Mould Calculus - On the secondary symmetries

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Abstract

Mould calculus is a powerful combinatorial tool which often provides some explicit formulae when there are no other available computational methods. It has a well-know interpretation/dictionary in term of Hopf algebras. But this dictionary does not provide any equivalent of formal moulds. Thus, we present here such an interpretation and give a generic way to prove mould symmetries of formal moulds.

Résumé

Le calcul moulien est un outil combinatoire puissant qui fournit souvent des formules explicites alors que d'autres moyens de calcul n'aboutissent pas. Il en existe une interprétation/un dictionnaire en termes d'algèbres de Hopf. Mais ce dictionnaire n'a pas été développé jusqu'aux moules formels. Nous présentons ici une telle interprétation et donnons alors une méthode générique permettant de prouver les symétries de moules formels.

Key words: Mould calculus, mould symmetries, formal moulds, Hopf algebra.

Version française abrégée

Dans tout ce qui suit, C désignera une algèbre commutative, Ω sera un ensemble et $\Omega^\star$ sera le monoïde libre sur Ω (*i.e.* l'ensemble des séquences ou mots construits sur Ω, voir [14]).

Ecalle définit souvent un *moule* comme étant "une fonction à un nombre variable de variables" (cf. [8], [9] par exemple, ou la préface de [6]). Plus précisément, on peut définir un moule comme une fonction définie sur $\Omega^\star$ et à valeurs dans C. Un moule générique est noté $M^\bullet$ alors que son évaluation sur une séquence $\underline{\omega}$ est notée $M^{\underline{\omega}}$.

Si $⧢$ (resp. $\uplus$) désignent le produit de mélange (resp. mélange contractant) des séquences (cf. [8] par exemple, ou [2], [3], [11], [13]), Ecalle définit la notion de moule symétr***a***l (resp. symétr***e***l) et altern***a***l (resp. altern***e***l) par (cf. [8], [9] et [10], ou encore [2], [3], [6] et [16]) :

$$\forall (\underline{\omega}^1\,;\,\underline{\omega}^2) \in (\Omega^\star)^2\ , \sum_{\substack{\underline{\omega} \text{ apparaît dans } \underline{\omega}^1 ⧢ \underline{\omega}^2 \\ (\text{resp. } \underline{\omega} \text{ apparaît dans } \underline{\omega}^1 \uplus \underline{\omega}^2)}} M^{\underline{\omega}} = \begin{cases} M^{\underline{\omega}^1} M^{\underline{\omega}^2} & \rightsquigarrow \text{ symétr}\underline{\boldsymbol{a}}\text{l (resp. symétr}\underline{\boldsymbol{e}}\text{l)}, \\ 0 & \rightsquigarrow \text{ altern}\underline{\boldsymbol{a}}\text{l (resp. altern}\underline{\boldsymbol{e}}\text{l)}. \end{cases} \quad (1)$$

Étant donné un moule $M^\bullet$, Ecalle considère souvent le moule des séries génératrices ordinaires de $M^\bullet$ (cf. [9], §8, ou [10], §1.2 par exemple) que nous appellerons ici $Mog^\bullet$ et définit par (13). Nous associerons aussi à $M^\bullet$ le moule $Meg^\bullet$ construit comme étant les séries génératrices exponentielles de $M^\bullet$.

Jean Ecalle affirme alors le résultat suivant, que nous complétons avec le théorème 2 :

Théorème 1. *Soit $M^\bullet$ un moule défini sur un alphabet dénombrable.*

*Alors : (i.) $M^\bullet$ est symétr**a**l $\Longleftrightarrow$ $Mog^\bullet$ est symétr**a**l . (iii.) $M^\bullet$ est symétr**e**l $\Longleftrightarrow$ $Mog^\bullet$ est symétr**i**l .*

*(ii.) $M^\bullet$ est altern**a**l $\Longleftrightarrow$ $Mog^\bullet$ est altern**a**l . (iv.) $M^\bullet$ est altern**e**l $\Longleftrightarrow$ $Mog^\bullet$ est altern**i**l .*

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Théorème 2. *Soit $M^\bullet$ un moule défini sur un alphabet dénombrable.*

Alors : (i.) $M^\bullet$ est symétr$\underline{\boldsymbol{a}}$l $\Longleftrightarrow$ $Meg^\bullet$ est symétr$\underline{\boldsymbol{a}}$l . (iii.) $M^\bullet$ est symétr$\underline{\boldsymbol{e}}$l $\Longleftrightarrow$ $Meg^\bullet$ est symétr$\underline{\boldsymbol{e}}$l .

(ii.) $M^\bullet$ est altern$\underline{\boldsymbol{a}}$l $\Longleftrightarrow$ $Meg^\bullet$ est altern$\underline{\boldsymbol{a}}$l . (iv.) $M^\bullet$ est altern$\underline{\boldsymbol{e}}$l $\Longleftrightarrow$ $Meg^\bullet$ est altern$\underline{\boldsymbol{e}}$l .

Nous proposons ici une preuve de ces deux résultats basée sur la notion de contraction moule/comoule formelle et de ses interprétations en termes d'algèbres de Hopf (cf. point (i) à (v) à la fin de la note). Une telle preuve est une machinerie très puissante pour obtenir instantanément des théorèmes similaires à ceux-ci.

English version

1. Definition of moulds

In all this note, C is a commutative algebra, O a C-algebra, Ω an alphabet and $\Omega^\star$ the free monoid over Ω (*i.e.*, the set of sequences or words over Ω, see [14]).

As a concrete definition, Ecalle often defines a mould as "a function with a variable number of variables" (see [8], [9] or the preface of [6]). They have been first introduced extensively in [7] and are also introduced in detail in [8], [9] or [2], [3], [6] or [16]. More precisely, the following definitions are equivalent.

Definition 1.1. *A* mould *is a function defined over the set $\Omega^\star$ of (finite) sequences (or words) over Ω (or sometimes over a subset of $\Omega^\star$) with values in the algebra C.*

Definition 1.2. *A* mould *is a collection of functions $(f_0, f_1, f_2, \cdots)$ where f_0 is a constant and for all integers n, f_n is a function of n variables defined on Ω^n (or over a subset of Ω^n) and valued in C.*

Since we want to mix easily index and exponent in notations and want to understand at first sight the type of object we are currently dealing with, we need some specific notations for moulds. The following ones turn out to be quite useful conventions:

(i) Sequences are always underlined, with an upper indexation if necessary. We call length of $\underline{\omega}$ and denote by $l(\underline{\omega})$ the number of elements of $\underline{\omega}$. The empty sequence (*i.e.* the sequence of length 0) is denoted by $\emptyset$. Note that the letter r is generically reserved to indicate the length of sequences.

(ii) A generic mould M is actually denoted by $M^\bullet$.

(iii) For a mould $M^\bullet$, we will prefer the notation $M^{\underline{\omega}}$ to the functional notation that would have been $M(\underline{\omega})$: it indicates the evaluation of the mould $M^\bullet$ on the sequence $\underline{\omega}$ of $\Omega^\star$.

Since a mould is a function, the classical operations on functions are extended to moulds (see [8]-[10]):

Proposition 1.3. *The set of all moulds defined over Ω and valued in C, endowed with the mould operations is a noncommutative, associative, unitary C-algebra, where the operations on moulds are defined by:*

$$\begin{cases} \text{Addition: } S^\bullet = M^\bullet + N^\bullet \iff \forall \underline{\omega} \in \Omega^\star \ , \ S^{\underline{\omega}} = M^{\underline{\omega}} + N^{\underline{\omega}} \ . \\ \text{Scalar multiplication: } (\lambda M)^\bullet = \lambda \cdot M^\bullet \iff \forall \underline{\omega} \in \Omega^\star \ , \ (\lambda M)^{\underline{\omega}} = \lambda \, M^{\underline{\omega}} \ . \\ \text{Mould multiplication: } P^\bullet = M^\bullet \times N^\bullet \iff \forall \underline{\omega} \in \Omega^\star \ , \ P^{\underline{\omega}} = \displaystyle\sum_{\substack{(\underline{\omega}^1;\, \underline{\omega}^2) \in (\Omega^\star)^2 \\ \underline{\omega}^1 \cdot \underline{\omega}^2 = \underline{\omega}}} M^{\underline{\omega}^1} N^{\underline{\omega}^2} \ . \end{cases} \tag{2}$$

2. Mould-comould contraction

For analytical reasons, moulds can be contracted with dual objects, called comoulds (see [1], [8] or [16]):

Definition 2.1. *A* comould *is an homomorphisms defined over $\Omega^\star$ (or over a subset of $\Omega^\star$), valued in a* C*-algebra* O.

It turns out that comoulds are actually functions with a variable number of variables and can be seen as some moulds. Nevertheless, we emphasize the slight differences with moulds using another name:

(i) moulds are valued in a commutative algebra C, while co-moulds are restrictively valued in a C-algebra O (which is possibly a noncommutative algebra)

(ii) moulds are interpreted as *coefficients* while co-moulds are interpreted as *operators*: the target algebra O of a comould is an algebra of a different type than the target algebra C of a mould.

(iii) a mould is any *map* $\Omega^\star \longmapsto \mathsf{C}$ while a comould is any *homomorphism* $\Omega^\star \longmapsto \mathsf{O}$.

Definition 2.2. *The* mould-comould contraction *of a mould $M^\bullet$ defined over Ω and valued in C, and a comould $B_\bullet$ defined over Ω and valued in O is defined by:*

$$\sum_\bullet M^\bullet B_\bullet := \sum_{\underline{\omega}\in\Omega^\star} M^{\underline{\omega}} B_{\underline{\omega}} \qquad \textit{(if the sum is well-defined)} \tag{3}$$

A mould-comould contraction might be understood to be an *algebra automorphism* or a *derivation* for analytical reasons (see [8] or [5], [15]). Consequently, all the definitions from mould calculus come from such an interpretation, in particular the definitions of mould operations and mould symmetries. As we do not want to make some analysis here, let us define a new notion, the one of *formal mould-comould contraction*, which is an element of a free Lie algebra (see [14]) that can be specialized when necessary to a mould-comould contraction:

Definition 2.3. *Let $M^\bullet$ be a mould defined over Ω valued in C.*
For each letter $\omega \in \Omega$, we define a symbol a_ω, such that the symbols $(a_\omega)_{\omega\in\Omega}$ do not commute. Let us extend the symbols a_ω to words by concatenation: $a_{\underline{\omega}} = a_{\omega_1\cdots\omega_r} := a_{\omega_1}\cdots a_{\omega_r}$ for all $\underline{\omega} = \omega_1\cdots\omega_r \in \Omega^\star$.
Then the formal mould-comould contraction *of a mould $M^\bullet$, defined over Ω and valued in C, is the formal series $s(M^\bullet) \in \mathbf{C}\langle\langle\mathsf{A}\rangle\rangle$, where $\mathsf{A} = \{a_\omega \ ; \ \omega\in\Omega\}$, defined by:*

$$s(M^\bullet) = \sum_{\underline{\omega}\in\Omega^\star} M^{\underline{\omega}}\ a_{\underline{\omega}} := \sum_\bullet M^\bullet\ a_\bullet\ . \tag{4}$$

Notice first that, in a formal mould-comould contraction, the words over A play the role of the comoulds: if $B_\bullet$ is a comould and φ the specialization morphism defined by $\varphi(a_{\underline{\omega}}) = B_{\underline{\omega}}$, for all word $\underline{\omega}\in\Omega^\star$, then

$$\varphi\Big(s(M^\bullet)\Big) = \sum_\bullet M^\bullet\ B_\bullet \text{ is a mould-comould contraction.} \tag{5}$$

Notice then that this definition is the necessary background to understand the mould definitions and operations. As an example, the mould product is defined to satisfy in $\mathbf{C}\langle\langle\mathsf{A}\rangle\rangle$ for all moulds $M^\bullet$ and $N^\bullet$:

$$\sum_\bullet (M^\bullet \times N^\bullet)\ a_\bullet = \Big(\sum_\bullet M^\bullet\ a_\bullet\Big)\Big(\sum_\bullet N^\bullet\ a_\bullet\Big)\ . \tag{6}$$

3. Primary symmetries

In practice (see for example [8] or [1], [16], [15]), a comould $B_\bullet$ often satisfies some "*Leibnitz rules*". The more commons are defined for all $\omega\in\Omega$ by $B_\omega(\varphi\psi) = B_\omega(\varphi)\psi + \varphi B_\omega(\psi)$ or $B_\omega(\varphi\psi) = \sum\limits_{\substack{\omega_1,\omega_2\in\Omega\\ \omega_1\perp\omega_2=\omega}} B_{\omega_1}(\varphi)B_{\omega_2}(\psi)$, if $(\Omega,\perp)$ is a semi-group. These rules conduct to two coproducts $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\underline{\sqcup\!\sqcup}}$ defined for all $a_\omega\in\mathsf{A}$ by:

$$\Delta_{\sqcup\!\sqcup}(a_\omega) = a_\omega\otimes 1 + 1\otimes a_\omega \qquad\qquad \Delta_{\underline{\sqcup\!\sqcup}}(a_\omega) = \sum_{\substack{\omega_1,\omega_2\in\Omega\\ \omega_1\perp\omega_2=\omega}} a_{\omega_1}\otimes a_{\omega_2} \text{ if } (\Omega,\perp) \text{ is a semi-group.} \tag{7}$$

We extend these coproducts as an algebra homomorphism. Consequently, $(\mathbf{C}\langle\langle \mathsf{A}\rangle\rangle, \cdot, \Delta)$ is a Hopf algebra when $\Delta = \Delta_{\sqcup\!\sqcup}$ or $\Delta_{\uplus}$. These two coproducts are respectively the dual of the shuffle product and stuffle product (see [8] for example, or [2], [3], [11], [13]):

$$\forall \underline{\omega} \in \Omega^\star \ , \ \Delta_{\sqcup\!\sqcup}(a_{\underline{\omega}}) = \sum_{\substack{\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\star \\ \underline{\omega} \text{ appears in } \underline{\omega}^1 \sqcup\!\sqcup\, \underline{\omega}^2}} a_{\underline{\omega}^1} \otimes a_{\underline{\omega}^2} \quad \text{and} \quad \Delta_{\uplus}(a_{\underline{\omega}}) = \sum_{\substack{\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\star \\ \underline{\omega} \text{ appears in } \underline{\omega}^1 \uplus \underline{\omega}^2}} a_{\underline{\omega}^1} \otimes a_{\underline{\omega}^2} \ . \tag{8}$$

There exist some sufficient conditions for a contraction to be an automorphism or a derivation of the algebra O.

Proposition 3.1. *Let us consider a mould $M^\bullet$ and a comould $B_\bullet$ both defined over $\Omega^\star$ and respectively valued in C and O.*
Let us suppose that the comould $B_\bullet$ satisfies one of the following "Leibniz rules":

$$B_\omega(\varphi\psi) = B_\omega(\varphi)\psi + \varphi B_\omega(\psi) \qquad \textit{or} \qquad B_\omega(\varphi\psi) = \sum_{\substack{\omega_1, \omega_2 \in \Omega \\ \omega_1 \perp \omega_2 = \omega}} B_{\omega_1}(\varphi) B_{\omega_2}(\psi)$$

and let us consider Φ the specialization morphism defined by $\Phi(\underline{\omega}) = B_{\underline{\omega}}$ for all $\underline{\omega} \in \Omega^\star$.
The coproduct $\Delta = \Delta_{\sqcup\!\sqcup}$ or $\Delta_{\uplus}$ being respectively associated to the Leibniz rule satisfied by $B_\bullet$, we have:

- *(i) if $s(M^\bullet)$ is a group-like element of $(\mathbf{C}\langle\langle \mathsf{A}\rangle\rangle, \cdot, \Delta)$, then $\Phi\left(s(M^\bullet)\right)$ is an automorphism.*
- *(ii) if $s(M^\bullet)$ is a primitive element of $(\mathbf{C}\langle\langle \mathsf{A}\rangle\rangle, \cdot, \Delta)$, then $\Phi\left(s(M^\bullet)\right)$ is a derivation.*

Proposition 3.2. *A mould $M^\bullet$ defined over $\Omega^\star$ and valued in C satisfies:*

- *(i) $s(M^\bullet)$ is a group-like element of $(\mathbf{C}\langle\langle \mathsf{A}\rangle\rangle, \cdot, \Delta)$ where $\Delta = \Delta_{\sqcup\!\sqcup}$ (resp. $\Delta = \Delta_{\uplus}$) if, and only if*

$$\forall (\underline{\omega}^1 \,;\, \underline{\omega}^2) \in (\Omega^\star)^2 \ , \sum_{\substack{\underline{\omega} \textit{ appears in } \underline{\omega}^1 \sqcup\!\sqcup\, \underline{\omega}^2 \\ (\textit{resp. } \underline{\omega} \textit{ appears in } \underline{\omega}^1 \uplus \underline{\omega}^2)}} M^{\underline{\omega}} = M^{\underline{\omega}^1} M^{\underline{\omega}^2} \ . \tag{9}$$

- *(ii) $s(M^\bullet)$ is a primitive element of $(\mathbf{C}\langle\langle \mathsf{A}\rangle\rangle, \cdot, \Delta)$ where $\Delta = \Delta_{\sqcup\!\sqcup}$ (resp. $\Delta = \Delta_{\uplus}$) if, and only if*

$$M^\emptyset = 0 \textit{ and for all } \underline{\omega}^1 \ , \underline{\omega}^2 \in \Omega^\star - \{\emptyset\} \ , \sum_{\substack{\underline{\omega} \textit{ appears in } \underline{\omega}^1 \sqcup\!\sqcup\, \underline{\omega}^2 \\ (\textit{resp.} \underline{\omega} \textit{ appears in } \underline{\omega}^1 \uplus \underline{\omega}^2)}} M^{\underline{\omega}} = 0 \ . \tag{10}$$

These two propositions give nice motivations to the following definition/terminology (see [8], [9], [10] or [2], [3], [6] or [16]). Other similar definitions exist (in particular for a symmetr***i̲***l mould, or an altern***i̲***l mould which are in a sense similar to symmetr***e̲***l and altern***e̲***l moulds, see [10] or [1], [6], [12]).

Definition 3.3. *A mould $M^\bullet$ defined over $\Omega^\star$ and valued in C is called:*

- *(i)* symmetr***a̲***l *(resp.* symmetr***e̲***l *) when:*

$$\forall (\underline{\omega}^1 \,;\, \underline{\omega}^2) \in (\Omega^\star)^2 \ , \sum_{\substack{\underline{\omega} \textit{ appears in } \underline{\omega}^1 \sqcup\!\sqcup\, \underline{\omega}^2 \\ (\textit{resp. } \underline{\omega} \textit{ appears in } \underline{\omega}^1 \uplus \underline{\omega}^2)}} M^{\underline{\omega}} = M^{\underline{\omega}^1} M^{\underline{\omega}^2} \ . \tag{11}$$

- *(ii)* altern***a̲***l *(resp.* altern***e̲***l *) when:*

$$\begin{cases} M^\emptyset = 0 \ . \\ \forall (\underline{\omega}^1 \,;\, \underline{\omega}^2) \in (\Omega^\star - \{\emptyset\})^2 \ , \displaystyle\sum_{\substack{\underline{\omega} \textit{ appears in } \underline{\omega}^1 \sqcup\!\sqcup\, \underline{\omega}^2 \\ (\textit{resp. } \underline{\omega} \textit{ appears in } \underline{\omega}^1 \uplus \underline{\omega}^2)}} M^{\underline{\omega}} = 0 \ . \end{cases} \tag{12}$$

4. Formal moulds

One can restrict Definition 1.2 to formal series to obtain the notion of formal moulds (see [2]), and then consider ordinary/exponential generating series.

Definition 4.1. *A* formal mould *is a collection of formal series* $(S_0, S_1, S_2, \cdots)$ *where* S_0 *is constant and for all integers* n*,* S_n *is a formal power series in* n *indeterminates constructed from the set* Ω *and valued in* C*.*

Definition 4.2. *If* $\Omega = \{\omega_0; \omega_1; \cdots\}$ *is a countable set and* $M^\bullet$ *is a mould defined over* $\Omega^\star$ *and valued in* C*, then we define the formal moulds* $Mog^\bullet$ *and* $Meg^\bullet$ *by:*

$$Mog^{X_1,\cdots,X_r} = \sum_{p_1,\cdots,p_r\in\mathbb{N}} M^{\omega_{p_1},\cdots,\omega_{p_r}} X_1^{p_1}\cdots X_r^{p_r} \qquad Meg^{X_1,\cdots,X_r} = \sum_{s_1,\cdots,s_r\in\mathbb{N}} M^{\omega_{p_1},\cdots,\omega_{p_r}} \frac{X_1^{p_1}}{p_1!}\cdots\frac{X_r^{p_r}}{p_r!} \quad (13)$$

Therefore, a formal mould is in particular a mould. But the main difference is the following. If $M^\bullet$ is a mould defined, for example, over the set $\Omega = \{a, b\}$, then there is a priori no link between $M^{a,b}$ and $M^{b,a}$. On the other hand, if $M^\bullet$ is a formal mould defined over the set $\Omega = \{X; Y\}$ then $M^{X,Y}$ and $M^{Y,X}$ are related by the *substitution* of the indeterminates and M^{X+Y} is also defined using the substitution of the sum of the indeterminates $X + Y$ in the formal series S_1 associated with $M^\bullet$. This remark turns out to be a fundamental one and directly leads to the proofs of Theorem 5.1 and Theorem 6.1.

Moreover, seen as a mould, a formal mould could have some symmetries. Since a formal mould is a special type of mould, we emphasize its particularity by using the following terminology:

Type	mould	formal mould
Name of the symmetries	*primary symmetries*	*secondary symmetries*

5. Main results

One of the results often used by Ecalle (see [9], [10]) is the following:

Theorem 5.1. *Let* $M^\bullet$ *be a mould defined over a countable alphabet valued in a commutative algebra. Then: (i.)* $M^\bullet$ *is symmetr*$\underline{\textbf{a}}$*l* $\Longleftrightarrow$ $Mog^\bullet$ *is symmetr*$\underline{\textbf{a}}$*l . (iii.)* $M^\bullet$ *is symmetr*$\underline{\textbf{e}}$*l* $\Longleftrightarrow$ $Mog^\bullet$ *is symmetr*$\underline{\textbf{i}}$*l .*

(ii.) $M^\bullet$ *is altern*$\underline{\textbf{a}}$*l* $\Longleftrightarrow$ $Mog^\bullet$ *is altern*$\underline{\textbf{a}}$*l . (iv.)* $M^\bullet$ *is altern*$\underline{\textbf{e}}$*l* $\Longleftrightarrow$ $Mog^\bullet$ *is altern*$\underline{\textbf{i}}$*l .*

Our main result is an interpretation in terms of Hopf algebras of the secondary symmetries, similar to the interpretation of the primary symmetries given in Proposition 3.2. This leads to a complete proof of Ecalle's previous statement, as well as a general method to prove similar results. The proof contains five steps:

(i) If $\mathbf{X} = \{X_1, X_2, \cdots\}$ is an infinite set of indeterminates, we consider a new set of indeterminates $\mathbf{Y}$:

$$\mathbf{Y} = \mathbb{N}\mathbf{X} = \left\{ \sum_{x\in\mathbf{X}} \lambda_x x \ ; \ (\lambda_x)_{x\in\mathbf{X}} \in \mathbb{N}^{\mathbf{X}} \text{ has finitely nonzero terms} \right\} .$$

(ii) For each element $y \in \mathbf{Y}$, we define a new symbol A_y such that the symbols $(A_y)_{y\in\mathbf{Y}}$ do not commute. Let us extends them to words by concatenation: $A_{\underline{y}} = A_{y_1\cdots y_r} := A_{y_1}\cdots A_{y_r}$ for all $\underline{y} = y_1\cdots y_r \in \mathbf{Y}^\star$. and consider the new alphabet $\mathcal{A} = \{A_{\underline{y}} \ ; \ \underline{y} \in \mathbf{Y}\}$.

(iii) We define a secondary formal mould/comould contraction, *i.e.* with a formal mould $FM^\bullet$ valued in a commutative algebra C, we associate the series $S(FM^\bullet) \in \mathsf{C}[\![\mathbf{X}]\!]\langle\!\langle\mathcal{A}\rangle\!\rangle$ defined by:

$$S(FM^\bullet) = \sum_{\underline{y}\in\mathbf{Y}^\star} FM^{\underline{y}}\ A_{\underline{y}} := \sum_\bullet FM^\bullet\ A_\bullet\ .$$

Notice that, for all $\underline{y} \in \mathbf{Y}^\star$, $FM^{\underline{y}}$ is well-defined using some substitution of indeterminates.

(iv) We define two coproducts $\Delta_{ш}$ and $\Delta_{\uplus}$ for all $\underline{A} \in \mathcal{A}^\star$ by:

$$\Delta_{ш}(\underline{A}) = \sum_{\substack{\underline{B},\underline{C}\in\mathcal{A}^\star \\ \underline{A}\text{ appears in }\underline{B}ш\underline{C}}} \underline{B}\otimes\underline{C} \qquad\qquad \Delta_{\uplus}(\underline{A}) = \sum_{\substack{\underline{B},\underline{C}\in\mathcal{A}^\star \\ \underline{A}\text{ appears in }\underline{B}\uplus\underline{C}}} \underline{B}\otimes\underline{C}\ .$$

(v) Now, we just have to adapt the proof of Proposition 3.2.

6. Conclusion

Let $M^\bullet$ be a mould, with some primary symmetry. Looking specifically at sequences of small length, it is easy to understand which secondary symmetry is satistied by a formal mould (like $Mog^\bullet$ or $Meg^\bullet$) associated with $M^\bullet$. Using a new set of indeterminates, and using them as the substitution of indeterminates on formal moulds (seen as formal series), we are able to give a proof of the previous statement.

This is done by following the five points of the previous proof. It turns out that these points are actually a nice machinery to obtain quasi-instantly theorems similar to Theorem 5.1, like (see [4]):

Theorem 6.1. *Let $M^\bullet$ be a mould defined over a countable alphabet valued in a commutative algebra C. Then: (i.) $M^\bullet$ is symmetr$\underline{\mathbf{a}}$l $\Longleftrightarrow$ $Meg^\bullet$ is symmetr$\underline{\mathbf{a}}$l . (iii.) $M^\bullet$ is symmetr$\underline{\mathbf{e}}$l $\Longleftrightarrow$ $Meg^\bullet$ is symmetr$\underline{\mathbf{e}}$l .*

(ii.) $M^\bullet$ is altern$\underline{\mathbf{a}}$l $\Longleftrightarrow$ $Meg^\bullet$ is altern$\underline{\mathbf{a}}$l . (iv.) $M^\bullet$ is altern$\underline{\mathbf{e}}$l $\Longleftrightarrow$ $Meg^\bullet$ is altern$\underline{\mathbf{e}}$l .

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