied to a, with $n_0 = 0$, gives us:

$$\begin{aligned} e^{GL}_N(a) &= \int_0^{+\infty} \left(\frac{-i}{2(t-i)^{N+1}L_N(i)} + \frac{i}{2(t+i)^{N+1}L_N(-i)} \right) t^N e^{-t}\, dt \\ &= \Re e \left(\frac{i}{L_N(-i)} \int_0^{+\infty} \frac{t^N\ e^{-t}}{(t+i)^{N+1}}\ dt \right) . \end{aligned} \tag{36}$$

As explicit examples, we have:

$$e^{GL}_1(a) = \frac{1}{2} \int_0^{+\infty} \frac{t^3+2t^2-t}{(1+t^2)^2} e^{-t}\ dt \tag{37}$$

$$e^{GL}_2(a) = \frac{1}{17} \int_0^{+\infty} \frac{8t^5+6t^4-24t^3-2t^2}{(1+t^2)^3} e^{-t}\ dt \tag{38}$$

$$e^{GL}_3(a) = \frac{3}{148} \int_0^{+\infty} \frac{17t^7-12t^6-102t^5+12t^4+17t^3}{(1+t^2)^4} e^{-t}\ dt \tag{39}$$

and we effectively have, using successive integration by parts:

$$e^{GL}_1(a) = \int_0^{+\infty} \frac{e^{-t}}{1+t^2}\ dt - \frac{1}{2}\ . \tag{40}$$

$$e^{GL}_2(a) = \int_0^{+\infty} \frac{e^{-t}}{1+t^2}\ dt - \frac{11}{17}\ . \tag{41}$$

3.2 The error of the Gauss-Laguerre-like quadrature

From now on, we want to change the weight function $x \longmapsto e^{-x}$ to $x \longmapsto \dfrac{1}{1+e^x}$ to prove an analogue of Theorem 1.2 for the Gauss-Laguerre-like quadrature.

So, we will look for precise estimations of $g_0 : z \longmapsto \dfrac{1}{1+e^{-z}}$ on circles centered in 0. Then, we will easily be able to state a derivation of Theorem 1.2 for the Gauss-Laguerre-like quadrature.

3.2.1 Estimations on circles centered in 0 of the weight function

If $z = Re^{i\theta} \in \mathcal{C}(R,0)$, where the radius R satisfies $R \in \mathbb{R}^+ - (2\mathbb{Z}+1)\pi$, we have:

$$|g_0(z)|^2 = \left| \frac{1}{1+e^{-z}} \right|^2 = \frac{1}{1+2e^{-R\cos\theta}\cos(R\sin\theta)+e^{-2R\cos\theta}}\ . \tag{42}$$

The denominator of the right-hand side of Equation (42) is not so simple to handle as a function of $\theta \in]-\pi;\pi]$. Nevertheless, the following Proposition gives us a uniform (in the variable R) upper bound of the function g_0. This result will be sufficient to prove that the hypothesis (3) are satisfied with $n_0 = 0$ for a large range of values of R:

Proposition 3.2. *Let* $\delta \in]0;\pi]$.
There exist a constant $C(\delta) > 0$*, depending only on* δ*, such that for all positive real numbers* R *satisfying* $dist(R,(2\mathbb{Z}+1)\pi) \geqslant \delta$*, we have:*

$$\forall z \in \mathcal{C}(R,0)\ ,\ \left|\frac{1}{1+e^{-z}}\right| \leqslant C(\delta)\ . \tag{43}$$

More precisely, $C(\delta)$ *could be chosen to be defined by:*

$$C(\delta) = \frac{1}{\min\left(0.93;\sin\delta;1-e^{-\delta};\sqrt{1-\cos^4\left(\frac{\delta}{2}\right)}\right)}\ . \tag{44}$$

The result would have been easy to prove if an integer n would have been fixed and $R \in [(2n-1)\pi+\delta;(2n+1)\pi-\delta]$, according to the continuity of $z \longmapsto (1+e^{-z})^{-1}$ of the compact annulus $\{z \in \mathbb{C};(2n-1)\pi+\delta \leqslant |z| \leqslant (2n+1)\pi-\delta\}$. Nevertheless, the theoretical upper bound found would have been a function of n...

Consequently, the proof of this Proposition is long, delicate, technical and non really informative, according to the subtle behaviour of the denominator of the right-hand side of Equation (42) as a function of the parameter θ. Therefore, it is merely postponed to Annex 6.

3.2.2 Application to the Gauss-Laguerre-like quadrature

Applying Theoreme 1.2 to the fonction $\widetilde{f} : z \longmapsto \dfrac{f(z)}{1+e^{-z}}$, where $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies the hypothesis of Theorem 1.2 gives us:

Theorem 3.3. *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be an holomorphic function on* $\mathbb{C} - \mathcal{S}(f)$ *such that:*

- $\mathcal{S}(f)$ *is a closed subset of* $\mathbb{C}$*, called the set of the isolated singular points of* f*;*
- f *has only isolated singularities on* $\mathbb{C}$ *and no singularities in* $]0;+\infty[$*;*
- *the integral* $\displaystyle\int_0^{+\infty} \frac{f(t)}{1+e^t}\,dt$ *is well-defined.*

Let us also consider $\delta \in]0;\pi]$ *and an increasing sequence* $(R_n)_{n\in\mathbb{N}}$ *of positive real numbers growing to* $+\infty$ *such that:*

- *for all integers n, $dist\big(R_n; (2\mathbb{Z}+1)\pi\big) \geqslant \delta$.*
- *no pole of f and no zero of L_n are located on the circles $\mathcal{C}(0, R_k)$, $k = 0$, $1, \cdots$;*
- *f does not grow faster than a polynomial on all the circles $\mathcal{C}(0, R_k)$, $k = 0$, $1, \cdots$:*

$$\exists n_0 \in \mathbb{N}\ ,\ \exists C > 0\ ,\ \forall k \in \mathbb{N}\ ,\ \forall z \in \mathbb{C}\ ,\ |z| = R_k \Longrightarrow |f(z)| \leqslant C|z|^{n_0}\ . \quad (45)$$

If $N > n_0$, the error $E_N(f)$ of the N-point Gauss-Laguerre-like quadrature is then given by:

$$E_N(f) \;=\; \int_0^{+\infty} \sum_{s \in \mathcal{S}} Res\left(z \longmapsto \frac{1}{(t-z)^{N+1} L_n(z)} \frac{f(z)}{1+e^{-z}}, s\right) \cdot t^n\ e^{-t}\ dt \ ,(46)$$

where $\mathcal{S}$ denotes the set of isolated singularities of the function $z \longmapsto \dfrac{f(z)}{1+e^{-z}}$.

Proof. From Equation (10), we know that the error of a Gauss-Laguerre-like quadrature is related to the error of a Gauss-Laguerre quadrature by:

$$E_N(f) = e_N^{GL}\left(t \longmapsto \frac{f(t)}{1+e^{-t}}\right)\ . \quad (47)$$

We will compute this last value using Theorem (1.2).

Let us also define $\widetilde{f} = f \cdot g_0$, *i.e.*:

$$\widetilde{f}(z) = \frac{f(z)}{1+e^{-z}}\ ,\ \text{for all } z \in \mathbb{C} - \mathcal{S}(\widetilde{f})\ , \quad (48)$$

where $\mathcal{S}(\widetilde{f}) = \mathcal{S}(f) \cup (2\mathbb{Z}+1)i\pi$.

Firstly, as a product of functions defined over $\mathbb{C}$, holomorphic respectively over $\mathbb{C}-\mathcal{S}(f)$ and $\mathbb{C}-(2\mathbb{Z}+1)i\pi$, with isolated singularities, $\widetilde{f}$ is an holomorphic functions over $\mathbb{C} - \mathcal{S}(\widetilde{f})$, with isolated singularities. Moreover, no pole of $\widetilde{f}$ is located on a circle $\mathcal{C}(R_k, 0)$, $k = 0, 1, \cdots$ and $\displaystyle\int_0^{+\infty} \frac{f(t)}{1+e^t}\, dt$ is assumed to be well-defined.

Finally, according to Proposition 3.2, we know that there exists a constant $C(\delta) > 0$ such that for all $z \in \mathbb{C}$, we have:

$$\text{dist}\big(|z|, (2\mathbb{Z}+1)\pi\big) \geqslant \delta \Longrightarrow |g_0(z)| \leqslant C(\delta)\ . \quad (49)$$

According to the assumption on the function f (see Equation (45)), we are now able to claim that for all $z \in \mathcal{C}(R_k, 0)$, we have:

$$|\widetilde{f}| = |f(z) \cdot |g_0(z)| \leqslant C|z|^{n_0} \cdot C(\delta)\ . \quad (50)$$

This proves that $\widetilde{f}$ satisfies Hypothesis (3).

Consequently, Theorem 1.2 can be applied to the function $\widetilde{f}$:

$$E_N(f) = \int_0^{+\infty} \sum_{s \in \mathcal{S}(\widetilde{f})} Res\left(z \longmapsto \frac{1}{(t-z)^{N+1} L_n(z)} \frac{f(z)}{1+e^{-z}}, s\right) \cdot t^n \ e^{-t} \ dt \ . \quad (51)$$

□

Let us emphasize that when f has only a finite number of poles, the residues in Equation (46) should be easy to compute explicitly. Using an estimation of Laguerre polynomials evaluated on iX (see Subsection 5.1), we can show that the series and the integral can be permuted.

This is exactly what we can now do, coming back to Example 2:

Example 7. Using $R_n = 2n\pi$ for all $n \in \mathbb{N}$ and $\delta = \pi$, Theorem 1.2 applied to the constant function 1 (with $n_0 = 0$) gives us:

$$E_N(1) = \int_0^{+\infty} \left(\sum_{k \in \mathbb{Z}} \frac{1}{L_N\big((2k+1)i\pi\big)} \frac{t^N \ e^{-t}}{\big(t-(2k+1)i\pi\big)^{N+1}} \right) \ dt \ , \quad (52)$$

for all $N > 0$.

Let us emphasize that even if this is an explicit result of the error of the quadrature, there is unfortunately, still a long way to go to quantify numerically this error...

This will be done in Section 5, especially in Subsection 5.4.

4 Asymptotic of the integral $\mathcal{U}_N(z)$ for $z \in \mathbb{C} - \mathbb{R}^+$

We shall now study precisely the fundamental integral $\displaystyle\int_0^{+\infty} \frac{t^N \ e^{-t}}{(t-z)^{N+1}} \ dt$, $z \in \mathbb{C} - \mathbb{R}^+$ and $N \in \mathbb{N}$, which naturarly appears in the explicit expression of the error $E_n(f)$ when f has a finite number of poles.

In particular, we want to be able to quantify each term of the summation $E_N(f)$. Consequently, the main goal of this section is to prove the following

Proposition 4.1. *For all $z \in \mathbb{C} - \mathbb{R}^+$, we have:*

$$\int_0^{+\infty} \frac{t^n}{(t-z)^{n+1}} e^{-t} \ dt \underset{n \longrightarrow +\infty}{\sim} \frac{\sqrt{\pi} e^{-\frac{z}{2}}}{(-nz)^{\frac{1}{4}}} e^{-2\sqrt{-nz}} \ . \quad (53)$$

using the Laplace method (for example, see [4], chapter 8 for an introduction to this asymptotic method).

To prove this result, let us denote the left-hand side of the equivalent sign in Equation (53) by $\mathcal{U}_n(z)$:

$$\mathcal{U}_n(z) = \int_0^{+\infty} \frac{t^n}{(t-z)^{n+1}} e^{-t} \ dt \ . \quad (54)$$

We will also denote respectively the principal branch of the logarithm and the square root by log and $\sqrt{\ }$. During this Section, we will consider a fixed complex number $z \in \mathbb{C} - \mathbb{R}^+$, so that $s = \sqrt{-z}$ is a well-defined complex number satisfying $\Re e\ s > 0$.

Finally, let us fix an integer n.

4.1 Modification of the integration path.

The function $t \longmapsto n \log(t) - (n+1)\log(t-z) - t$ has a saddle node located at $\dfrac{z - 1 + \sqrt{(1-z)^2 - 4nz}}{2}$, *i.e.* near $\sqrt{-nz}$. In this Subsection, we are moving the initial integration path, $[0, +\infty[$, to a path going through $\sqrt{-nz}$, *i.e.* quite close to the previously indicated saddle node.

If $\theta_0 = Arg(\sqrt{-nz})$ and $R > |z|$, we successively have:

$$\begin{aligned} \left| \int_{\mathcal{C}(0,R)_{|0 \leqslant \theta \leqslant \theta_0}} \frac{\chi^n}{(\chi - z)^{n+1}} e^{-\chi}\, d\chi \right| &\leqslant \int_0^{\theta_0} \frac{R^{n+1}}{|Re^{i\theta} - z|^{n+1}} e^{-R\cos\theta}\, d\theta \\ &\leqslant \int_0^{\theta_0} \frac{R^{n+1}}{\big|R - |z|\big|^{n+1}} e^{-R\cos\theta}\, d\theta \\ &\leqslant \frac{\theta_0}{\left|1 - \frac{|z|}{R}\right|^{n+1}} e^{-R\cos\theta_0}\ . \end{aligned} \tag{55}$$

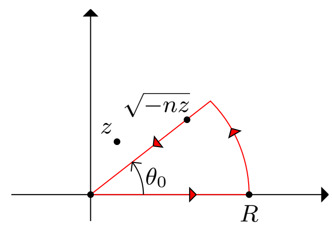


Figure 1: Paths used to modify the path of integration of $\mathcal{U}_n(z)$.

From $z \in \mathbb{C} - \mathbb{R}^+$, we know that $\theta_0 \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$, *i.e.* $\cos\theta_0 > 0$. Consequently, we deduce that:

$$\left| \int_{\mathcal{C}(0,R)_{|0 \leqslant \theta \leqslant \theta_0}} \frac{\chi^n}{(\chi - z)^{n+1}} e^{-\chi}\, d\chi \right| \underset{R \longrightarrow +\infty}{\longrightarrow} 0\ . \tag{56}$$

According to the residue theorem applied to the path described on Figure 1, we conclude that:

$$\mathcal{U}_n(z) = \int_0^{\sqrt{-nz}\cdot\infty} \frac{t^n}{(t-z)^{n+1}} e^{-t}\, dt\ . \tag{57}$$

It turns out that the integral on the right-hand side of Equation (57) can also be written as:

$$\int_0^{+\infty} \frac{t^n}{(t+s)^{n+1}} e^{-st}\, dt\ , \text{ with } s = \sqrt{-z}\ . \tag{58}$$

From now on, for all $z \in \mathbb{C} - \mathbb{R}^+$, we denote $s = \sqrt{-z}$. Therefore, for all $s \in \mathbb{C}$ such that $\Re e\ s > 0$, we define by $\mathcal{I}_n(s)$ the integral (58):

$$\mathcal{I}_n(s) = \int_0^{+\infty} \frac{t^n}{(t+s)^{n+1}} e^{-st}\, dt \ , \text{ for all } s \in \mathbb{C} \ , \ \Re e\ s > 0 \ , \tag{59}$$

so that we have:

$$\mathcal{U}_n(z) = \mathcal{I}_n(s), \text{ with } s = \sqrt{-z} \ . \tag{60}$$

4.2 A first elementary estimation

When s is a positive real number, we have:

$$\begin{aligned} 0 \ \leqslant \ \mathcal{I}_n(s) \ &\leqslant \ \frac{1}{s} \int_0^{+\infty} \Big(1 + \frac{s}{x}\Big)^{-n} e^{-sx}\, dx \\ &\leqslant \ \frac{1}{s} \int_0^{+\infty} \exp\left(- s\Big(x + \frac{n}{s+x}\Big)\right) dx \ , \end{aligned} \tag{61}$$

according to $\ln(1+x) \geqslant \dfrac{x}{1+x}$ for all $x \geqslant 0$. The integral on the right-hand side of Inequation (61) can be generalised slightly, in order to find out an explicit upper bound of $\mathcal{I}_n(s)$:

Lemma 4.2. *Let $u > 0$, $v \geqslant 0$ and n a positive integer.*
Therefore, the integral $\widetilde{\mathcal{I}}_n(u,v)$ defined by

$$\widetilde{\mathcal{I}}_n(u,v) = \int_0^{+\infty} \exp\left(-u\Big(x + \frac{n}{x+v}\Big)\right)\ dx \tag{62}$$

satisfies:

$$\widetilde{\mathcal{I}}_n(u,v) \leqslant \begin{cases} e^{-u\sqrt{n}}\left(\dfrac{1}{u} + \dfrac{\sqrt[4]{n}\cdot\sqrt{\pi}}{\sqrt{u}}\right) & , \text{ if } 0 \leqslant v \leqslant \sqrt{n} \ . \\[2ex] \dfrac{e^{-u\frac{\sqrt{n}}{\sqrt{n+\pi}}}}{u}\left(3 + \dfrac{2n}{\pi}\right) & , \text{ if } v \in \left]\sqrt{n}; \sqrt{n+\pi}\right] \ . \\[2ex] \dfrac{2e^{-\frac{un}{v}}}{u} & , \text{ if } v > \sqrt{n+\pi} \ . \end{cases} \tag{63}$$

Proof. Let us remark that when $v = 0$, the integral $\widetilde{I}_n(u,0)$ is still a convergent one for all $u > 0$.

• For all $u > 0$, the function $v \longmapsto \widetilde{\mathcal{I}}(u,v)$ is increasing on $[0; +\infty[$. Consequently, if $u > 0$ and $v \in [0; \sqrt{n}]$, we have:

$$\widetilde{\mathcal{I}}(u,0) \leqslant \widetilde{\mathcal{I}}(u,v) \leqslant \widetilde{\mathcal{I}}(u,\sqrt{n}) \ . \tag{64}$$

• Let us now emphasize that $y \longmapsto \dfrac{y-2v+\sqrt{y^2-4n}}{2}$ is an increasing function on $[2\sqrt{n}; +\infty[$ from $\sqrt{n}-v$ to $+\infty$. So, 0 is in its image when $v \geqslant \sqrt{n}$.

With this condition, using first the substitution $x = \dfrac{y-2v+\sqrt{y^2-4n}}{2}$, *i.e.* $y - v = x + \dfrac{n}{x+v}$, and then $y = z + 2\sqrt{n}$, in the integral $\widetilde{I}_n(u,v)$, we successively have for all $u > 0$ and $v \geqslant \sqrt{n}$:

$$\begin{aligned}
\widetilde{\mathcal{I}}_n(u,v) &= \frac{e^{u(v-2\sqrt{n})}}{2} \int_{(\sqrt{v}-\sqrt{\frac{n}{v}})^2}^{+\infty} e^{-uz} \left(1 + \frac{z+2\sqrt{n}}{\sqrt{z^2+4z\sqrt{n}}}\right) dz \\
&\leqslant \frac{e^{u(v-2\sqrt{n})}}{2} \int_{(\sqrt{v}-\sqrt{\frac{n}{v}})^2}^{+\infty} e^{-uz} \left(1 + \sqrt{1+\frac{4\sqrt{n}}{z}}\right) dz \\
&\leqslant e^{u(v-2\sqrt{n})} \int_{(\sqrt{v}-\sqrt{\frac{n}{v}})^2}^{+\infty} e^{-uz} \left(1 + \frac{\sqrt[4]{n}}{\sqrt{z}}\right) dz \\
&= \frac{e^{-\frac{un}{v}}}{u} + \sqrt[4]{n} e^{u(v-2\sqrt{n})} \int_{(\sqrt{v}-\sqrt{\frac{n}{v}})^2}^{+\infty} e^{-uz} \frac{dz}{\sqrt{z}} \ . \qquad (65)
\end{aligned}$$

Using now the substitution $t = \sqrt{uz}$, we finally obtain, if $v \geqslant \sqrt{n}$:

$$\widetilde{\mathcal{I}}_n(u,v) \leqslant \frac{e^{-\frac{un}{v}}}{u} + \frac{2\sqrt[4]{n}}{\sqrt{u}} e^{u(v-2\sqrt{n})} \int_{\sqrt{u}(\sqrt{v}-\sqrt{\frac{n}{v}})}^{+\infty} e^{-t^2} dt \ . \qquad (66)$$

To conclude the proof, we just have to check whether the lower bound of the Gaussian integral is null or not.

<u>**Case 1:**</u> $0 \leqslant v \leqslant \sqrt{n}$

Using the explicit values of Gaussian integral, Equation (66) becomes:

$$\widetilde{\mathcal{I}}_n(u,v) \leqslant \widetilde{\mathcal{I}}_n(u,\sqrt{n}) \leqslant e^{-u\sqrt{n}} \left(\frac{1}{u} + \frac{\sqrt[4]{n}\cdot\sqrt{\pi}}{\sqrt{u}}\right) \ , \qquad (67)$$

for all $u > 0$.

<u>**Case 2:**</u> $v > \sqrt{n}$

Yet, an easy integration by parts shows that for all $\alpha > 0$, we have:

$$\int_{\alpha}^{+\infty} e^{-t^2}\ dt \leqslant \frac{e^{-\alpha^2}}{2\alpha} \ . \qquad (68)$$

Therefore, if $v > \sqrt{n}$, Equations (66) and (68) gives:

$$\widetilde{\mathcal{I}}_n(u,v) \leqslant \frac{e^{-\frac{un}{v}}}{u} \left(1 + \frac{\sqrt[4]{n}\cdot\sqrt{v}}{v-\sqrt{n}}\right) \ . \qquad (69)$$

If $v \in \left]\sqrt{n}; \sqrt{n+\pi}\right]$, we deduce from Equation (69) that for all $u > 0$:

$$\begin{aligned}\widetilde{I}_n(u,v) &\leqslant \widetilde{I}_n(u,\sqrt{n+\pi}) \leqslant \frac{e^{-\frac{un}{\sqrt{n+\pi}}}}{u}\left(1+\frac{\sqrt[4]{n}\cdot\sqrt[4]{n+\pi}}{\sqrt{n+\pi}-\sqrt{n}}\right) \\ &\leqslant \frac{e^{-u\frac{\sqrt{n}}{\sqrt{n+\pi}}}}{u}\left(1+\frac{2}{\pi}(n+\pi)\right) \ . \end{aligned} \tag{70}$$

If $v > \sqrt{n+\pi}$, we deduce now from Equation (69) that for all $u > 0$:

$$\begin{aligned}\widetilde{I}_n(u,v) &\leqslant \frac{e^{-\frac{un}{v}}}{u}\left(1+\frac{\sqrt[4]{n}}{\sqrt{v}-\sqrt{\frac{n}{v}}}\right) \\ &\leqslant \frac{e^{-\frac{un}{v}}}{u}\left(1+\frac{\sqrt[4]{n}}{\sqrt{n+\pi}-\frac{\sqrt{n}}{\sqrt{n+\pi}}}\right) \\ &\leqslant \frac{e^{-\frac{un}{v}}}{u}\left(1+\frac{\sqrt[4]{n}}{\sqrt{n+\pi}-1}\right) \leqslant \frac{2e^{-\frac{un}{v}}}{u} \ . \end{aligned} \tag{71}$$

□

From that result, we can deduce an elementary estimation of $\mathcal{U}_n(z)$, $z \in \mathbb{C} - \mathbb{R}^+$. It will be useful in Section 5 to estimate explicitly and simply some terms in error's expression of Examples 2 and 6.

We can in particular emphasize that this elementary upper bound gives us an exponentially decreassing character in the variable s. Nevertheless, it is not an optimal one, according to Proposition 4.1 which will be proved in the following Subsections.

Proposition 4.3. *For all $z \in \mathbb{C} - \mathbb{R}^+$ and all positive integer n, let us denote $s = \Re e\ (\sqrt{-z})$. Therefore, we have:*

$$\left|\int_0^{+\infty} \frac{x^n}{(x-z)^{n+1}} e^{-x}\ dx\right| \leqslant \begin{cases} \dfrac{e^{-s\sqrt{n}}}{s\sqrt{s}}\left(\dfrac{1}{\sqrt{s}}+\sqrt[4]{n}\sqrt{\pi}\right) & ,\ \textit{if } 0 \leqslant s \leqslant \sqrt{n}\ . \\ \dfrac{e^{-s\frac{\sqrt{n}}{\sqrt{n+\pi}}}}{s^2}\left(3+\dfrac{2n}{\pi}\right) & ,\ \textit{if } s \in]\sqrt{n};\sqrt{n+\pi}]\ . \\ \dfrac{2e^{-n}}{s^2} & ,\ \textit{if } s > \sqrt{n+\pi}\ . \end{cases} \tag{72}$$

Proof. Equations (58) and (60) as well as Inequation (61) can be used together and rewritten as:

$$|\mathcal{U}_n(z)| \leqslant \mathcal{I}_n(s) \leqslant \frac{1}{s}\cdot\widetilde{\mathcal{I}}_n(s,s)\ . \tag{73}$$

According to

$$z \in \mathbb{C} - \mathbb{R}^+ \Longrightarrow Arg(\sqrt{-z}) \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\Longrightarrow s > 0 \ , \tag{74}$$

Lemma 4.2 now concludes the proof. □

4.3 Notations and Laplace's method implementation

The integrand function inside the integral of Equation (59) has a paek near $\sqrt{n}$ and is nearly null outside of it. Consequently, only a small interval around $\sqrt{n}$ contributes to the whole integral $\mathcal{I}_n(s)$.

Let us fix $\theta \in \left]\frac{1}{4}; \frac{1}{3}\right[$, so that our small interval around $\sqrt{n}$ will be $\left[\sqrt{n} - n^\theta; \sqrt{n} + n^\theta\right]$. Then, let us now decompose $\mathcal{I}_n(s)$ into three parts:

$$\mathcal{I}_n(s) = \mathcal{J}_n(s) + \mathcal{K}_n(s) + \mathcal{L}_n(s) \ , \tag{75}$$

where:

$$\mathcal{J}_n(s) = \int_0^{\sqrt{n}-n^\theta} \frac{t^n}{(t+s)^{n+1}} e^{-st} \ dt \ , \tag{76}$$

$$\mathcal{K}_n(s) = \int_{\sqrt{n}-n^\theta}^{\sqrt{n}+n^\theta} \frac{t^n}{(t+s)^{n+1}} e^{-st} \ dt \ , \tag{77}$$

$$\mathcal{L}_n(s) = \int_{\sqrt{n}+n^\theta}^{+\infty} \frac{t^n}{(t+s)^{n+1}} e^{-st} \ dt \ . \tag{78}$$

Let us also define the function f_n by

$$f_n(\zeta) = n \log(\zeta + \sqrt{n}) - (n+1) \log(\zeta + s + \sqrt{n}) - s \cdot \zeta \tag{79}$$

for all $\zeta \in \mathbb{C}$ such that $\Re e \ \zeta > 0$.

Finally, let us denote by ζ_n the saddle-node points of f_n (*i.e.* a point ζ where $f_n'(\zeta) = 0$) defined by:

$$\zeta_n = \frac{\sqrt{(1+s^2)^2 + 4s^2 n} - (1 + 2s\sqrt{n} + s^2)}{2s} \ . \tag{80}$$

4.4 Tail pruning.

Using similar substitutions to those used in Lemma 4.2, the main goal of this Subsection is to show the following:

Lemma 4.4. *For all $s \in \mathbb{C}$ such that $\Re e \ s > 0$, we have:*

$$\mathcal{J}_n(s) \underset{n \longrightarrow +\infty}{=} o\left(n^{-\frac{1}{4}} e^{-2s\sqrt{n}}\right) \ \textit{and} \ \mathcal{L}_n(s) \underset{n \longrightarrow +\infty}{=} o\left(n^{-\frac{1}{4}} e^{-2s\sqrt{n}}\right) \ . \tag{81}$$

Proof. • Firstly, let us focus on the case of the integral $\mathcal{J}_n(s)$. Since n goes to $+\infty$, we can assume that $n^\theta \geqslant \Re e\ (s)$.

We successively have:

$$\begin{aligned}
|\mathcal{J}_n(s)| &\leqslant \int_0^{\sqrt{n}-n^\theta} \frac{t^n}{(t+\Re e\ s)^{n+1}} e^{-t\Re e\ s}\ dt \\
&\leqslant \int_0^{\sqrt{n}-n^\theta} \left(1-\frac{\Re e\ s}{t+\Re e\ s}\right)^n \frac{e^{-t\Re e\ s}}{t+\Re e\ s}\ dt \\
&\leqslant \frac{1}{\Re e\ s}\int_0^{\sqrt{n}-n^\theta} \exp\left(-\Re e\ s\cdot\left(t+\frac{n}{t+\Re e\ s}\right)\right)\ dt\ .
\end{aligned} \tag{82}$$

The functions $\varphi : [2\sqrt{n}-\Re e\ (s); +\infty[\longrightarrow]-\Re e\ (s); \sqrt{n}-\Re e\ (s)]$ and $\psi :]-\Re e\ (s); \sqrt{n}-\Re e\ (s)] \longrightarrow [2\sqrt{n}-\Re e\ (s); +\infty[$ defined by

$$\varphi(u) = \frac{1}{2}\left(u-\Re e\ (s)-\sqrt{\big(u+\Re e\ (s)\big)^2-4n}\right) \tag{83}$$

$$\psi(t) = t+\frac{n}{t+\Re e\ (s)} \tag{84}$$

are inverse functions.

Therefore, using the substitution $t=\varphi(u)$, we have:

$$\begin{aligned}
&\int_0^{\sqrt{n}-n^\theta} \exp\left(-\Re e\ s\cdot\left(t+\frac{n}{t+\Re e\ s}\right)\right)\ dt \\
&\qquad = \int_{\psi(0)}^{\psi(\sqrt{n}-n^\theta)} e^{-u\Re e\ s}\varphi'(u)\ du \\
&\qquad = \frac{1}{2}\int_{\frac{n}{\Re e\ (s)}}^{\sqrt{n}-n^\theta+\frac{n}{\sqrt{n}-n^\theta+\Re e\ (s)}} e^{-u\Re e\ s}\left(1-\frac{u+\Re e\ (s)}{\sqrt{\big(u+\Re e\ (s)\big)^2-4n}}\right) du
\end{aligned} \tag{85}$$

According to the decreassing caracter of ψ on $]-\Re e\ (s); \sqrt{n}-\Re e\ (s)]$, we deduce that $\dfrac{n}{\Re e\ (s)} \geqslant \sqrt{n} - n^{\theta} + \frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}$. Therefore, we successively have:

$$\begin{aligned}
|\mathcal{J}_n(s)| &\leqslant \frac{1}{2\Re e\ (s)} \int_{\sqrt{n}-n^{\theta}+\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}}^{\frac{n}{\Re e\ (s)}} e^{-u\Re e\ (s)} \frac{u+\Re e\ (s)}{\sqrt{\big(u+\Re e\ (s)\big)^2-4n}}\, du \\
&\leqslant \frac{\exp\left(-\Re e\ (s)\cdot\big(\sqrt{n}-n^{\theta}+\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}\big)\right)}{2\Re e\ (s)} \times \\
&\qquad\qquad \left[\sqrt{\big(u+\Re e\ (s)\big)^2-4n}\right]_{\sqrt{n}-n^{\theta}+\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}}^{\frac{n}{\Re e\ (s)}} \\
&\leqslant \frac{\exp\left(-\Re e\ (s)\cdot\big(\sqrt{n}-n^{\theta}+\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}\big)\right)}{2\Re e\ (s)} \left|\frac{n}{\Re e\ (s)}-\Re e\ (s)\right| \\
&\leqslant \frac{n}{2\Re e\ (s)^2}\cdot\exp\left(-\Re e\ (s)\cdot\big(\sqrt{n}-n^{\theta}+\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ (s)}\big)\right)\ , \qquad (86)\\
&\qquad\qquad \text{according to } \Re e\ (s)\leqslant n^{\theta}\leqslant\sqrt{n}\ .
\end{aligned}$$

Consequently, we finally have:

$$\begin{aligned}
\left|n^{\frac{1}{4}}e^{2s\sqrt{n}}\mathcal{J}_n(s)\right| &\leqslant \frac{n^{\frac{5}{4}}}{(\Re e\ s)^2}\exp\left(\Re e\ s\Big(\sqrt{n}+n^{\theta}-\frac{n}{\sqrt{n}-n^{\theta}+\Re e\ s}\Big)\right) \\
&\leqslant \frac{n^{\frac{5}{4}}}{(\Re e\ s)^2}\exp\left(-\Re e\ s\ n^{2(\theta-\frac{1}{4})}\frac{1-\frac{\Re e\ s}{n^{2(\theta-\frac{1}{4})}}-\frac{\Re e\ s}{n^{\theta}}}{1-\frac{1}{n^{\frac{1}{2}-\theta}}+\frac{\Re e\ s}{\sqrt{n}}}\right) \qquad (87)
\end{aligned}$$

which show that $\mathcal{J}_n(s) \underset{n\longrightarrow+\infty}{=} o\left(n^{-\frac{1}{4}}e^{-2s\sqrt{n}}\right)$, according to $\theta > \dfrac{1}{4}$.

• Let us now focus on the integral $\mathcal{L}_n(s)$, using one more time a similar substitution to this used in Lemma 4.2.

We successively have:

$$\begin{aligned}
|\mathcal{L}_n(s)| &\leqslant \int_{\sqrt{n}+n^{\theta}}^{+\infty} \frac{t^n}{(t+\Re e\ s)^{n+1}} e^{-t\Re e\ s}\, dt \\
&\leqslant \int_{\sqrt{n}+n^{\theta}}^{+\infty} \left(1-\frac{\Re e\ s}{t+\Re e\ (s)}\right)^n \frac{e^{-t\Re e\ s}}{t+\Re e\ s}\, dt \\
&\leqslant \frac{1}{\sqrt{n}+n^{\theta}+\Re e\ (s)} \int_{\sqrt{n}+n^{\theta}}^{+\infty} \exp\left(-\Re e\ (s)\Big(t+\frac{n}{t+\Re e\ (s)}\Big)\right)\, dt \qquad (88)
\end{aligned}$$

Denoting $N(n,s,\theta)=\sqrt{n}+n^{\theta}+\Re e\ (s)$ and using successively the substitutions $t=\frac{1}{2}\left(u-\Re e\ (s)+\sqrt{\left((u+\Re e\ (s)\right)^2-4n}\right)$, $v=u+s-2\sqrt{n}$ and then $w=v\sqrt{s}$, we have:

$$
\begin{aligned}
|\mathcal{L}_n(s)| &\leqslant \frac{1}{2N(n,s,\theta)}\int_{N(n,s,\theta)-s+\frac{n}{N(n,s,\theta)}}^{+\infty} e^{-u\Re e\ (s)}\left(1+\frac{u+s}{\sqrt{\left(u+\Re e\ (s)\right)^2-4n}}\right)\ du\ .\\
&= \frac{e^{\Re e\ (s)\cdot\left(\Re e\ (s)-2\sqrt{n}\right)}}{2N(n,s,\theta)}\int_{\left(\sqrt{N(n,s,\theta)}-\sqrt{\frac{n}{N(n,s,\theta)}}\right)^2}^{+\infty} e^{-v\Re e\ (s)}\left(1+\frac{v+2\sqrt{n}}{\sqrt{v^2+4v\sqrt{n}}}\right)\ dv\\
&\leqslant \frac{e^{\Re e\ (s)\cdot\left(\Re e\ (s)-2\sqrt{n}\right)}}{N(n,s,\theta)}\int_{\left(\sqrt{N(n,s,\theta)}-\sqrt{\frac{n}{N(n,s,\theta)}}\right)^2}^{+\infty} e^{-v\Re e\ (s)}\left(1+\frac{\sqrt[4]{n}}{\sqrt{v}}\right)\ dv\\
&= \frac{e^{\Re e\ (s)\cdot\left(\Re e\ (s)-2\sqrt{n}\right)}}{N(n,s,\theta)}\left(\frac{1}{\Re e\ (s)}\exp\left(-\Re e\ (s)\left(\sqrt{N(n,s,\theta)}-\sqrt{\frac{n}{N(n,s,\theta)}}\right)^2\right)+\right.\\
&\qquad\left.\frac{2\sqrt[4]{n}}{\sqrt{\Re e\ (s)}}\int_{\sqrt{\Re e\ (s)}\left(\sqrt{N(n,s,\theta)}-\sqrt{\frac{n}{N(n,s,\theta)}}\right)}^{+\infty} e^{-w^2}\ dw\right),\\
&\qquad\qquad\text{according to } N(n,s,\theta)\geqslant n\\
&\leqslant \frac{e^{\Re e\ (s)\cdot\left(\Re e\ (s)-2\sqrt{n}\right)}}{N(n,s,\theta)}\cdot\exp\left(-\Re e\ (s)\left(\sqrt{N(n,s,\theta)}-\sqrt{\frac{n}{N(n,s,\theta)}}\right)^2\right)\\
&\quad\times\frac{1}{\Re e\ (s)}\cdot\left(1+\sqrt[4]{n}\frac{\sqrt{N(n,s,\theta)}}{N(n,s,\theta)-\sqrt{n}}\right)\\
&\leqslant \frac{1}{\Re e\ (s)N(n,s,\theta)}\cdot\exp\left(-\Re e\ (s)\left(\sqrt{n}+n^{\theta}+\frac{n}{N(n,s,\theta)}\right)\right)\cdot\left(1+\frac{N(n,s,\theta)}{N(n,s,\theta)-\sqrt{n}}\right)\\
&\leqslant \frac{1}{\Re e\ (s)}\cdot\exp\left(-\Re e\ (s)\left(\sqrt{n}+n^{\theta}+\frac{n}{N(n,s,\theta)}\right)\right)\cdot\left(\frac{1}{\sqrt{n}}+\frac{1}{n^{\theta}}\right)\ . \qquad (89)
\end{aligned}
$$

We finally have:

$$
\begin{aligned}
\left|n^{\frac{1}{4}}e^{2s\sqrt{n}}\mathcal{L}_n(s)\right| &\leqslant \frac{1}{\Re e\ (s)}\cdot\exp\left(-\Re e\ (s)\left(n^{\theta}-\sqrt{n}+\frac{n}{N(n,s,\theta)}\right)\right)\cdot\left(\frac{1}{\sqrt[4]{n}}+\frac{1}{n^{\theta-\frac{1}{4}}}\right)\\
&\leqslant \frac{2}{\Re e\ (s)}\cdot\exp\left(-\Re e\ (s)\frac{n^{2\theta}-\Re e\ (s)(\sqrt{n}-n^{\theta})}{\sqrt{n}+n^{\theta}+\Re e\ (s)}\right)\ . \qquad (90)
\end{aligned}
$$

This concludes the proof of $\mathcal{L}_n(s)\underset{n\longrightarrow+\infty}{=}o\left(n^{-\frac{1}{4}}e^{-2s\sqrt{n}}\right)$, according to $\theta>\frac{1}{4}$. □

4.5 On the integral $\mathcal{K}_n(s)$.

The main goal of this Subsection is to show that $\mathcal{K}_n(s) \underset{n \longrightarrow +\infty}{\sim} \frac{\sqrt{\pi} e^{-\frac{z}{2}}}{(-nz)^{\frac{1}{4}}} e^{-2s\sqrt{n}}$.
Using the substitution $u = t - \sqrt{n}$, we have:

$$\begin{aligned} \mathcal{K}_n(s) &= e^{-s\sqrt{n}} \int_{-n^\theta}^{+n^\theta} \frac{(u+\sqrt{n})^n}{(u+s+\sqrt{n})^{n+1}} e^{-su} \, du \\ &= e^{-s\sqrt{n}} \int_{-n^\theta}^{+n^\theta} e^{f_n(u)} \, du \ , \end{aligned} \tag{91}$$

where f_n is defined by Equation (79).

We will prove the wanted equivalent using two steps, first a uniform central approximation, and then showing that $\mathcal{K}_n(s)$ is essentially a Gauss integral.

Uniform central approximation. In this Paragraph, we will show the following

Lemma 4.5. *Let θ be a positive real number.*
There exists a positive constant $C(s)$ and a positive integer N_0 such that for all integers $n \geqslant N_0$, we have:

$$\sup_{t \in [-n^\theta; n^\theta]} \left| f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n) \right| \leqslant \frac{C(s)}{n^{1-3\theta}} \ . \tag{92}$$

Proof. • The sequence $(\zeta_n)_{n \in \mathbb{N}}$ of saddle nodes is a converging sequence to $\zeta_\infty = -\frac{1+s^2}{2s}$. Consequently, there exist two positive integers N_1 and N_2 such that:

$$\begin{cases} n \geqslant N_1 \Longrightarrow |\zeta_n - \zeta_\infty| \leqslant 1 \ . \\ n \geqslant N_2 \Longrightarrow |\zeta_\infty| \leqslant n^\theta \ . \end{cases} \tag{93}$$

Consequently, if $n \geqslant \max(N_1, N_2)$, $t \in [-n^\theta; n^\theta]$ and $u \in [t, \zeta_n]$, we successively have:

$$
\begin{aligned}
&\Re e\ u \\
&= \Re e\ \big(\kappa t + (1-\kappa)\zeta_n\big)\ ,\text{ for a } \kappa \in [0;1] \\
&= \kappa t + (1-\kappa)\Re e\ (\zeta_n - \zeta_\infty) + (1-\kappa)\Re e\ (\zeta_\infty) \\
&\geqslant \kappa t - (1-\kappa)|\zeta_n - \zeta_\infty| - (1-\kappa)|\zeta_\infty| \\
&\geqslant \kappa t - (1-\kappa)(1+n^\theta) \\
&\geqslant -\big(\kappa n^\theta + (1-\kappa)(1+n^\theta)\big) \\
&\geqslant -\big(1-\kappa+n^\theta\big) \geqslant -(1+n^\theta)\ . \qquad (94)
\end{aligned}
$$

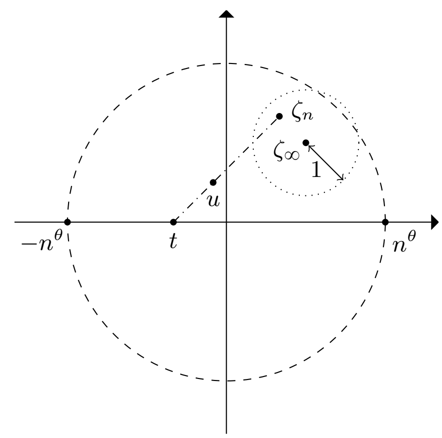


Figure 2: Illustration of the used variables in the proof

Let us remark that $x \longmapsto x^3 - x^2 - 1$ is an increasing function on $\left[\frac{2}{3};+\infty\right[$ with a unique zero located in $x_0 \approx 1.46557$. From $10 \geqslant x_0^6$, we therefore have for all $n \geqslant \max(10, N_1, N_2)$:

$$\sqrt{n} - n^\theta - 1 > \sqrt{n} - \sqrt[3]{n} - 1 \geqslant \sqrt{10} - \sqrt[3]{10} - 1 > 0\ . \qquad (95)$$

• Let us now fix an integer $n \geqslant \max(10, N_1, N_2)$. Then, the function f_n is $\mathcal{C}^\infty$ over $\mathbb{C}-]-\infty;-\sqrt{n}]$ and we have for all $u \in \mathbb{C}-]-\infty;-\sqrt{n}]$:

$$
\begin{aligned}
f_n'''(u) &= \frac{2n}{(u+\sqrt{n})^3} - \frac{2(n+1)}{(u+s+\sqrt{n})^3} \\
&= 2n \cdot \frac{3s(u+\sqrt{n})^2 + 3s^2(u+\sqrt{n}) + s^3}{(u+\sqrt{n})^3(u+s+\sqrt{n})^3} - \frac{2}{(u+s+\sqrt{n})^3}\ . \qquad (96)
\end{aligned}
$$

So, for all $t \in [-n^\theta; n^\theta]$ and $u \in [t, \zeta_n]$, we successively have:

$$
\begin{aligned}
|f_n'''(u)| &\leqslant \left(\frac{3|s|}{|\Re e\ u + \sqrt{n}|} + \frac{3|s|^2}{|\Re e\ u + \sqrt{n}|^2} + \frac{|s|^3}{|\Re e\ u + \sqrt{n}|^3}\right) \cdot \frac{2n}{|\Re e\ (u) + \Re e\ (s) + \sqrt{n}|^3} \\
&\quad + \frac{2}{|\Re e\ (u) + \Re e\ (s) + \sqrt{n}|^3} \\
&\leqslant \left(\frac{3|s|}{\sqrt{n} - n^\theta - 1} + \frac{3|s|^2}{(\sqrt{n} - n^\theta - 1)^2} + \frac{|s|^3}{(\sqrt{n} - n^\theta - 1)^3}\right) \cdot \frac{2n}{(\sqrt{n} - n^\theta - 1)^3} \\
&\quad + \frac{2}{(\sqrt{n} - n^\theta - 1)^3}\ ,\text{ according to Equation (94)} \\
&\leqslant \left(3|s| + \frac{3|s|^2}{\sqrt{10} - \sqrt[3]{10} - 1} + \frac{|s|^3}{(\sqrt{10} - \sqrt[3]{10} - 1)^2}\right) \cdot \frac{2n}{(\sqrt{n} - \sqrt[3]{n} - 1)^4} \\
&\quad + \frac{2}{(\sqrt{n} - \sqrt[3]{n} - 1)^3}\ ,\text{ according to Equation (95) and } \Re e\ (s) > 0\ . \qquad (97)
\end{aligned}
$$

Moreover, there exists two integers N_3 and N_4 such that:

$$\left\{\begin{array}{lcl} n \geqslant N_3 & \Longrightarrow & \dfrac{2n}{(\sqrt{n}-\sqrt[3]{n}-1)^4} \leqslant \dfrac{3}{n} \ . \\ n \geqslant N_4 & \Longrightarrow & \dfrac{2}{(\sqrt{n}-\sqrt[3]{n}-1)^3} \leqslant \dfrac{3}{n\sqrt{n}} \leqslant \dfrac{3}{n} \ . \end{array}\right. \tag{98}$$

Consequently, for all $n \geqslant \max(10, N_1, N_2, N_3, N_4)$, $t \in \big[-n^\theta; n^\theta\big]$ and $u \in [t, \zeta_n]$, we have:

$$|f_n'''(u)| \ \leqslant \ \frac{3}{n}\left(3|s| + \frac{3|s|^2}{\sqrt{10}-\sqrt[3]{10}-1} + \frac{|s|^3}{(\sqrt{10}-\sqrt[3]{10}-1)^2} + 1\right) \ . \tag{99}$$

Defining $N_0 = \max(10, N_1, N_2, N_3, N_4)$ and

$$c(s) = 3|s| + \frac{3|s|^2}{\sqrt{10}-10^{\frac{1}{3}}-1} + \frac{|s|^3}{(\sqrt{10}-10^{\frac{1}{3}}-1)^2} + 1 \ , \tag{100}$$

we can conclude that if $n \in \mathbb{N}$ satisfies $n \geqslant N_0$, we have:

$$\forall t \in [-n^\theta; n^\theta] \ , \ \forall u \in [t, \zeta_n] \ , \ |f_n'''(u)| \leqslant \frac{c(s)}{n} \ . \tag{101}$$

• Finally, let us fix an integer $n \geqslant N_0$ and a real number $t \in [-n^\theta, n^\theta]$.

We remind that $f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n)$ can be expressed as an iterated integral, according to $f_n'(\zeta_n) = 0$:

$$\begin{aligned} & f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n) \\ = \ & \int_t^{\zeta_n}\left(\int_{u_2}^{\zeta_n}\Big(\int_{u_1}^{\zeta_n} f_n'''(u_0)\ du_0\Big)\ du_1\right)\ du_2 \\ = \ & (\zeta_n - t)^3 \times \\ & \int_0^1\left(\int_0^1\Big[\int_0^1 f_n'''\Big(t + \big[t_2 + \big(t_1 + t_0(1-t_1)\big)(1-t_2)\big](\zeta_n - t)\Big)\ dt_0\Big]\right. \\ & \left. \cdot(1-t_1)\ dt_1\right)(1-t_2)^2\ dt_2 \end{aligned} \tag{102}$$

For all real numbers t_0, t_1 and t_2 between 0 and 1, it is clear that we have:

$$0 \leqslant t_2 + \big(t_1 + t_0(1-t_1)\big)(1-t_2) \leqslant 1 \ . \tag{103}$$

Therefore, according to Equations (101) and (102), we deduce that:

$$\left| f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n)\right| \leqslant \frac{c(s)}{6n}|\zeta_n - t|^3 \ . \tag{104}$$

To conclude, it is enough to remark that:

$$|t-\zeta_n| \leqslant |t| + |\zeta_n - \zeta_\infty| + |\zeta_\infty| \leqslant n^\theta + 1 + n^\theta \leqslant 3n^\theta \ , \tag{105}$$

i.e.

$$\left| f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n) \right| \leqslant \frac{9\,c(s)}{2n^{1-3\theta}} \ . \tag{106}$$

Therefore, the constant $C(s)$ can be defined by:

$$C(s) = \frac{9c(s)}{2} \ . \tag{107}$$

□

Tail completion. In this Paragraph, we are shoving the following

Lemma 4.6. *For all* $\theta > \frac{1}{4}$, $\mathcal{M}_n(s) = \int_{-n^\theta}^{+n^\theta} \exp\left(\frac{1}{2}(\chi-\zeta_n)^2 f_n''(\zeta_n)\right) \, d\chi$ *satisfies*

$$\mathcal{M}_n(s) \underset{n\longrightarrow+\infty}{\sim} n^{\frac{1}{4}} \sqrt{\frac{\pi}{s}} \ . \tag{108}$$

Proof. • According to Equation 7.4.32 of [1], for all complex numbers a, b and c such that $a \neq 0$, we know that:

$$\int e^{-(ax^2+2bx+c)} \ dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-ac}{a}\right) \operatorname{erf}\left(x\sqrt{a} + \frac{b}{\sqrt{a}}\right) + Cst \ , \tag{109}$$

where the function erf is defined for all complex numbers z by:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \ dt \ . \tag{110}$$

Therefore, we have:

$$\int_{-R}^{R} e^{-a(x-b)^2} \ dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \Big(\operatorname{erf}\Big(\sqrt{a}(R+b)\Big) + \operatorname{erf}\Big(\sqrt{a}(-R+b)\Big) \Big) \ . \tag{111}$$

Consequently, we deduce an explicit expression of $\mathcal{M}_n(s)$ from Equation (111) with $R = n^\theta$, $a = -\frac{1}{2} f_n''(\zeta_n)$ and $b = \zeta_n$. From now on, we will denote R_n, a_n and b_n instead of R, a and b.

• According to Equation 7.1.16 of [1], we know that $\lim\limits_{\substack{z\longrightarrow+\infty \\ |\arg(z)|<\frac{\pi}{4}}} \operatorname{erf}(z) = 1$. We will use that result to conclude the proof.

First, we remind that the sequence $(\zeta_n)_{n\in\mathbb{N}}$ of saddle nodes is a converging sequence to $\zeta_\infty = -\frac{1+s^2}{2s}$. Moreover, we also have:

$$f_n''(\zeta_n) = \frac{n+1}{(\zeta_n + \sqrt{n} + s)^2} - \frac{n}{(\zeta_n + \sqrt{n})^2} \underset{n\longrightarrow+\infty}{\sim} -\frac{2s}{\sqrt{n}} \ , \tag{112}$$

i.e. $a_n \underset{n\longrightarrow+\infty}{\sim} \frac{s}{\sqrt{n}}$.

Thus, $\sqrt{a_n}(R_n+b_n) \underset{n\longrightarrow+\infty}{\sim} n^{\theta-\frac{1}{4}}\sqrt{s}$ and $\sqrt{a_n}(-R_n+b_n) \underset{n\longrightarrow+\infty}{\sim} -n^{\theta-\frac{1}{4}}\sqrt{s}$. From $z \in \mathbb{C}-\mathbb{R}^+$, we know that $\arg(\sqrt{s}) \in \left]-\frac{\pi}{4};\frac{\pi}{4}\right[$.

Consequently, we have, since $\theta > \frac{\pi}{4}$:

$$\operatorname{erf}\Big(\sqrt{a_n}(R_n+b_n)\Big) + \operatorname{erf}\Big(\sqrt{a_n}(-R_n+b_n)\Big) \underset{n\longrightarrow+\infty}{\longrightarrow} 2\ , \tag{113}$$

which finally gives us

$$\mathcal{M}_n(s) \underset{n\longrightarrow+\infty}{\sim} \sqrt{\frac{\pi}{a_n}} \underset{n\longrightarrow+\infty}{\sim} n^{\frac{1}{4}}\sqrt{\frac{\pi}{s}}\ . \tag{114}$$

□

4.6 End of the proof of Proposition 4.1.

Let us come back to the integral $\mathcal{K}_n(s)$. The following lemma will be useful:

Lemma 4.7. *Let $(s_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ be two converging sequences of complex numbers. Let also $\theta \leqslant \frac{1}{2}$ be a real number.*
Then, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 such that for all $t \in [-n^\theta; n^\theta]$, we have:

$$\Re e\ \left(-\frac{s_n}{\sqrt{n}}(t-\zeta_n)^2\right) \leqslant \alpha_n - \frac{t^2}{\sqrt{n}}\Re e\ (s_n) \tag{115}$$

Proof. We have for all $t \in [-n^\theta; n^\theta]$:

$$\begin{aligned}\Re e\ \left(-\frac{s_n}{\sqrt{n}}(t-\zeta_n)^2\right) &= -\frac{t^2}{\sqrt{n}}\Re e\ (s_n) + 2t\frac{\Re e\ (s_n\zeta_n)}{\sqrt{n}} - \frac{\Re e\ (s_n\zeta_n^2)}{\sqrt{n}}\\ &\leqslant -\frac{t^2}{\sqrt{n}}\Re e\ (s_n) + 2|t|\frac{|\Re e\ (s_n\zeta_n)|}{\sqrt{n}} + \frac{|\Re e\ (s_n\zeta_n^2)|}{\sqrt{n}}\\ &\leqslant -\frac{t^2}{\sqrt{n}}\Re e\ (s_n) + 2|\Re e\ (s_n\zeta_n)|\frac{n^\theta}{\sqrt{n}} + \frac{|\Re e\ (s_n\zeta_n^2)|}{\sqrt{n}}\ .\end{aligned}$$

Defining α_n by $\alpha_n = 2|\Re e\ (s_n\zeta_n)|\frac{n^\theta}{\sqrt{n}} + \frac{|\Re e\ (s_n\zeta_n^2)|}{\sqrt{n}}$ proves the lemma. □

Let us denote by $\widetilde{f_n}$ the function defined by

$$\widetilde{f_n}(t) = f_n(t) - f_n(\zeta_n) - \frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n) \tag{116}$$

for all $t \in [-n^\theta, n^\theta]$, as well as $s_n = -\dfrac{\sqrt{n}}{2} f_n''(\zeta_n)$. Therefore, we have:

$$e^{f_n(t)-f_n(\zeta_n)} - e^{\frac{1}{2}(t-\zeta_n)^2 f_n''(\zeta_n)} = \left(e^{\widetilde{f_n}(t)} - 1\right) \cdot \exp\left(-\frac{s_n}{\sqrt{n}}(t-\zeta_n)^2\right) \ . \quad (117)$$

$$e^{s\sqrt{n}-f_n(\zeta_n)}\mathcal{K}_n(s) - \mathcal{M}_n(s) = \int_{-n^\theta}^{+n^\theta} \left(e^{\widetilde{f_n}(t)} - 1\right) \cdot \exp\left(-\frac{s_n}{\sqrt{n}}(t-\zeta_n)^2\right) \ . \quad (118)$$

According to Lemmas 4.5 and 4.7, we therefore have for all $n \geqslant N_0$:

$$\begin{aligned}
&\left|e^{s\sqrt{n}-f_n(\zeta_n)}\mathcal{K}_n(s) - \mathcal{M}_n(s)\right| \\
&\qquad \leqslant \left(\exp\left(\frac{C(s)}{n^{1-3\theta}}\right) - 1\right) \int_{-n^\theta}^{+n^\theta} \exp\left(\Re e\ \left(-\frac{s_n}{\sqrt{n}}(t-\zeta_n)^2\right)\right) \\
&\qquad \leqslant \left(\exp\left(\frac{C(s)}{n^{1-3\theta}}\right) - 1\right) \int_{-n^\theta}^{+n^\theta} \exp\left(\alpha_n - \frac{t^2}{\sqrt{n}}\Re e\ (s_n)\right) \\
&\qquad \leqslant \left(\exp\left(\frac{C(s)}{n^{1-3\theta}}\right) - 1\right) \frac{e^{\alpha_n} n^{\frac{1}{4}}}{\sqrt{\Re e\ (s_n)}} \sqrt{\pi} \ . \qquad (119)
\end{aligned}$$

From $\theta < \dfrac{1}{3}$, we consequently have:

$$e^{s\sqrt{n}-f_n(\zeta_n)}\mathcal{K}_n(s) - \mathcal{M}_n(s) \underset{n\longrightarrow+\infty}{=} o(\sqrt[4]{n}) \ , \quad (120)$$

i.e. according to the tail completion part of the proof:

$$\mathcal{K}_n(s) \underset{n\longrightarrow+\infty}{\sim} e^{-s\sqrt{n}+f_n(\zeta_n)} n^{\frac{1}{4}} \sqrt{\frac{\pi}{s}} \ . \quad (121)$$

From $f_n(\zeta_n) \underset{n\longrightarrow+\infty}{=} -\dfrac{\ln n}{2} - s\sqrt{n} + \dfrac{s^2}{2} + o(1)$, we finally deduce that

$$\mathcal{K}_n(s) \underset{n\longrightarrow+\infty}{\sim} e^{-2s\sqrt{n}} \frac{e^{\frac{s^2}{2}}\sqrt{\pi}}{n^{\frac{1}{4}}\sqrt{s}} \ , \quad (122)$$

which implies that

$$\mathcal{I}_n(s) = \mathcal{J}_n(s) + \mathcal{K}_n(s) + \mathcal{L}_n(s) \underset{n\longrightarrow+\infty}{\sim} e^{-2s\sqrt{n}} \frac{e^{\frac{s^2}{2}}\sqrt{\pi}}{n^{\frac{1}{4}}\sqrt{s}} \ . \quad (123)$$

With $s = \sqrt{-z}$, we therefore have proven Proposition 4.1, *i.e.* for all $z \in \mathbb{C} - \mathbb{R}^+$, we have:

$$\mathcal{U}_n(z) = \int_0^{+\infty} \frac{t^n}{(t-z)^{n+1}} e^{-t}\ dt = \mathcal{I}_n(s) \underset{n\longrightarrow+\infty}{\sim} \frac{\sqrt{\pi} e^{-\frac{z}{2}}}{(-nz)^{\frac{1}{4}}} e^{-2\sqrt{-nz}} \ . \quad (124)$$

5 Error estimations

It is well-known that Perron's formula gives the asymptotic behaviour of Laguerre polynomials when $n \longrightarrow +\infty$ in the cut complex plane $\mathbb{C} - \mathbb{R}^+$ (see [15], Theorem 8.22.3):

$$L_n(z) \underset{n \longrightarrow +\infty}{\sim} \frac{e^{\frac{z}{2}}}{2\sqrt{\pi}} \frac{e^{2\sqrt{-nz}}}{(-nz)^{\frac{1}{4}}} \ . \tag{125}$$

Therefore, according to Proposition 4.1, we deduce:

Proposition 5.1. *For all $z \in \mathbb{C} - \mathbb{R}^+$, we have:*

$$\frac{1}{L_n(z)} \int_0^{+\infty} \frac{t^n \ e^{-t}}{(t-z)^{n+1}} dt \underset{n \longrightarrow +\infty}{\sim} 2\pi e^{-z} e^{-4\sqrt{-nz}} \ . \tag{126}$$

Perron's formula shows that Laguerre's polynomials are responsible for "half" of the exponential character of Equation (126).

Unfortunately, we can not deduce from this an asymptotic result on the real part of $\frac{1}{L_n(z)} \int_0^{+\infty} \frac{t^n \ e^{-t}}{(t-z)^{n+1}} dt$.

In this section, we will first find a lower bound of Laguerre's polynomials on $i\mathbb{R}$, losing the asymptotic exponential character of Equation (125). Then, in the next subsection, we will develop an explicit upper bound of $\mathcal{U}_n(z)$ for $z \in \mathbb{C} - \mathbb{R}^+$. Finally, we will review our two key examples (Examples 1 and 2) to find out upper bounds of the quadrature error involved in each example.

5.1 A lower bound of the modulus of Laguerre polynomials on $i\mathbb{R}$

The n-th Laguerre polynomials can be explicitly defined by the following sum:

$$L_n(X) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-X)^k}{k!} \ . \tag{127}$$

Therefore, we have the following explicit expression for $L_n(iX)|^2$:

Proposition 5.2. *Let n be a non-negative integer.*
Then:

$$|L_n(iX)|^2 = \sum_{d=0}^{n} \binom{n}{d} \binom{n+d}{d} \frac{X^{2d}}{(2d)!} \ . \tag{128}$$

Proof. Let us remind that the modified Bessel function $\mathcal{I}_0$ is defined for all complex number z by:

$$\mathcal{I}_0(z) = \sum_{k \geqslant 0} \frac{1}{k!^2} \left(\frac{z}{2}\right)^{2k} \ . \tag{129}$$

According to the Hardy-Hille formula (see [15], Theorem 5.1, p. 102), we successively have:

$$\begin{aligned}
\sum_{n\geqslant 0} |L_n(iX)|^2 \, Y^n &= \frac{1}{1-Y}\mathcal{I}_0\left(\frac{2X\sqrt{Y}}{1-Y}\right) = \sum_{d\geqslant 0}\frac{1}{d!^2}\frac{X^{2d}Y^d}{(1-Y)^{2d+1}} \\
&= \sum_{d\geqslant 0}\sum_{n\geqslant 0}\frac{1}{d!^2}\binom{n+2d}{n}X^{2d}\,Y^{n+d} \\
&= \sum_{d\geqslant 0}\sum_{n\geqslant d}\frac{1}{d!^2}\binom{n+d}{n-d}X^{2d}\,Y^{n} \\
&= \sum_{n\geqslant 0}\left(\sum_{d=0}^{n}\frac{1}{d!^2}\binom{n+d}{n-d}X^{2d}\right)Y^n \ , \qquad (130)
\end{aligned}$$

which directly leads to the announced expression of $|_n(iX)|^2$, $n \in \mathbb{N}$. □

As a direct consequence, we deduce:

Corollary 5.3. *Let n and d be non-negative integers.*
Let us denote by $[X^d]P$ the d-th coefficient of the polynomial P.
Then:

1. $[X^{2d+1}]|L_n(iX)|^2 = 0$.
2. $[X^{2d}]|L_n(iX)|^2 = \dfrac{1}{(2d)!}\dbinom{n}{d}\dbinom{n+d}{d} \geqslant 0$, *if* $0 \leqslant d \leqslant n$.
3. $[X^{2d}]|L_n(iX)|^2 = 0$ *if* $d > n$.

Corollary 5.4. *Let n and d be two non-negative integers such that $0 \leqslant d \leqslant n$.*
Let us define the constant $C_{N,d}$ by:

$$C_{N,d} = \sqrt{\frac{(2d)!}{\binom{N}{d}\binom{N+d}{N}}} \ . \qquad (131)$$

Then, for all real numbers t, we have:

$$|L_n(it)| \geqslant \frac{|t|^d}{C_{N,d}} \ . \qquad (132)$$

As explicit examples, we therefore have:

$$\forall n \geqslant 0 \ , \ |L_n(it)| \ \geqslant \ 1 \ . \qquad (133)$$

$$\forall n \geqslant 1 \ , \ |L_n(it)| \ \geqslant \ \sqrt{\frac{n(n+1)}{2}}|t| \geqslant \frac{n|t|}{\sqrt{2}} \ . \qquad (134)$$

$$\forall n \geqslant 2 \ , \ |L_n(it)| \ \geqslant \ \sqrt{\frac{(n-1)n(n+1)(n+2)}{96}}t^2 \geqslant \frac{(n-1)^2t^2}{4\sqrt{6}} \ . \qquad (135)$$

$$\forall n \geqslant 0 \ , \ |L_n(it)| \ \geqslant \ \frac{|t|^n}{n!} \ . \qquad (136)$$

As a final remark, let us mention that the sequence $\left([X^{2d}]|L_n(iX)|^2\right)_{0\leqslant d\leqslant n}$ is log-concave for all integers n:

$$
\begin{aligned}
&\frac{\left([X^{2d}]|L_n(iX)|^2\right)^2}{[X^{2d}]|L_{n-1}(iX)|^2\cdot[X^{2d}]|L_{n+1}(iX)|^2} \\
&\qquad = \frac{(d+1)(2d+1)}{(2d-1)(d-1)}\cdot\frac{(d+1)^2}{d^2}\cdot\frac{n-d+1}{n-d}\cdot\frac{n+d}{n+d+1} \\
&\qquad = \frac{(2d+1)(d+1)^3}{(2d-1)(d-1)d^2}\cdot\frac{n^2-d^2+n+d}{n^2-d^2+n-d} \geqslant 1 \qquad (137)
\end{aligned}
$$

Then, $\left([X^{2d}]|L_n(iX)|^2\right)_{0\leqslant d\leqslant n}$ is a unimodal sequence (*i.e.* is first increasing and then decreasing). So, Corollary 5.4 can be used with the value $d = d_0(n)$ for which this sequence is maximum.

The following Proposition gives us some characterization to compute easily and rapidly these values $d_0(n)$ for any power of 2. From the increassing character, this can be done for any non-negative integer n by exploration. For example, Table 3 gives some values $d_0(2^k)$ for $k \in [\![0;.13]\!]$.

Proposition 5.5. *Let us denote by $d_0(n)$ for all non-negative integer n, the maximum value of the unimodal sequence $\left(b_{n,d}\right)_{0\leqslant d\leqslant n}$, with*

$$b_{n,d} = \frac{1}{(2d)!}\binom{n}{d}\binom{n+d}{n} . \qquad (138)$$

Let also k be a non-negative integer.
Then,

1. $d_0(2^{2k+1}) = 2^k$.
2. $d_0(2^{2k}) \geqslant \lfloor 2^n \ln(2)\rfloor$.
3. $d_0(p) \leqslant d_0(q)$ *if* $p \leqslant q$.

Proof. We have for all non-negative integer n and d such that $0 \leqslant d \leqslant n$:

$$\frac{b_{n,d+1}}{b_{n,d}} - 1 = \frac{n^2+n-(d+1)(4d^3+10d^2+9d+2)}{2(d+1)^3(2d+1)} . \qquad (139)$$

Let us denote by P the polynomial $P(n,d) = n^2+n-(d+1)(4d^3+10d^2+9d+2)$. Therefore, we have:

$$d \leqslant d_0(n) - 1 \Longleftrightarrow P(n,d) \geqslant 0 . \qquad (140)$$

$$d \geqslant d_0(n) \Longleftrightarrow P(n,d) \leqslant 0 . \qquad (141)$$

• Let us now fix $n = 2^{2k+1}$ for a non-negative integer k.
If $d \leqslant 2^k - 1$, we have $2(d+1)^2 \leqslant n$, so that $P(n,d) \geqslant (d+1)(2d^2+5d+4) \geqslant 0$.

n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
$d_0(n)$	0	1	1	2	2	4	5	8	11	16	22	32	45	64

Table 3: Values of $d_0(n)$ when n is the first power of 2.

Therefore, we deduce from Equation (140) that $d_0(2^{2k+1}) \geqslant 2^k$.
If $d \geqslant 2^k$, we have $2d^2 \geqslant n$, so that $P(n,d) \leqslant -(14d^3 + 17d^2 + 11d + 2) \leqslant 0$. Therefore, we deduce from Equation (141) that $d_0(2^{2k+1}) \leqslant 2^k$, which concludes the proof of the first point.

• Using Equation (140), we deduce that $d_0(2^{2k}) \geqslant \lfloor 2^n \ln(2) \rfloor$ if, and only if $P\big(2^{2k}, e_k - 1\big) \geqslant 0$, where $e_k = \lfloor 2^k \ln(2) \rfloor$.

But, from $P(n, d-1) = n^2 + n - 4d^4 + 2d^3 - d^2 + d \geqslant (n^2 - 4d^4) + (n - d^2)$, $2^{4k} - 4e_k^4 \geqslant 2^{4k}\big(1 - 4\ln(2)^4\big) \geqslant 0$ and $2^{2k} - e_k^2 \geqslant 2^{2k}\big(1 - \ln(2)^2\big) \geqslant 0$, we deduce that $P\big(2^{2k}, e_k - 1\big) \geqslant 0$ for all non-negative integer k.

• We have $d_0(n+1) \geqslant d_0(n)$ if, and only if, for all integer d such that $d < d_0(n)$, $P(n+1, d) \geqslant 0$, which is clear according to $P(n+1, d) = P(n, d) + 2n + 2$ and Equation (140). This proves the increasing character of the sequence $\big(d_0(n)\big)_{n \in \mathbb{N}}$. □

5.2 A computational upper bound of the integral $\mathcal{U}_n(z)$

In this Section, we will adapt from Section 4 and sum up all the necessary Equations to have a computational and explicit upper bound of $\mathcal{U}_n(z)$ for alls $z \in \mathbb{C} - \mathbb{R}^+$.

Let us emphasize that, during the proof of Lemma 4.4, we have proven that, if $n^\theta \geqslant \Re e\ (s)$ (see Equations (86) and (89)):

$$|\mathcal{J}_n(s)| \leqslant \frac{n}{2\Re e\ (s)^2} \exp\left(-\Re e\ (s) \cdot \left(\sqrt{n} - n^\theta + \frac{n}{\sqrt{n} - n^\theta + \Re e\ (s)}\right)\right) , \quad (142)$$

$$|\mathcal{L}_n(s)| \leqslant \frac{1}{\Re e\ (s)} \cdot \exp\left(-\Re e\ (s) \cdot \left(\sqrt{n} + n^\theta + \frac{n}{\sqrt{n} + n^\theta + \Re e\ (s)}\right)\right) \cdot \left(\frac{1}{\sqrt{n}} + \frac{1}{n^\theta}\right) . \quad (143)$$

This can also be rewritten, or weakened, into the simpler upper bounds:

$$|\mathcal{J}_n(s)| \leqslant \frac{n}{2\Re e\ (s)^2} \exp\left(-\Re e\ (s) \cdot \left(2\sqrt{n} - \Re e\ (s) + \frac{(n^\theta - \Re e\ (s))^2}{\sqrt{n} - n^\theta + \Re e\ (s)}\right)\right) , \quad (144)$$

$$|\mathcal{L}_n(s)| \leqslant \frac{2}{\Re e\ (s)} \cdot \exp\left(-\Re e\ (s) \cdot \left(2\sqrt{n} + n^\theta - \frac{n^{\theta + \frac{1}{2}} + s\sqrt{n}}{\sqrt{n} + n^\theta + s}\right)\right) . \quad (145)$$

Moreover, using elementary techniques, we can obtain an upper bound of $|\mathcal{K}_n(s)|$. Of course, this upper bound will not behave as the equivalent obtained in Equation (122), but the term $\exp(-2\Re e\ (s)\cdot\sqrt{n})$ will nevertheless naturally appear.

Lemma 5.6. *For all $s \in \mathbb{C}$ such that $\Re e\ (s) > 0$, $\theta \in \left]\frac{1}{4};\frac{1}{3}\right[$ and $n \geqslant 2$, we have:*

$$|\mathcal{K}_n(s)| \leqslant \frac{2n^\theta e^{\Re e\ (s)\ \cdot\ c_n\left(\Re e\ (s),\theta\right)}}{\sqrt{n}-n^\theta+\Re e\ (s)} \exp\left(-\Re e\ (s)\left(2\sqrt{n}-2n^\theta+n^{2\theta-\frac{1}{2}}-\frac{\Re e\ (s)}{2}\right)\right) , \tag{146}$$

where $c_n(s,\theta) = \dfrac{n^{3\theta-\frac{1}{2}}}{\sqrt{n}+n^\theta} + \dfrac{s}{n^{\frac{1}{2}-\theta}-1} + \dfrac{s}{2(n^{\frac{1}{2}-\theta}-1)^2} \underset{n\longrightarrow+\infty}{=} \mathcal{O}(1)$.

Proof. • First of all, let us remind that $\mathcal{K}_n(s)$ is defined as

$$\mathcal{K}_n(s) = e^{-s\sqrt{n}}\int_{-n^\theta}^{+n^\theta} \frac{(u+\sqrt{n})^n}{(u+\sqrt{n}+s)^{n+1}} e^{-su}\ du , \tag{147}$$

so that we have: $|\mathcal{K}_n(s)| \leqslant \mathcal{K}_n\left(\Re e\ (s)\right)$.

• From now on, let us assume that $s \in \mathbb{R}_+^*$. For all $u \in [-n^\theta; n^\theta]$, we successively have:

$$\begin{aligned}
&\frac{(u+\sqrt{n})^n}{(u+\sqrt{n}+s)^{n+1}}e^{-su} \\
&\qquad = \left(1+\frac{s}{u+\sqrt{n}}\right)^{-n} \frac{e^{-su}}{u+\sqrt{n}+s} \\
&\qquad \leqslant \exp\left(-\frac{ns}{u+\sqrt{n}}+\frac{ns^2}{2(u+\sqrt{n})^2}\right)\frac{e^{-su}}{u+\sqrt{n}+s} , \\
&\qquad\qquad\qquad \text{according to } \ln(1+x) \geqslant x-\frac{x^2}{2} \text{ for all } x \geqslant 0 \\
&\qquad \leqslant \exp\left(-\frac{ns}{\sqrt{n}+n^\theta}+\frac{ns^2}{2(\sqrt{n}-n^\theta)^2}\right)\frac{e^{sn^\theta}}{u+\sqrt{n}+s} . \qquad (148)
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
&\left(-\frac{ns}{\sqrt{n}+n^\theta}+\frac{ns^2}{2(\sqrt{n}-n^\theta)^2}+sn^\theta\right)-\left(-s\sqrt{n}+2sn^\theta-sn^{2\theta-\frac{1}{2}}+\frac{s^2}{2}\right) \\
&\qquad = \frac{sn^{3\theta-\frac{1}{2}}}{\sqrt{n}+n^\theta}+\frac{s^2}{n^{\frac{1}{2}-\theta}-1}+\frac{s^2}{2(n^{\frac{1}{2}-\theta}-1)^2} = s\cdot c_n(s,\theta) . \qquad (149)
\end{aligned}$$

Consequently, we have:

$$\begin{aligned}
&\int_{-n^\theta}^{n^\theta} \frac{(u+\sqrt{n})^n}{(u+\sqrt{n}+s)^{n+1}} e^{-su}\,du \\
&\quad\leqslant\ \ln\left(1+\frac{2n^\theta}{\sqrt{n}-n^\theta+s}\right)\cdot \exp\left(-s\sqrt{n}+2sn^\theta - sn^{2\theta-\frac{1}{2}}+\frac{s^2}{2}+s\cdot c_n(s,\theta)\right) \\
&\quad\leqslant\ \frac{2n^\theta}{\sqrt{n}-n^\theta+s}\cdot \exp\left(-s\sqrt{n}+2sn^\theta - sn^{2\theta-\frac{1}{2}}+\frac{s^2}{2}+s\cdot c_n(s,\theta)\right)\ , \qquad (150)
\end{aligned}$$

which concludes the proof of the Lemma. □

Let us finally summarize the situation in order to have a computational upper bound of the integral $\mathcal{U}_n(z) = \displaystyle\int_0^{+\infty} \frac{t^n}{(t-z)^{n+1}} e^{-t}\,dt$:

Corollary 5.7. *Let $z \in \mathbb{C} - \mathbb{R}^+$ and n be a positive integer and $\theta \in \left]\frac{1}{4};\frac{1}{3}\right[$. Therefore, the integral $\mathcal{U}_n(z)$ satisfies*

$$|\mathcal{U}_n(z)| \leqslant \exp\Big(-2\Re e\ (s)\cdot(\sqrt{n}-n^\theta)\Big)\cdot \mathcal{M}_n(s,\theta) \qquad (151)$$

if $n^\theta \geqslant \Re e\ (s)$, where:

$$\begin{aligned}
s &= \sqrt{-z}\ . && (152)\\
\mathcal{M}_n(s,\theta) &= \frac{n\cdot e^{j_n(s,\theta)}}{2\Re e\ (s)^2} + \frac{2n^\theta\cdot e^{k_n(s,\theta)}}{\sqrt{n}-n^\theta+\Re e\ (s)} + \frac{2}{\Re e\ (s)} e^{l_n(s,\theta)}\ . && (153)\\
j_n(s,\theta) &= \Re e\ (s)\cdot\left(\Re e\ (s) - \frac{\big(n^\theta-\Re e\ (s)\big)^2}{\sqrt{n}-n^\theta+\Re e\ (s)} - 2n^\theta\right)\ . && (154)\\
k_n(s,\theta) &= \Re e\ (s)\cdot\Big(-n^{2\theta-\frac{1}{2}}+\frac{\Re e\ (s)}{2}+c_n\big(\Re e\ (s),\theta\big)\Big)\ . && (155)\\
l_n(s,\theta) &= \Re e\ (s)\cdot\left(-3n^\theta + \frac{n^{\theta+\frac{1}{2}}+\Re e\ (s)\sqrt{n}}{\sqrt{n}+n^\theta+\Re e\ (s)}\right)\ . && (156)\\
c_n(s,\theta) &= \frac{n^{3\theta-\frac{1}{2}}}{\sqrt{n}+n^\theta} + \frac{s}{n^{\frac{1}{2}-\theta}-1} + \frac{s}{2(n^{\frac{1}{2}-\theta}-1)^2}\ . && (157)
\end{aligned}$$

Remark 2. *Let us emphasize that $j_n(s,\theta) \underset{n\longleftrightarrow+\infty}{\longrightarrow} -\infty$, $k_n(s,\theta) \underset{n\longleftrightarrow+\infty}{\longrightarrow} -\infty$, and $l_n(s,\theta) \underset{n\longleftrightarrow+\infty}{\longrightarrow} -\infty$, so that $\mathcal{M}_n(s,\theta) \underset{n\longleftrightarrow+\infty}{=} \mathcal{O}(1)$.*

Proof. The proof is straightforward, using Equations (60) and (75), as well as the upper bounds given by Equations (144) - (146). □

Let us emphasize that Proposition 4.3 and Corollary 5.7 pursue the same goal: having a upper bound of $\mathcal{U}_n(z)$ for $z \in \mathbb{C} - \mathbb{R}^+$ and $n \in \mathbb{N}^*$. Proposition 4.3 is not a sharp upper bound, while Corollary 5.7 is, but it has an advantage on Corollary 5.7: the upper bound is a decreassing function of $s = \Re e\left(\sqrt{-z}\right)$. We will use this fact in Proposition 5.9 of Subsection 5.4.

5.3 Back to the Example 1

We have proven in Example 6, *i.e.* the continuation of Example 1, a simple and explicit error formula:

$$e_N^{GL}(a) = \Re e\left(\frac{i}{L_N(-i)} \int_0^{+\infty} \frac{t^N\ e^{-t}}{(t+i)^{N+1}}\ dt\right) = \Re e\left(i\ \frac{\mathcal{U}_N(-i)}{L_N(-i)}\right)\ , \qquad (158)$$

where the function a is defined by $a(z) = \dfrac{1}{1+z^2}$ for all $z \in \mathbb{C} - \{i; -i\}$.

According to Proposition 5.1, we conjecture that:

Conjecture 1. *1.* $e_N^{GL}(a) \underset{N\longrightarrow+\infty}{\sim} 4\pi e^{-2\sqrt{2N}}$.

2. $|e_N^{GL}(a)| \leqslant 4\pi e^{-2\sqrt{2N}}$ *for all non-negative integers* N.

If there is still a long way to go to reach it, we have created a path in its direction in Section 4 and Subsections 5.1 and 5.2.

The error $e_N^{GL}(a)$ containing a unique term $\mathcal{U}_N(z)$, we therefore prefer use Corollary 5.7 to Proposition 4.3. This gives directly the following upper bound:

$$\begin{aligned} |e_N^{GL}(a)| &\leqslant \left|\frac{\mathcal{U}_N(-i)}{L_N(-i)}\right| \\ &\leqslant \frac{1}{|L_N(-i)|} e^{-\sqrt{2}(\sqrt{N}-N^\theta)} \mathcal{M}_N(\sqrt{-i},\theta)\ . \end{aligned} \qquad (159)$$

We can now compute the N-th Laguerre polynomial explicitly, using its explicit definition given by Equation (127), or using Corollary 5.4 to obtain a weaker (but easier and more rapid to compute) upper bound of the error:

$$|e_N^{GL}(a)| \leqslant c_N e^{-\sqrt{2}(\sqrt{N}-N^\theta)} \mathcal{M}_N(\sqrt{-i},\theta)\ , \qquad (160)$$

where

$$c_N = \frac{1}{C_{N,d_0(N)}} = \sqrt{\frac{\Big(2d_0(N)\Big)!}{\begin{pmatrix} N \\ d_0(N)\end{pmatrix}\begin{pmatrix} N+d_0(N) \\ N\end{pmatrix}}}\ . \qquad (161)$$

Table 4 shows the exact values of $e_N^{GL}(a)$, its explicit upper bound given by Equation (160) for $\theta = \dfrac{7}{24}$, as well as its minimal value for $\theta \in \left]\dfrac{1}{4};\dfrac{1}{3}\right[$ for all $N \in \{2^k; [\![0;13]\!]\}$.

N	Exact value of $e_N^{GL}(a)$		Value of the explicit upper bound of $\lvert e_n^{GL}(a)\rvert$, for $\theta = \dfrac{7}{24}$		Minimal value of the explicit upper bound of $\lvert e_n^{GL}(a)\rvert$, for $\theta = \left]\dfrac{1}{4};\dfrac{1}{3}\right[$	
1	0.	121450				
2	−0.(1)	256092	815066.	124892	14201.	822991
4	−0.(1)	149774	13.	215660	3.	889484
8	0.(2)	137463	0.	377890	0.	197386
16	−0.(4)	568860	0.(1)	159339	0.(1)	105591
32	0.(6)	342774	0.(3)	257229	0.(3)	193731
64	0.(9)	230821	0.(6)	807091	0.(6)	645529
128	−0.(13)	257593	0.(9)	199900	0.(9)	159066
256	0.(19)	418492	0.(14)	118497	0.(15)	867921
512	0.(27)	194960	0.(22)	358943	0.(22)	216674
1024	0.(38)	302784	0.(33)	592136	0.(33)	245275
2048	0.(54)	155439	0.(48)	252906	0.(49)	521822
4096	−0.(77)	123392	0.(70)	357770	0.(71)	223753
8192	−0.(110)	211831	0.(101)	318619	0.(103)	333039

Table 4: Explicit upper bounds of the error $e_N^{GL}(a)$

. .

Consequently, we see here that:

1. Equations (159) and (160) give explicit upper bounds of the error, so that we can now compute the integral $\displaystyle\int_0^{+\infty} \frac{e^{-t}}{1+t^2}\, dt$ by the Gauss-Laguerre quadrature to find out a predefined numbers of exact digits ;
2. the error $e_N(a)$ *nearly* converges to 0 as rapidly as $e^{-2\sqrt{2N\pi}}$.
3. the upper bound in Equation (160) is quite sharp, only a few digits are lost compared to the exact values of the error $e_N^{GL}(a)$.

5.4 Back to the Example 2

Let us now focus on our second main example, *i.e.* Example 2 which is pursued in Example 7: the example of application of the Gauss-Laguerre-like quadrature to the constant function 1.

5.4.1 A new expression of $E_N(1)$

We have proven in Example 7 an explicit error formula given by Equation (52):

$$E_N(1) = \Re e \left(\int_0^{+\infty} \Big(\sum_{k\geqslant 0} \frac{2}{L_N\big((2k+1)i\pi\big)} \frac{t^N \; e^{-t}}{\big(t-(2k+1)i\pi\big)^{N+1}} \Big) \, dt \right) . \quad (162)$$

We are now able to prove the permutation of the symbols $\sum$ and $\int$ in Equation (52) using term-by-term integration.

Let us denote by f_k the function defined over $\mathbb{R}^+$ by

$$f_k(t) = \frac{1}{L_N\big((2k+1)i\pi\big)} \frac{t^N\ e^{-t}}{\big(t-(2k+1)i\pi\big)^{N+1}} \ . \tag{163}$$

According to Equation (134), we have for all non-negative integers k and all non-negative real numbers t:

$$|f_k(t)| \leqslant \frac{\sqrt{2}e^{-t}}{N(2k+1)^2\pi^2} \leqslant \frac{\sqrt{2}}{N(2k+1)^2\pi^2} \ . \tag{164}$$

Therefore, the Weierstrass' M test shows that the series $\sum_k f_k$ is normally convergent on $\mathbb{R}^+$ and its sum is a continuous function on $\mathbb{R}^+$.

Moreover, we have:

$$\int_0^{+\infty} |f_k(t)|\ dt \leqslant \frac{\sqrt{2}}{N(2k+1)^2\pi^2} \int_0^{+\infty} e^{-t}\ dt = \frac{\sqrt{2}}{N(2k+1)^2\pi^2} \ , \tag{165}$$

so that $\sum_k \int_0^{+\infty} |f_k(t)|\ dt$ is a convergent series.

Consequently, it is possible to permute the symbol $\sum$ and $\int$ in Equation (52), which gives the following:

Proposition 5.8. *For all positive integer N, we have:*

$$E_N(1) = \sum_{k\geqslant 0} \Re e \left(\frac{2}{L_N\big((2k+1)i\pi\big)} \int_0^{+\infty} \frac{t^N\ e^{-t}}{\big(t-(2k+1)i\pi\big)^{N+1}}\ dt \right) \tag{166}$$

$$= \sum_{k\geqslant 0} \Re e \left(\frac{\mathcal{U}_N\big((2k+1)i\pi\big)}{L_N\big((2k+1)i\pi\big)} \right) \ . \tag{167}$$

5.4.2 Upper bounds of $E_N(1)$

According to Propositions 5.1 and 5.8, we conjecture the following upper bound of the error:

Conjecture 2. *1.* $E_N(1) \underset{N\longrightarrow+\infty}{\sim} \Re e \left(2\ \frac{|\mathcal{U}_N(i\pi)|}{|L_N(i\pi)|} \right)$.

2. $|E_N(1)| \leqslant 4\pi e^{-2\sqrt{2N\pi}}$ *for all non-negative integers N.*

In this direction, we can prove the following explicit upper bound of $E_N(1)$:

Proposition 5.9. *Let $N \geqslant 2$ be an integer, and let us denote $fl_N = \left\lfloor \frac{N}{\pi} - \frac{1}{2} \right\rfloor$ and*

$$c_N = \frac{1}{C_{N,d_0(N)}} = \sqrt{\frac{\Big(2d_0(N)\Big)!}{\binom{N}{d_0(N)}\binom{N+d_0(N)}{d_0(N)}}} \ . \tag{168}$$

For all $K \geqslant 0$, we have:

$$\left| \sum_{k \geqslant K} \Re e \left(\frac{2}{L_N\Big((2k+1)i\pi\Big)} \int_0^{+\infty} \frac{t^N \ e^{-t}}{\Big(t-(2k+1)i\pi\Big)^{N+1}} \ dt \right) \right|$$

$$\leqslant \begin{cases} \dfrac{2^{\frac{3}{4}} c_N}{\big((2K+1)\pi\big)^{d_0(N)+\frac{3}{4}}} \left(\sqrt[4]{\dfrac{2}{\pi}} + \sqrt[4]{N}\sqrt{\pi} \right) \left(1 + \dfrac{1}{\sqrt{N\pi}} \right) \exp\left(-\sqrt{N\pi\left(K+\frac{1}{2}\right)} \right) \\ \qquad + \dfrac{c_N}{(2N)^{d_0(N)}} \left(\left(\dfrac{1}{\pi} + \dfrac{3}{2N} \right) \exp\left(-\frac{N\sqrt{2N}}{\sqrt{N+\pi}} \right) + \dfrac{2}{\pi d_0(N)} e^{-N} \right) & , \ \textit{if } K \leqslant fl_N \ . \\ \dfrac{c_N}{(2N)^{d_0(N)}} \left(\left(\dfrac{1}{\pi} + \dfrac{3}{2N} \right) \exp\left(-\frac{N\sqrt{2N}}{\sqrt{N+\pi}} \right) + \dfrac{2}{\pi d_0(N)} e^{-N} \right) & , \ \textit{if } K = fl_N + 1 \ . \\ \dfrac{2}{\pi d_0(N)} \dfrac{c_N \ e^{-N}}{\big((2K+1)\pi\big)^{d_0(N)}} & , \ \textit{if } K > fl_N + 1 \ . \end{cases} \tag{169}$$

Proof. To prove the Proposition, let us use the elementary upper bound of Proposition 4.3 instead of this given by Corollary 5.7, according to its decreassing caracter in $s = \Re e\ (\sqrt{-z})$. Consequently, the remainder of the series defining $E_N(1)$ will be estimated using an upper bound on the $\mathcal{U}_N$ part, the convergence of the remainder being guaranteed using Corollary 5.4.

Consequently, we cut $E_N(1)$ in three parts, relatively to the integer $fl_N = \left\lfloor \frac{N}{\pi} - \frac{1}{2} \right\rfloor$:

$$E_N^{(1)}(1) = \sum_{k=0}^{fl_N} \left| \frac{\mathcal{U}_N\big((2k+1)i\pi\big)}{L_N\big((2k+1)i\pi\big)} \right| \tag{170}$$

$$E_N^{(2)}(1) = \left| \frac{\mathcal{U}_N\big((2fl_N+3)i\pi\big)}{L_N\big((2fl_N+3)i\pi\big)} \right| \tag{171}$$

$$E_N^{(3)}(1) = \sum_{k \geqslant fl_N+2} \left| \frac{\mathcal{U}_N\big((2k+1)i\pi\big)}{L_N\big((2k+1)i\pi\big)} \right| \tag{172}$$

Let us also assume that $N \geqslant 2$, so that $d_0(N) > 0$. We will prove the Proposition only when $K = 0$, the extension to the other values being left to the reader.

The $E_N^{(1)}(1)$ part. The first case of Proposition 4.3 gives us:

$$E_N^{(1)}(1) \leqslant \sum_{k=0}^{fl_N} \frac{1}{|L_N\Big((2k+1)i\pi\Big)} \cdot \frac{\exp\left(-\sqrt{N\pi\left(k+\frac{1}{2}\right)}\right)}{\left(\left(k+\frac{1}{2}\right)\pi\right)^{\frac{3}{4}}} \left(\frac{1}{\left(k+\frac{1}{2}\right)^{\frac{1}{4}}\sqrt[4]{\pi}} + \sqrt[4]{N}\sqrt{\pi}\right)$$

$$\leqslant \left(\sqrt[4]{\frac{2}{\pi}} + \sqrt[4]{N}\sqrt{\pi}\right) \frac{c_N}{2^{d_0(N)}} \sum_{k=0}^{fl_N} \frac{\exp\left(-\sqrt{N\pi\left(k+\frac{1}{2}\right)}\right)}{\left(\left(k+\frac{1}{2}\right)\pi\right)^{d_0(N)+\frac{3}{4}}}\ , \tag{173}$$

according to Corollary 5.4 used with $d = d_0(N)$.

Therefore, we successively have, according to $d_0(N) > 0$:

$$E_N^{(1)}(1) \leqslant \frac{c_N}{2^{d_0(N)}} \left(\sqrt[4]{\frac{2}{\pi}} + \sqrt[4]{N}\sqrt{\pi}\right) \left(\frac{\exp\left(-\sqrt{\frac{N\pi}{2}}\right)}{\left(\frac{\pi}{2}\right)^{d_0(N)+\frac{3}{4}}} + \int_{\frac{1}{2}}^{+\infty} \frac{e^{-\sqrt{N\pi t}}}{(\pi t)^{d_0(N)+\frac{3}{4}}}\ dt\right)$$

$$\leqslant \frac{c_N}{2^{d_0(N)}} \left(\sqrt[4]{\frac{2}{\pi}} + \sqrt[4]{N}\sqrt{\pi}\right) \left(\frac{\exp\left(-\sqrt{\frac{N\pi}{2}}\right)}{\left(\frac{\pi}{2}\right)^{d_0(N)+\frac{3}{4}}} + \frac{\exp\left(-\sqrt{\frac{N\pi}{2}}\right)}{\left(\frac{\pi}{2}\right)^{d_0(N)+\frac{3}{4}}}\sqrt{\frac{1}{N\pi}}\right)$$

$$\leqslant \frac{2^{\frac{3}{4}} c_N}{\pi^{d_0(N)+\frac{3}{4}}} \left(\sqrt[4]{\frac{2}{\pi}} + \sqrt[4]{N}\sqrt{\pi}\right) \left(1 + \frac{1}{\sqrt{N\pi}}\right) \exp\left(-\sqrt{\frac{N\pi}{2}}\right)\ . \tag{174}$$

The $E_N^{(2)}(1)$ part. The second case of Proposition 4.3 gives us:

$$E_N^{(2)}(1) \leqslant \frac{\exp\left(-\sqrt{\left(fl_N+\frac{3}{2}\right)\pi \cdot \frac{N}{\sqrt{N+\pi}}}\right)}{\left(fl_N+\frac{3}{2}\right)\pi} \frac{3+\frac{2N}{\pi}}{\Big|L_N\Big((2fl_N+3)i\pi\Big)\Big|}\ . \tag{175}$$

Corollary 5.4, used with $d = d_0(N)$, gives now:

$$E_N^{(2)}(1) \leqslant c_N\left(3+\frac{2N}{\pi}\right) \frac{2\exp\left(-\sqrt{\left(fl_N+\frac{3}{2}\right)\pi\cdot\frac{N}{\sqrt{N+\pi}}}\right)}{\Big((2fl_N+3)\pi\Big)^{d_0(N)+1}}$$

$$\leqslant c_N\left(3+\frac{2N}{\pi}\right) \frac{\exp\left(-\frac{N\sqrt{2N}}{\sqrt{N+\pi}}\right)}{(2N)^{d_0(N)+1}}\ , \tag{176}$$

according to $(2fl_N+3)\pi \geqslant 2\pi\left(\frac{N}{\pi}-\frac{1}{2}\right)+\pi = 2N$.

The $E_N^{(3)}(1)$ part. The third case of Proposition 4.3 and Corollary 5.4, used with $d = d_0(N) > 0$ gives:

$$\begin{aligned} E_N^{(3)}(1) &\leqslant 2e^{-N} \sum_{k \geqslant fl_N+2} \frac{1}{\left|L_N\Big((2k+1)i\pi\Big)\right|} \cdot \frac{1}{\left(k+\frac{1}{2}\right)\pi} \\ &\leqslant \frac{2}{\pi} \frac{c_N e^{-N}}{(2\pi)^{d_0(N)}} \sum_{k \geqslant fl_N+2} \frac{1}{\left(k+\frac{1}{2}\right)^{d_0(N)+1}} \\ &\leqslant \frac{2}{\pi} \frac{c_N e^{-N}}{d_0(N)} \frac{1}{(2\pi)^{d_0(N)}} \frac{1}{\left(fl_N+\frac{3}{2}\right)^{d_0(N)}} \\ &\leqslant \frac{2}{\pi} \frac{c_N e^{-N}}{d_0(N)} \frac{1}{(2N)^{d_0(N)}} \ . \end{aligned} \tag{177}$$

□

5.4.3 Numerical computations

Now, we can estimate the error $E_N(1)$ and then compute the integral $\displaystyle\int_0^{+\infty} \frac{dt}{1+e^t}$ up to d digits, where the integer d is defined in advance: using Proposition 5.9 with $K = 0$, we just have to find out the lowest value of N which gives the required precision, according to the inequation $|E_N(1)| \leqslant 10^{-d}$ and then perform the N-points Gauss-Laguerre-like quadrature associated with the constant function 1.

Once this is done, we obtain Table 5 which shows the difference of values between the exact value of $|E_N(1)|$ and the estimation given by Proposition 5.9. It also gives the calculation time of the quadrature of degree N.

Consequently, we see in Table 5 that:

1. the dominant part of an equivalent of the error $E_N(1)$ seems to be in $e^{-c\sqrt{N}}$.

2. the upper bound of $|E_{2N}(1)|$ given by Proposition 5.9 is not really sharp, but is in the same order of magnitude as the exact value of $|E_N(1)|$.

3. the calculation time of the quadrature of degree $2N$ seems to be two time longer than the calculation time of the quadrature of degree N, which is quite predictable according to the double numbers of nodes and weights for the first quadrature in comparison to the second one.

Another program to estimate the error $E_N(1)$ more precisely could be used, since each term of the summation is relatively small in comparison to the previous term. Consequently, $E_N(1)$ can be very well estimated by computing a few of its first terms.

N	Upper bound of $\|E_N(1)\|$ from Prop. 5.9 and Eq. (132).		Exact value of $\|E_N(1)\|$		Calculation time of the quadrature of degree N	
2	0.(1)	66667	0.(2)	30447	0.	738 ms
4	0.(2)	75577	0.(4)	23385	1.	332 ms
8	0.(3)	30836	0.(5)	44075	2.	659 ms
16	0.(4)	17694	0.(8)	19843	5.	213 ms
32	0.(7)	26138	0.(11)	49516	10.	257 ms
64	0.(10)	20262	0.(16)	38798	17.	487 ms
128	0.(15)	12205	0.(23)	26126	32.	623 ms
256	0.(22)	18595	0.(34)	38050	55.	362 ms
512	0.(32)	16620	0.(48)	60513	110.	334 ms
1024	0.(46)	30823	0.(68)	25357	223.	837 ms
2048	0.(66)	18811	0.(97)	27463	489.	749 ms
4096	0.(94)	16064	0.(138)	51246	916.	177 ms
8192	0.(134)	14522	0.(196)	20657	1853.	715 ms

Table 5: Explicit upper bounds of the error $E_N(1)$

. .

Then, the integral $\displaystyle\int_0^{+\infty} \frac{dt}{1+e^t}$ is computed up to d digits by determining two integers, N and K: N names the order of the Gauss-Laguerre-like quadrature we will have to perform to estimate the integral, while K names the number of terms we will use to estimate $E_N(1)$.

On the first hand, according to Proposition 4.3, the integer K is chosen in such a way that the $(K+1)$th-remainder of the series $E_N(1)$ be small relatively to the order of magnitude of the first term of the sum. On the other hand, N will be the first positive integer we find out satisfying $|E_N(1)| \leqslant 10^{-d}$.

Now, we have to compute the first K terms of the sum $E_N(1)$. This is done by computing the terms $\mathcal{U}_N\big((2k+1)i\pi\big)$ using a Gauss-Laguerre quadrature, similar to this used in Example 1!

Even if the upper bound of the remainder of $|E_N(1)|$ given in Equation (169) is not so precise, this program reduces the order of the Gauss-Laguerre-like quadrature to perform, to reach the required precision. This has a drastic effect on the computation time.

5.4.4 A quasi equivalent of $|E_N(1)|$

Using Proposition 4.3, we are now able to prove the following

Proposition 5.10. *We have:*

$$E_N(1) \underset{N \longrightarrow +\infty}{=} \Re e \left(\frac{\mathcal{U}_N(i\pi)}{L_N(i\pi)} \right) + o\left(e^{-2\sqrt{2N\pi}} \right) \quad . \tag{178}$$

This is a significative advance in the direction of the first point of Conjecture 2.

Proof. According to Proposition 5.9, we know that the Kth-remainder of $E_N(1)$, denoted here by $\mathcal{R}_{N,K}$ satisfies:

$$|\mathcal{R}_{N,K}| \underset{N \longrightarrow +\infty}{=} \mathcal{O}(e^{-N}) \ , \tag{179}$$

when $K > \left\lfloor \frac{N}{\pi} - \frac{1}{2} \right\rfloor$. Therefore, we have:

$$|\mathcal{R}_{N,K}| \underset{N \longrightarrow +\infty}{=} \wr(e^{-2\sqrt{2N\pi}}) \ . \tag{180}$$

From Proposition 5.1, we also easily see that for all integers k, we have:

$$e^{2\sqrt{2N\pi}} \cdot \left| \frac{\mathcal{U}_N\big((2k+1)i\pi\big)}{L_N\big((2k+1)i\pi\big)} \right| \underset{N \longrightarrow +\infty}{\sim} 2\pi \exp\Big(-2\sqrt{2N\pi}\ cdot(\sqrt{2k+1} - 1)\Big) \ , \tag{181}$$

so that we have for all positive integers k:

$$\left| \frac{\mathcal{U}_N\big((2k+1)i\pi\big)}{L_N\big((2k+1)i\pi\big)} \right| \underset{N \longrightarrow +\infty}{=} o\left(e^{-2\sqrt{2N\pi}}\right) \ . \tag{182}$$

Consequently, we have proven that $|\mathcal{R}_{N,1}| \underset{N \longrightarrow +\infty}{=} \wr(e^{-2\sqrt{2N\pi}})$, which concludes the proof of the Proposition. ☐

6 Annex A: proof of Proposition 3.2

In this Annex, we will prove the delicate Proposition 3.2, where great caution will be given to the possible values of R.

If $R \in \mathbb{R}$, let us denote by f_R and g_R the 2π-periodic and even functions defined for all $\theta \in [0; 2\pi]$ by:

$$\begin{aligned} g_R(\theta) &= 1 + 2e^{-R\cos\theta}\cos(R\sin\theta) + e^{-2R\cos\theta} \qquad (183) \\ &= \left(\cos(R\sin\theta) + e^{-R\cos\theta}\right)^2 + \sin^2(R\sin\theta) \ . \\ f_R(\theta) &= \sqrt{g_R(\theta)} \ . \qquad (184) \end{aligned}$$

We shall study g_R on $[0; \pi]$ to find out a uniform (in the variable R) lower bound, valid for all $R \in \mathbb{R}_+^* - \bigcup_{n \in \mathbb{N}} \left[(2n+1)\pi - \delta; (2n+1)\pi + \delta\right]$.

In this direction, let us fix once and for all $\delta \in]0; \pi[$, $n \in \mathbb{N}^*$ and $R \in](2n-1)\pi + \delta; (2n+1)\pi - \delta[\cap \mathbb{R}_+^*$.

General behaviour of the functions f_R. Figure 3 shows a few graphs of functions f_R for a few values of $R \in \{1; 5; 2\pi; 9.3; 10; 50\}$. We can see here that f_R seems to have an approximative flat section, centered in 0, larger and larger with increasing values of R. It also seems to reach its maximum in π as an extremely intense peak with huge values of R.

In reality, the "flat section" is not so simple... On $\left[0; \frac{\pi}{2}\right]$, the flat section turns out to be an oscillatory part, with very small amplitudes, but increasing in amplitudes with θ (see Fig. 4a to Fig. 4g). However, the easy part is on $\left[\frac{\pi}{2}; \pi\right]$: f_R is an increasing function on this interval (see Fig. 4h to Fig. 4i).

Let us emphasize that the graphs of Figures 3 and 4 have been obtained using the Python programming language (see [16]) as well as the MATPLOTLIB library (see [9]). The quantity $1e^{-}n + u$ on top of the y-axis means that the scale has to be multiplied by 10^{-n}, and then, we have to add u to the graphed quantity (for example, $1e - 13 + 1$ in Fig. 4a means $1 + 10^{-13}$). Therefore, according to Figure 4, we see that $f_{30}(\theta) \approx 1$ for all $\theta \in [0; 1.45]$ up to two digits.

The increasing part of f_R on $\left[\frac{\pi}{2}; \pi\right]$. On this subinterval, it is sufficient to prove the following:

Lemma 6.1. *The function g_R, as well as the function f_R, are increasing functions on $\left[\frac{\pi}{2}; \pi\right]$.*

Proof. The fonction g_R is $\mathcal{C}^\infty$ on $[0; 2\pi]$ and for all θ, we have:

$$\begin{aligned} g_R'(\theta) &= 2R\sin\theta e^{-R\cos\theta}\cos(R\sin\theta) - 2R\cos\theta e^{-R\cos\theta}\sin(R\sin\theta) \\ &\quad +2R\sin\theta e^{-2R\cos\theta}\ . \end{aligned} \tag{185}$$

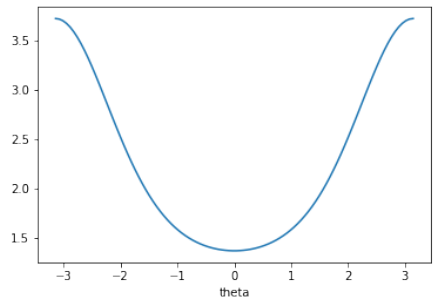
Let us define $C = R\cos\theta$ and $S = R\sin\theta$, so that $C \leqslant 0$ and $S \geqslant 0$. Therefore, $g_R'(\theta)$ can be expressed as:

$$\begin{aligned} g_R'(\theta) &= 2Se^{-C}\left(\cos(S) + e^{-C}\right) - 2Ce^{-C}\sin S \\ &= 2Se^{-C}\left(\left(1+\cos(S)\right) - C\cdot\left(1 + \frac{\sin S}{S}\right) + \left(e^{-C} - 1 + C\right)\right) \end{aligned} \tag{186}$$

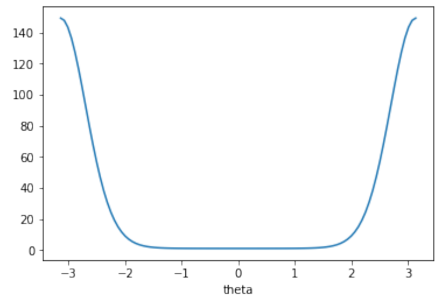
Each quantity inside the parenthesis of the right-hand side of Equation (186) is actually a positive one. So, $g_R'(\theta)$ is positive and g_R is an increasing function in $\left[\frac{\pi}{2}; \pi\right]$, as well as the function f_R. □

In contrast with Lemma 6.1, let us emphasize that the increasing part of f_R can begin before $\frac{\pi}{2}$ (see for example Fig. 4f).

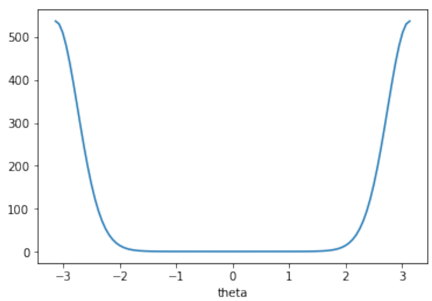
Now, we can easily derive a uniform bound on the sub-interval $\left[\frac{\pi}{2}; \pi\right]$.



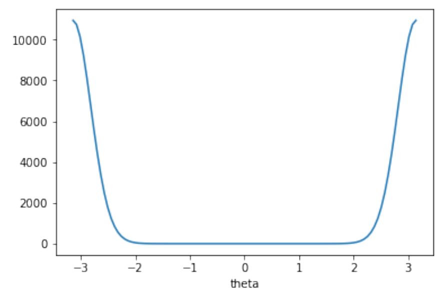
(a) Function f_1



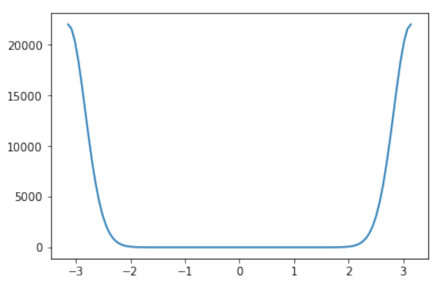
(b) Function f_5



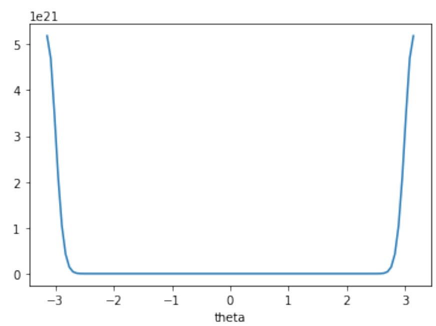
(c) Function $f_{2\pi}$



(d) Function $f_{9.3}$

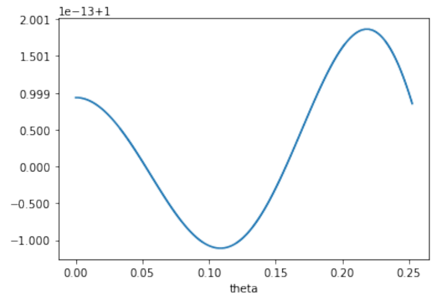


(e) Function f_{10}

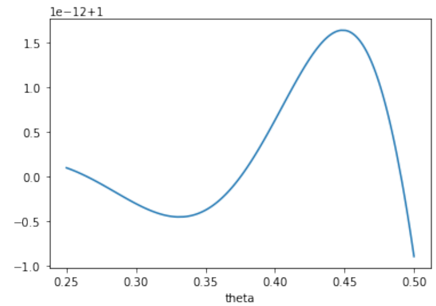


(f) Function f_{50}

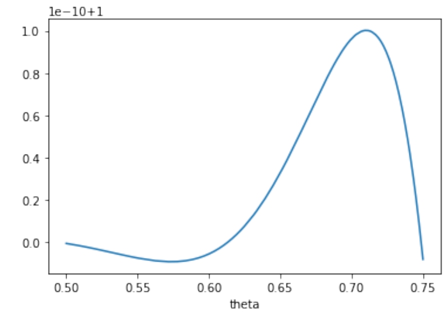
Figure 3: Graphs of function f_R for $R \in \{1; 5; 2\pi; 9.2; 10; 50\}$.



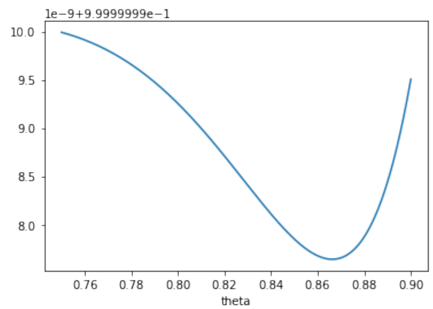
(a) Function f_{30} on $[0; 0.25]$



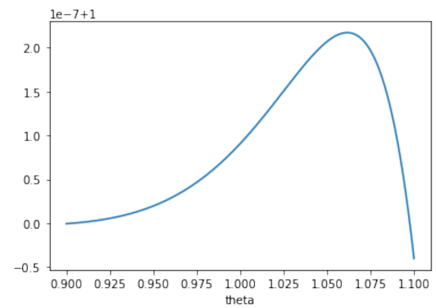
(b) Function f_{30} on $[0.25; 0.5]$



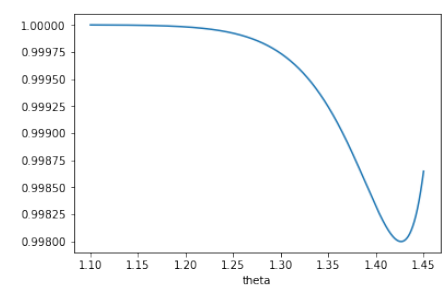
(c) Function f_{30} on $[0.5; 0.75]$



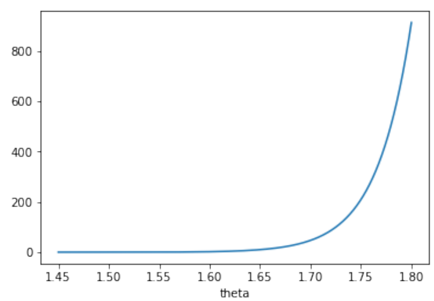
(d) Function f_{30} on $[0.75; 0.9]$



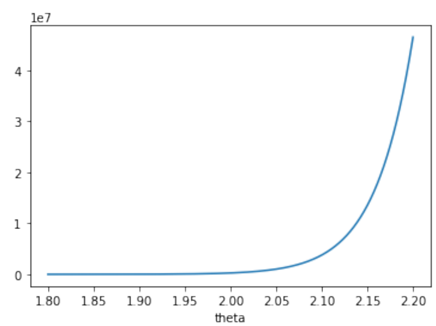
(e) Function f_{30} on $[0.9; 1.1]$



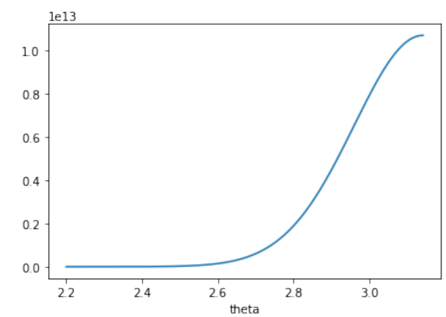
(f) Function f_{30} on $[1.1; 1.45]$



(g) Function f_{30} on $[1.45; 1.8]$



(h) Function f_{30} on $[1.8; 2.2]$



(i) Function f_{30} on $[2.2; \pi]$

Figure 4: The oscillatory part of f_{30} on $\left[0; \frac{\pi}{2}\right]$ and the increasing part on $\left[\frac{\pi}{2}; \pi\right]$

Corollary 6.2. *Let* $R \in \mathbb{R}^+ - (2\mathbb{Z}+1)\pi$ *and* $\delta \in]0;\pi]$.
Let us also consider the integer n *such that:*

$$(2n-1)\pi + \delta \leqslant R \leqslant (2n+1)\pi - \delta \ . \tag{187}$$

Therefore, if $z \in \mathcal{C}(R,0)$ *is such that* $\Re e\ (z) \leqslant 0$, *then, we have:*

$$\left|\frac{1}{1+e^{-z}}\right| \leqslant \frac{1}{2\sin\left(\dfrac{\delta}{2}\right)} \ . \tag{188}$$

Proof. We remind that, if $z = Re^{i\theta}$, $\theta \in \left[-\pi; \dfrac{-\pi}{2}\right] \cup \left[\dfrac{\pi}{2};\pi\right]$, we have:

$$\begin{aligned}\left|\frac{1}{1+e^{-z}}\right| &= \frac{1}{\sqrt{g_R(\theta)}} \leqslant \frac{1}{\sqrt{g_R\left(\dfrac{\pi}{2}\right)}} = \frac{1}{\sqrt{2(1+\cos R)}} \\ &\leqslant \frac{1}{\sqrt{2(1+\cos(\pi-\delta))}} = \frac{1}{2\sin\left(\dfrac{\delta}{2}\right)} \ ,\end{aligned} \tag{189}$$

according to the 2π-periodicity of g_R, its parity and its increasing caracter on $\left[\dfrac{\pi}{2};\pi\right]$. □

Subdivision of the interval $\left[0;\dfrac{\pi}{2}\right]$**.** To have a more precise idea of the behaviour of f_R on $\left[0;\dfrac{\pi}{2}\right]$, we will split $I = \left[0;\dfrac{\pi}{2}\right]$ in sub-intervals such that $\theta \longmapsto \cos\big(R\sin(\theta)\big)$ has a constant sign on each sub-interval.

- If $R \in \left[0;\dfrac{\pi}{2}\right]$, we just have to define $I_0 = \left[0;\dfrac{\pi}{2}\right] = I$.

- If $R > \dfrac{\pi}{2}$, we first define the positive integer $k_{max}(R)$ by

$$k_{max}(R) = \max\left\{k \in \mathbb{N}\,;\ \left(k+\frac{1}{2}\right)\frac{\pi}{R} \leqslant 1\right\} + 1 = \left\lfloor \frac{R}{\pi} - \frac{1}{2}\right\rfloor + 1 \ . \tag{190}$$

Now, we can split $\left[0;\dfrac{\pi}{2}\right]$:

$$I_0 = \left[0;\arcsin\left(\frac{\pi}{2R}\right)\right] \ . \tag{191}$$

$$I_k = \left[\arcsin\left(\left(k-\frac{1}{2}\right)\frac{\pi}{R}\right);\arcsin\left(\left(k+\frac{1}{2}\right)\frac{\pi}{R}\right)\right] \ ,\ k \in [\![1, k_{max}(R)-1]\!] \ . \tag{192}$$

$$I_{k_{\max}} = \left[\arcsin\left(\left(k_{\max}(R) - \frac{1}{2}\right)\frac{\pi}{R}\right);\frac{\pi}{2}\right] \ . \tag{193}$$

A lower bound of f_R in easy subintervals I_k. With these definitions, $\theta \longmapsto \cos\big(R\sin(\theta)\big)$ is positive on each sub-interval I_k with even indices k, and negative with odd indices k. Consequently, half part of the oscillatory part is easy:

Lemma 6.3. *Let $n \in \mathbb{N}$ and $R \in \,](2n-1)\pi; (2n+1)[\, \cap \mathbb{R}_+^*$.*
Then, $f_R(\theta) \geqslant 1$ for all $\theta \in \bigcup_{\substack{k\in\mathbb{N}\\ 0\leqslant 2k\leqslant k_{max}(R)}} I_{2k}$.

Proof. With the previous notations, it is sufficient to remark that the function $\cos(Rsin)$ is positive on I_{2k} for all integers k such that $0 \leqslant 2k \leqslant k_{max}$. □

Let us now focus when $\theta \in I_1,\ I_3,\ \cdots$, *i.e.* when the function $\cos(R\sin)$ is negative.

Lemma 6.4. *Let $n \in \mathbb{N}$ and $R \in \,](2n-1)\pi; (2n+1)\pi[\, \cap \mathbb{R}_+^*$.*
Then, for all non-negative integers k such that $2k+1 < k_{max}(R)$ and all $\theta \in I_{2k+1}$, we have:

$$f_R(\theta) \geqslant 0.93\ . \tag{194}$$

Proof. Let us fix $n \in \mathbb{N}$, $R \in \,](2n-1)\pi; (2n+1)[\, \cap \mathbb{R}_+^*$, an integer k satisfying $2k+1 < k_{max}(R)$ and $\theta \in I_{2k+1}$.

Therefore, $\cos\big(R\sin(\theta)\big) \leqslant 0$ and $\cos(\theta) \geqslant 0$, so that:

$$\begin{aligned} f_R(\theta) &\geqslant \sqrt{\left(1-e^{-R\cos\theta}\right)^2} = 1-e^{-R\cos\theta} \\ &\geqslant 1-e^{-R\cos\left(\arcsin\left(\left(2k+\frac{3}{2}\right)\frac{\pi}{R}\right)\right)} \\ &= 1-e^{-\sqrt{R^2-\left(2k+\frac{3}{2}\right)^2\pi^2}} \\ &\geqslant 1-e^{-\sqrt{R^2-\left(k_{max}(R)-\frac{1}{2}\right)^2\pi^2}}\ . \end{aligned} \tag{195}$$

We can now remark that we can assume that $n \in \mathbb{N}^*$. Otherwise, we would have $n = 0$ and $R \in]0;\pi[$. So that $k_{max} \in \{0;1\}$. Consequently, there would be no non-negative integer k such that $2k+1 < k_{max}$.

We therefore have, according to $n \in \mathbb{N}^*$:

- if $R \in \left](2n-1)\pi; \left(2n-\frac{1}{2}\right)\pi\right[$, $k_{max} = 2n-2$ and

$$R^2 - \left(k_{max}-\frac{1}{2}\right)^2\pi^2 \ \geqslant\ \frac{3\pi^2}{2}\left(4n-\frac{7}{2}\right) \geqslant \frac{3\pi^2}{4}\ . \tag{196}$$

- if $R \in \left[\left(2n - \frac{1}{2}\right)\pi; \left(2n + \frac{1}{2}\right)\pi\right[$, $k_{max} = 2n - 1$ and

$$R^2 - \left(k_{max} - \frac{1}{2}\right)^2 \pi^2 \geqslant (4n-2)\pi^2 \geqslant 2\pi^2 \ . \tag{197}$$

- if $R \in \left[\left(2n + \frac{1}{2}\right)\pi; (2n+1)\pi\right[$, $k_{max} = 2n$ and

$$R^2 - \left(k_{max} - \frac{1}{2}\right)^2 \pi^2 \geqslant 4n\pi^2 \geqslant 4\pi^2 \ . \tag{198}$$

Consequently, we have $R^2 - \left(k_{max} - \frac{1}{2}\right)^2 \pi^2 \geqslant \frac{3\pi^2}{4}$, *i.e.*:

$$f(\theta) \geqslant 1 - e^{-\frac{\pi\sqrt{3}}{2}} \geqslant 0.93 \ . \tag{199}$$

□

The remaining subintervals of $\left[0; \frac{\pi}{2}\right]$ to study. If $R \in](2n-1)\pi; (2n+1)\pi[$, the remaining subintervals I_k's of I where we do not already have studied a lower bound of g_R are :

I_{2n-1} if $R \in \left](2n-1)\pi; \left(2n - \frac{1}{2}\right)\pi\right[$ and $n \in \mathbb{N}^*$.

I_{2n+1} if $R \in \left[\left(2n + \frac{1}{2}\right)\pi; (2n+1)\pi\right[$ and $n \in \mathbb{N}$.

Some preliminaries lemma To cover these last two cases, let us state without proof the following two easy lemmas:

Lemma 6.5. *Let* $\alpha \in]-1; 0[$.
Then, the function $x \longmapsto e^{-x}(\cosh x + \alpha)$ *is a decreasing function on* $[0; -\ln|\alpha|]$ *from* $1 + \alpha$ *to* $\frac{1}{2}(1 - \alpha^2)$*, and then is an increasing function on* $[-\ln|\alpha|; +\infty[$ *approaching* $\frac{1}{2}$ *near* $+\infty$.

Lemma 6.6. *The function* $x \longmapsto x - \arccos\left(\frac{1 - \cos x}{2}\right)$ *is an increasing function on* $\mathbb{R}$.

We will also need the following

Lemma 6.7. *For all non-negative integers n, the function*

$$x \longmapsto x^2 - \left(n\pi + \arccos\left(\frac{1-\cos x}{2}\right)\right)^2 \tag{200}$$

is an increasing function on $\left[n\pi; \left(n+\frac{1}{2}\right)\pi\right]$.

Proof. • Let n be an integer. Let us also denote $sg(x)$ the sign of the real number x.

One can easily show that $A : x \longmapsto \arccos\left(\sin^2\left(\frac{x}{2}\right)\right)$ is an increasing function on $[(2n-1)\pi; 2n\pi]$ and decreassing on $[2n\pi; (2n+1)\pi]$.

The opposite of the derivative of A, defined by $Q : x \longmapsto \mathrm{sg}\left(\cos\left(\frac{x}{2}\right)\right) \dfrac{\sin\left(\frac{x}{2}\right)}{\sqrt{1+\sin^2\left(\frac{x}{2}\right)}}$, also turns out to be increasing on $](4n-1)\pi; (4n+1)\pi[$ and on $](4n+1)\pi; (4n+3)\pi[$, according to the increasing caracter of $x \longmapsto \dfrac{x}{\sqrt{1+x^2}}$ on $\mathbb{R}$.

• Let us now denote by f_n the function defined on $\left[n\pi; \left(n+\frac{1}{2}\right)\right]$ by:

$$f_n(x) = x^2 - \left(n\pi + \arccos\left(\sin^2\left(\frac{x}{2}\right)\right)\right)^2 = x^2 - \left(n\pi + A(x)\right)^2 . \tag{201}$$

This function is $\mathcal{C}^1$ on $\left[n\pi; \left(n+\frac{1}{2}\right)\right]$ for all nonnegative integers n, and we have, if $x \notin \frac{\pi}{2} + \pi\mathbb{Z}$:

$$f_n'(x) = 2x + 2\left(n\pi + A(x)\right) Q(x) . \tag{202}$$

$$= 2\left(n\pi + A(x)\right)\left(\frac{x}{n\pi + A(x)} + Q(x)\right) . \tag{203}$$

On $\left[2n\pi; \left(2n+\frac{1}{2}\right)\pi\right[$, A is a decreasing function while Q is an increasing one. Therefore, according to Equation (203), we have on this interval:

$$f_{2n}'(x) \geqslant 2\left(2n\pi + A(x)\right)\left(\frac{2n\pi}{n\pi + A(2n\pi)} + Q(2n\pi)\right)$$

$$= 2\left(2n\pi + A(x)\right)\frac{2n\pi}{n\pi + \frac{\pi}{2}} \geqslant \frac{8n^2\pi^2}{n\pi + \frac{\pi}{2}} \geqslant 0 , \tag{204}$$

for all $x \in \left[2n\pi; \left(2n+\frac{1}{2}\right)\pi\right[$.

On $\left[(2n+1)\pi; \left(2n+\frac{3}{2}\right)\pi\right[$, A and Q are increasing function. Therefore, according to Equation (202), we have on this interval:

$$\begin{aligned} f'_{2n+1}(x) &\geqslant 2x + 2\Big((2n+1)\pi + A\big((2n+1)\pi\big)\Big) \cdot \lim_{t \longrightarrow \left((2n+1)\pi\right)^+} Q(t) \\ &= 2(2n+1)\pi - \sqrt{2}(2n+1)\pi \geqslant 0 \ . \end{aligned} \tag{205}$$

Consequently, for all non-negative integers n, f'_n is positive on $\left[n\pi; \left(n+\frac{1}{2}\right)\pi\right]$, so that f_n is finally increasing on this interval. □

Therefore, we have:

Corollary 6.8. *Let* $\delta \in \left]0; \frac{\pi}{2}\right[$.

1. *If* $n \in \mathbb{N}^*$, $R \in \left[(2n-1)\pi + \delta; \left(2n-\frac{1}{2}\right)\pi\right[$ *and* $\theta \in I_{2n-1}$, *we then have:*

$$f_R(\theta) \geqslant \min\left(1 - e^{-\delta}; \sqrt{1 - \cos^4\left(\frac{\delta}{2}\right)}\right) \ . \tag{206}$$

2. *If* $n \in \mathbb{N}$, $R \in \left]\left(2n+\frac{1}{2}\right)\pi; (2n+1)\pi - \delta\right]$ *and* $\theta \in I_{2n+1}$, *we then have:*

$$f_R(\theta) \geqslant \sin\delta \ . \tag{207}$$

Proof. 1. Let us consider first $n \in \mathbb{N}$ and $R \in \left]\left(2n+\frac{1}{2}\right)\pi; (2n+1)\pi - \delta\right]$.

For all $\theta \in I_{2n+1} = \left[\arcsin\left(\left(2n+\frac{1}{2}\right)\frac{\pi}{R}\right); \frac{\pi}{2}\right]$, we therefore have:

$$2n\pi + \frac{\pi}{2} \leqslant R\sin\theta \leqslant R < (2n+1)\pi \ , \tag{208}$$

which directly implies that

$$\cos R \leqslant \cos(R\sin\theta) \leqslant 0 \ . \tag{209}$$

Consequently, we successively have:

$$\begin{aligned} g_R(\theta) &= 2e^{-R\cos\theta}\big(\cosh(R\cos\theta) + \cos(R\sin\theta)\big) \\ &\geqslant 2e^{-R\cos\theta}\big(\cosh(R\cos\theta) + \cos(R)\big) \\ &\geqslant 1 - \cos^2 R, \text{ according to Lemma 6.5, since } R\cos\theta \geqslant 0 \text{ and} \\ &\qquad -1 < \cos R < 0 \ . \end{aligned}$$

Therefore, $f_R(\theta) \geqslant \sin R \geqslant \sin\delta$ for all $\theta \in I_{2n+1}$.

2. Then, let us consider $n \in \mathbb{N}^*$ and $R \in \left[(2n-1)\pi + \delta; \left(2n - \frac{1}{2}\right)\pi\right[$.

   The method, here, is to use the simple idea developed in the previous part of the proof. Nevertheless, we will have to split in three part the sub-interval $I_{2n-1} = \left[\arcsin\left(\left(2n - \frac{3}{2}\right)\frac{\pi}{R}\right); \frac{\pi}{2}\right]$.

   On the interval $J = \left[\arcsin\left((2n-1)\frac{\pi}{R}\right); \frac{\pi}{2}\right]$, $\cos(R\sin)$ is an increasing function from -1 to $\cos R < 0$. Let $\theta_0 \in J \subset I_{2n-1}$ be the unique solution on J of the equation $\cos\big(R\sin(\theta)\big) = \frac{-1+\cos R}{2} = -\sin^2\left(\frac{R}{2}\right)$:

$$\theta_0 = \arcsin\left(\frac{1}{R}\Big((2n-1)\pi + \arccos\big(\sin^2\frac{R}{2})\Big)\right) \ , \tag{210}$$

   which is a well defined quantity, according to Lemma 6.6.

   - For $\theta \in \left[\arcsin\left(\left(2n-\frac{3}{2}\right)\frac{\pi}{R}\right); \arcsin\left(\left(2n-1\right)\frac{\pi}{R}\right)\right]$, we have:

$$\begin{aligned} R\cos\theta &\geqslant \sqrt{R^2 - R^2\sin^2\theta} \\ &\geqslant \sqrt{R^2 - (2n-1)^2\pi^2} \\ &\geqslant \sqrt{\big((2n+1)\pi+\delta\big)^2 - (2n+1)^2\pi^2} \geqslant \delta \ . \end{aligned} \tag{211}$$

   Therefore, we have:

$$f_R(\theta) \geqslant \sqrt{(1-e^{-R\cos\theta})^2} \geqslant 1 - e^{-\delta} \ . \tag{212}$$

   - For $\theta \in \left[\arcsin\left(\left(2n-1\right)\frac{\pi}{R}\right); \theta_0\right]$, we have:

$$\begin{aligned} R\cos\theta &\geqslant R\cos\theta_0 = \sqrt{R^2 - R^2\sin^2\theta_0} \\ &= \sqrt{R^2 - \left((2n-1)\pi + \arccos\left(\sin^2\big(\frac{R}{2}\big)\right)\right)^2} \\ &\geqslant \sqrt{((2n-1)\pi+\delta)^2 - \left((2n-1)\pi + \arccos\left(\sin^2\big(n\pi - \frac{\pi}{2} + \frac{\delta}{2}\big)\right)\right)^2} \ , \\ &\qquad \text{according to Lemma 6.7} \\ &= \sqrt{((2n-1)\pi+\delta)^2 - \left((2n-1)\pi + \frac{\delta}{2}\right)^2} = \sqrt{(2n-1)\delta\pi + \frac{3\delta^2}{4}} \\ &\geqslant \sqrt{\delta\pi} \geqslant \sqrt{\delta^2} = \delta \ . \end{aligned} \tag{213}$$

Therefore, we have:

$$f_R(\theta) \geqslant \sqrt{\left(1-e^{-R\cos\theta}\right)^2} = 1-e^{-R\cos\theta} \geqslant 1-e^{-\delta} \ . \qquad (214)$$

- As in the first part of the proof, for all $\theta \in \left[\theta_0; \frac{\pi}{2}\right]$, we have:

$$-\sin^2\left(\frac{R}{2}\right) \leqslant \cos\left(R\sin\theta\right) \leqslant \cos R < 0 \ . \qquad (215)$$

Consequently, we successively have:

$$\begin{aligned} g_R(\theta) &= 2e^{-R\cos\theta}\big(\cosh(R\cos\theta)+\cos(R\sin\theta)\big) \\ &\geqslant 2e^{-R\cos\theta}\big(\cosh(R\cos\theta)-\sin^2\left(\frac{R}{2}\right)\big) \\ &\geqslant 1-\sin^4\left(\frac{R}{2}\right), \text{ according to Lemma 6.5, since } \begin{cases} R\cos\theta \geqslant 0 \\ -1 < -\sin^2\left(\frac{R}{2}\right) < 0 \end{cases} \\ &\geqslant 1-\cos^4\left(\frac{\delta}{2}\right) \ . \end{aligned}$$

Therefore, we have: $f_R(\theta) \geqslant \sqrt{1-\cos^4\left(\frac{\delta}{2}\right)}$ for all $\theta \in \left[\theta_0; \frac{\pi}{2}\right]$.

Finally, unifying all the results on the three sub-intervals of I_{2n-1}, we have proven that

$$f_R(\theta) \geqslant \min\left(1-e^{-\delta}; \sqrt{1-\cos^4\left(\frac{\delta}{2}\right)}\right) \text{ for all } \theta \in I_{2n-1} \ . \qquad (216)$$

□

The inequalities proven in Corollary 6.8 are more and more sharper when R approaches π modulo 2π. We refer to Figure 5 to see how these inequalities are accurate for values of $R = 3\pi - \delta$ (first array) and $R = 3\pi + \delta$ (second array) and conclude that Inequality (207) is more precise than Inequality (206).

The oscillatory part of f_R. Nevertheless, Inequality (206) will be sufficient to derive a uniform (in the parameter R) upper bound of the function f_R on the sub-interval $\left[0; \frac{\pi}{2}\right]$, *i.e.* on its oscillatory part.

Corollary 6.9. *Let $R \in \mathbb{R}^+ - (2\mathbb{Z}+1)\pi$, and $\delta \in \,]0;\pi]$.*
Let us also consider the integer n such that:

$$(2n-1)\pi + \delta \leqslant R \leqslant (2n+1)\pi - \delta \ . \qquad (217)$$

R	7.9	8.5	9	9.3	9.42
δ	1.5248	0.9248	0.4248	0.1248	0.0048
$\inf_{[0;\frac{\pi}{2}]} f_R$	0.9992	0.8069	0.4126	0.1246	0.00477796
$\sin(\delta)$	0.9989	0.7985	0.4121	0.1245	0.00477794
relative error	0.03 %	1.04 %	0.12 %	0.08 %	0.0004 %

R	10.9	10.3	9.8	9.5	9.43
δ	1.4752	0.8752	0.3752	0.07522	0.005222039
$\inf_{[0;\frac{\pi}{2}]} f_R$	0.9709	0.7611	0.3662	0.07520	0.005222033
$min\left(1-e^{-\delta};\sqrt{1-\cos^4\frac{\delta}{2}}\right)$	0.7713	0.5718	0.2615	0.0532	0.0037
relative error	20.55 %	24.88 %	28.59 %	29.26 %	28.85 %

Figure 5: Examples of tests of inequalities obtained in the cases 1 and 2 of Corollary 6.8.

Therefore, if $z \in \mathcal{C}(R,0)$ is such that $\Re e\ (z) \geqslant 0$, then, we have:

$$\left|\frac{1}{1+e^{-z}}\right| \leqslant \frac{1}{\min\left(0.93; \sin\delta; 1-e^{-\delta}; \sqrt{1-\cos^4\left(\frac{\delta}{2}\right)}\right)} \ . \tag{218}$$

Proof. • Let us assume that $\delta \in]0;\pi]$.

If $(2n-1)\pi+\delta \leqslant R < \left(2n-\frac{1}{2}\right)\pi$, $\left[0;\frac{\pi}{2}\right]$ is subdivided using the intervals I_0, I_1, $\cdots$, I_{2n-1}. From Lemmas 6.3 and 6.4, as well as the first part of Corollary 6.8, we have for all $\theta \in \left[0;\frac{\pi}{2}\right]$:

$$f_R(\theta) \geqslant \min\left(0.93; 1-e^{-\delta}; \sqrt{1-\cos^4\left(\frac{\delta}{2}\right)}\right) \ . \tag{219}$$

If now $\left(2n-\frac{1}{2}\right)\pi \leqslant R < \left(2n+\frac{1}{2}\right)\pi$, $\left[0;\frac{\pi}{2}\right]$ is subdivided using the intervals I_0, I_1, $\cdots$, I_{2n-1}, I_{2n}. From Lemmas 6.3 and 6.4, we have for all $\theta \in \left[0;\frac{\pi}{2}\right]$:

$$f_R(\theta) \geqslant 0.93 \ . \tag{220}$$

Finally, if $\left(2n+\frac{1}{2}\right)\pi \leqslant R < (2n+1)\pi-\delta$, $\left[0;\frac{\pi}{2}\right]$ is subdivided using the intervals I_0, I_1, $\cdots$, I_{2n+1}. From Lemmas 6.3 and 6.4, as well as the first part of Corollary 6.8, we have for all $\theta \in \left[0; \frac{\pi}{2}\right]$:

$$f_R(\theta) \geqslant \min\left(0.93; \sin(\delta)\right) \ . \tag{221}$$

• As a conclusion, grouping the results of Equations (219) to (221) gives us for all $\delta \in]0;\pi]$ and all $\theta \in \left[0; \frac{\pi}{2}\right]$:

$$f_R(\theta) \geqslant \min\left(0.93; \sin\delta; 1 - e^{-\delta}; \sqrt{1 - \cos^4\left(\frac{\delta}{2}\right)}\right) \ . \tag{222}$$

As a conclusion of the proof, if $z = Re^{i\theta}$, $\Re e \ z > 0$, we have $\theta \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$. Therefore, we have:

$$\begin{aligned} \left|\frac{1}{1+e^{-z}}\right| &= \frac{1}{|f_R(\theta)|} = \frac{1}{\left|f_R(|\theta|)\right|} \\ &\leqslant \frac{1}{\min\left(0.93; \sin\delta; 1 - e^{-\delta}; \sqrt{1 - \cos^4\left(\frac{\delta}{2}\right)}\right)} \ . \end{aligned} \tag{223}$$

□

Conclusion of the proof of Proposition 3.2. To conclude the proof of Proposition 3.2, we need now to put together Corollary 6.9 and 6.2. All we have to do is now to remark that, for $x \in [0;\pi]$:

$$2\sin\left(\frac{x}{2}\right) \geqslant \sin x \ . \tag{224}$$

References

[1] M. Abramowitz, I. Stegun (eds.) : *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1972.

[2] O. Bouillot : *Phase Portraits of Bi-dimensional Zeta Values*, *in*: A. M. Bigatti, J. Carette, J. H. Davenport, M. Joswig, T. de Wolff (eds.), *ICMS* 2020., LNCS, vol. 12097, p. 393 − 405, Springer, Cham (2020).

[3] P. J. David, P. Rabinowitz : *Methods of Numerical Integration*, 2nd edition, Academic Press, New York, 1984.

[4] P. Flajolet, R. Sedgewick : *Analytic Combinatorics*, Cambridge University Press, New York, NY, USA, 1st edition, 2009.

[5] W. Gautschi : *A Survey of Gauss-Christoffel Quadrature Formulae*, In: P. L. Butzer, F. Fehér (Eds), *E. B. Christoffel: The Influence of his Work on Mathematics and the Physical Sciences*, Birkhäuser, Basel, 1981, p. 72-147.

[6] E. L. Lindelöf, : *Le calcul des résidus et ses applications à la théorie des fonctions*, Gauthier-Villars, Paris, 1905.

[7] G. Mastroianni, G. Monegato : *Convergence of product integration rules over* $(0, \infty)$ *for functions with weak singularities at the origin*, Math. Comp. 64, (1995), p. 237-249.

[8] G. Mastroianni, J. Szabados : *Polynomial approximation on the real semiaxis with generalized Laguerre weights*, Stud. Univ. Babes-Bolyai Math. , 52 (2007), n°4, p. 105-128.

[9] J. D. Hunter : *Matplotlib: A 2D graphics environment*, *in* Computing in Science & Engineering, vol. 9, n°3, p. 90-95, May-June 2007.

[10] D. S. Lubinsky : *Geometric convergence of Lagrangian interpolation and numerical integration rules over unbounded contours and intervals*, J. Approx. Theory 39 (1983), p. 338-360.

[11] J. McNamee : *Error-bounds for the Evaluation of Integrals by the Euler-Maclaurin Formula and by Gauss-type Formulae*, Math. Comp., 18 (1964), p. 368-381.

[12] J. V. Uspenksy : *On the convergence of quadrature formulas related to an infinite interval*, Trans. Amer. Math. Soc. 30 (1928), p. 542-559.

[13] T. J. Stieljes : *Quelques recherches sur la théorie des quadratures dites mécaniques*, Annales scientifiques de l'É.N.S. 3e série, tome 1 (1884), p. 409-426.

[14] Stroud A. H., Chen K. W. : *Peano Error Estimates for Gauss-Laguerre Quadrature Formulas* SIAM J. Numer. Anal., Vol. 9, n°2, 1972.

[15] G. Szegö : *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1975.

[16] G. van Rossum : *Python tutorial, Technical Report CS-R9526*, Centrum voor Wiskunde en Informatica (CWI), Amsterdam, May 1995.

[17] S. Xiang : *Asymptotics on Laguerre or Hermite polynomial expansions and their applications in Gauss quadrature*, J. Math. Anal. Appl., 393 (2), 2012, p. 434-444.