On Hurwitz Multizeta Functions

Olivier Bouillot[1]

Laboratoire d'informatique Gaspard-Monge
Cité Descartes, Bât. Copernic
5, boulevard Descartes
Champs sur Marne
F-77454 Marne-la-Valle Cedex 2

Abstract

We give two new properties of Hurwitz multizeta functions. The first one shows that each Hurwitz multizeta function is an example of a simple resurgent function. The second property deals with the algebra $\mathcal{H}mzf_{cv}$ over $\mathbb{C}$ spanned by the Hurwitz multizeta functions. It turns out that this algebra is a polynomial algebra with no other algebraic relations than those coming from the stuffle product, and thus is isomorphic to $QSym$.

Keywords: Hurwitz multizeta functions, Multizetas values, Resurgence theory, Quasi-symmetric functions, Mould calculus.
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Email address: `olivier.bouillot@u-pem.fr` (Olivier Bouillot)
URL: `http://www-igm.univ-mlv.fr/~bouillot/` (Olivier Bouillot)

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1. Introduction

For any sequence $(s_1, \cdots, s_r) \in (\mathbb{N}^*)^r$, of length $r \in \mathbb{N}^*$ such that $s_1 \geq 2$, we define the Hurwitz multiple zeta function $\mathcal{H}e_+^{s_1, \cdots, s_r}$ over $\mathbb{C} - (-\mathbb{N})$, by

$$\mathcal{H}e_+^{s_1, \cdots, s_r} : z \longmapsto \sum_{0 < n_r < \cdots < n_1} \frac{1}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \ . \tag{1}$$

We also define $\mathcal{H}e_+^{\emptyset} : z \longmapsto 1$.

The condition $s_1 \geq 2$ ensures the convergence of the series and the set of sequences of positive integers satisfying this condition is denoted by $\mathcal{S}^\star$:

$$\mathcal{S}^\star = \{\underline{\mathbf{s}} \in \mathbb{N}_1^\star \, ; \, s_1 \geq 2\} \ , \tag{2}$$

where $\mathbb{N}_1 = \mathbb{N}^*$. The set of sequences of elements $e \in E$ is denoted by $E^\star$.

Let us also recall that a mould is a function defined over a free monoid, or a subset of a free monoid like $\mathcal{S}^\star$ (See Section 2 for more details). Consequently, Equation (1) defines a mould, denoted by $\mathcal{H}e_+^\bullet$, defined over $\mathcal{S}^\star$ and called the mould of Hurwitz multiple zeta functions (Hurwitz multizeta functions for short).

These functions are a generalization of the classical Hurwitz function

$$\zeta(s, z) = \sum_{n > 0} \frac{1}{(n+z)^s} \ , \tag{3}$$

and, up to the knowlegde of the author, appeared in the literature at the beginning of the XXIth century. The reader can find them under many others names, like "mono-center Hurwitz polyzeta" in [25], [42], "Hurwitz type of Euler-Zagier multiple zeta functions" (or also "Hurwitz-Lerch type of Euler-Zagier multiple zeta functions") in [27], [44], [45], and [46].

Today, these functions appear in different mathematical fields like:

1. the theory of special functions (see [24] for the particular case $r = 2$, [14] for a functional equation related to that of the classical Hurwitz function, [34] for the links between Hurwitz multizeta functions and the Gamma function, [6] for the links with multitangent functions);
2. the multiple Dirichlet series, because they have an easy analytical continuation in the variables s_i's (see [1] when $r = 2$, [35], [41], [43] for a wider class of functions);
3. holomorphic dynamics (see [4] and [6] for the links with analytical invariants);
4. the study of the Riemann Hypothesis (see [44], [45] for the localization of the zeros of the Hurwitz multiple zeta functions);
5. renormalisation theory (see [27], [29], [40] for their values at negative integers and [8] for the link with Bernoulli polynomials);

6. quantum field theory (see [36]).

We also define $\mathcal{H}e_-^{\emptyset} : z \longmapsto 1$ and for any sequence $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$ such that $\overleftarrow{\underline{\mathbf{s}}} = (s_r, \cdots, s_1) \in \mathcal{S}^\star$ the negative Hurwitz multizeta function $\mathcal{H}e_-^{\underline{\mathbf{s}}}$ by:

$$\mathcal{H}e_-^{s_1,\cdots,s_r} : z \longmapsto \sum_{n_r<\cdots<n_1<0} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ . \tag{4}$$

1.1. Classical properties of Hurwitz multizeta functions

For any nonempty sequence $\underline{\mathbf{s}} \in \mathcal{S}^\star$, the function $\mathcal{H}e_+^{s_1,\cdots,s_r}$ is holomorphic over $\mathbb{C} - (-\mathbb{N}^*)$, with derivative given by

$$\frac{d\ \mathcal{H}e_+^{s_1,\cdots,s_r}}{dz} = -\sum_{k=1}^{r} s_k \mathcal{H}e_+^{s_1,\cdots,s_{k-1},s_k+1,s_{k+1},\cdots,s_r} \ . \tag{5}$$

Such a function is also an evaluation of a monomial quasi-symmetric function $M_{s_1,\cdots,s_r}$ (See [28], or [3] for a more recent presentation). Therefore, this family spans an algebra $\mathcal{H}mzf_{cv}$ over $\mathbb{C}$, where the product is the product of *QSym*, known as the stuffle product (also called augmented shuffle, contracting shuffle, quasi-shuffle, ...: see [32], [22], [18], ...).

More precisely, the stuffle, denoted by $\uplus$, is the product of the basis M of monomial quasi-symmetric functions. It is recursively defined on words, and then extended by linearity to non-commutative polynomials or series over an alphabet Ω, which is assumed to have a commutative semi-group structure, denoted by $\dot{+}$:

$$\begin{cases} \varepsilon \uplus u &= u \uplus \varepsilon \ = \ u \ . \\ au \uplus bv &= a\,(u \uplus bv) + b\,(ua \uplus v) + (a\dot{+}b)(u \uplus v) \ . \end{cases} \tag{6}$$

Consequently, we have:

$$\mathcal{H}e_+^{\underline{\mathbf{s}}^1}\mathcal{H}e_+^{\underline{\mathbf{s}}^2} = \sum_{\underline{\mathbf{s}}\in\mathcal{S}^\star} \langle \underline{\mathbf{s}}^1 \uplus \underline{\mathbf{s}}^2 | \underline{\mathbf{s}} \rangle \mathcal{H}e_+^{\underline{\mathbf{s}}} \ . \tag{7}$$

A mould which satisfies such a multiplication rules is said to be *symmetr*$\underline{e}$*l* . So, $\mathcal{H}e_+^\bullet$, as well as $\mathcal{H}e_-^\bullet$, is a symmetr$\underline{e}$l mould. With another vocabulary, we can say that $\mathcal{H}e_+^\bullet$ and $\mathcal{H}e_-^\bullet$ are group-like elements of the Hopf algebra whose product is the concatenation product and whose coproduct is the dual of the stuffle). See Section 2 for an introduction to mould terminology, as well as [4], [6], [15], [20], [21] or [53].

1.2. Connection with multizeta values

The family of Hurwitz multizeta functions is (at first sight) a functional analogue of multizeta values, which are defined by $\mathcal{Z}e^{s_1,\cdots,s_r} = \mathcal{H}e_+^{s_1,\cdots,s_r}(0)$ (See [12], [55] and [56] for surveys on mulutizeta values), that is, for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star$:

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots {n_r}^{s_r}} \ . \tag{8}$$

The story of these numbers goes back to 1775, to Euler's article [26] which had first introduced these numbers in length 2 (*i.e.* $r = 2$). His main goal was to obtain an explicit formula for the evaluation at odd integers of the zeta Riemann function, in order to complete his computation for even integers. Actually, he had been the first to discover the interest of these numbers, proving surprising relations such as $\mathcal{Z}e^{2,1} = \mathcal{Z}e^3$, or more generally:

$$\forall p \in \mathbb{N}^* \ , \ \sum_{k=1}^{p-1} \mathcal{Z}e^{p+1-k,k} = \mathcal{Z}e^{p+1} \ . \tag{9}$$

These result have been forgotten during the XIX$^{\text{th}}$ century and for most of the XX$^{\text{th}}$ century, but in the late 70's, these numbers have been reintroduced by Jean Ecalle in holomorphic dynamics under the name "moule zetaïque" (See [19], vol. 1, p. 137). He used them as auxiliary coefficients in order to construct some geometrical and analytical objects. At first sight, he had not seen all the algebraic interest of these numbers, even if he probably knew their integral representation which is attributed to Kontsevich and is precisely, today, the main point of their algebraic study.

But he has been the first one to show some connection between these numbers and another theory, namely resurgence theory (See [19] in general; for others connections with resurgence theory and analytical invariants, see [4] and [5], as well as Section 4 of this article, see also the introductory course [54] to resurgence theory). Other connections have then been discovered during the late 80's (see p. 151 of [6]) and mathematicians have been definitely convinced of the interest of these numbers which eventually began to be studied for themselves.

One of the questions about these numbers is the algebraic description of their algebra over $\mathbb{Q}$ (denoted by $\mathcal{M}zv_{cv}$). For instance, is $\mathcal{M}zv_{cv}$ a graded algebra? If yes, could we obtain a formula for the dimensions of homogeneous components? But, it turns out that answering questions like that require arithmetical properties of multizeta values, difficult questions

[1]See Remark 4, p. 431 of the first volume of [19], where he refers the reader to Exercice *12e4*

on which only a little is known on this side, in spite of some recent research [9], [10].

Here, the question is a difficult one because multizeta values can be multiplied in two different ways, using first the stuffle (which has been previously defined), but also using another independent product of the stuffle, a shuffle product on binary words, coming from the Kontsevich's integral representation. Of course, the results of the two computations are numerically the same, but have different expressions. Consequently, $\mathcal{M}zv_{cv}$ is an algebra with a rich algebraic structure. By making some shortcuts, Jean Ecalle has shown at the begining of the XXIth century that, formally, this algebra is a polynomial algebra where the indeterminates are what he has called "irreducible multizeta values" (see [20] and [23]).

1.3. Results proven in this article

In this article, we prove two new results on Hurwitz multizeta functions which are consequences of an easy lemma. Hurwitz multiple functions satisfy a 1-order difference equation:

Lemma 1. *For all sequences $\underline{\mathbf{s}} = (s_1, \cdots, s_r) \in \mathcal{S}^\star$, we have:*

$$\mathcal{H}e_+^{\underline{\mathbf{s}}}(z-1) - \mathcal{H}e_+^{\underline{\mathbf{s}}}(z) = \mathcal{H}e_+^{s_1,\cdots,s_{r-1}}(z) \cdot \frac{1}{z^{s_r}} \ . \tag{10}$$

Section 2 will introduce these notations which will be used later on and Section 3 will deal with this Lemma.

The first result is part of resurgence theory, and can be stated in an easy way as

Theorem A. *The Hurwitz multizeta function $\mathcal{H}e_+^{\underline{\mathbf{s}}}$, with $\underline{\mathbf{s}} \in \mathcal{S}^\star$, is a resurgent function. It is the same for the function $\mathcal{H}e_-^{\underline{\mathbf{s}}}$, with $\underline{\mathbf{s}}$ such that $\overleftarrow{\underline{\mathbf{s}}} = (s_r, \cdots, s_1) \in \mathcal{S}^\star$.*

Section 4 is devoted to the proof of this theorem. It begins with a self-contained introduction to simple resurgent functions, as well as references of this subject and an easy, but quite useful example on a generic 1- order difference equation.

The second result has been announced in [7] and concerns the algebraic structure of $\mathcal{H}mzf_{cv}$. In general, elucidating the algebraic structure of an algebra, like $\mathcal{M}zv_{cv}$ or $\mathcal{H}mzf_{cv}$, is a difficult question since it is often an equivalent question to the linear independence of elements of this algebra. For instance, linear relations could come from a second natural product hidden somewhere inside the algebraic structure itself, which is exactly the case of $\mathcal{M}zv_{cv}$. They could also come from another process (see the algebra of multitangent functions, for instance, see Section 4.4.1 and [6]). Even if we can imagine some similarities between $\mathcal{M}zv_{cv}$ and $\mathcal{H}mzf_{cv}$, the situation of Hurwitz multizeta functions is considerably simpler because we can prove the linear independence of Hurwitz multizeta function over $\mathbb{C}(z)$. Consequently, the stuffle product, coming from the identification of the Hurwitz multizeta functions with a specialisation of monomial quasi-symmetric functions, is actually the only product needing to be taken in account in $\mathcal{H}mzf_{cv}$.

Summarizing this, the result can be stated as

Theorem B. *The algebra $\mathcal{H}mzf_{cv}$ is a polynomial algebra, isomorphic to a subalgebra of the quasi-symmetric functions. Thus, all the algebraic relations between Hurwitz multizeta functions, with coefficients in $\mathbb{C}$, come from the expansion of products, using the stuffle product.*

Section 5 is devoted to the proof of this theorem. More precisely, we will prove first that Hurwitz multizeta functions are linearly independent over the algebra of rational functions, and then deduce the theorem from this fact.

In the last section, we introduce divergent Hurwitz multizeta functions, *i.e.* the functions $\mathcal{H}e_+^{\underline{\mathbf{s}}}$, $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$ with $s_1 = 1$. Then, we extend the two previous results to these functions.

Finally, let us notice that there exists a lot of generalization of multizeta values as well as of Hurwitz multizeta functions (already mentionned in the previous citations). Let us mention another one, namely the colored Hurwitz multizeta functions.

For any sequence $\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r} \in \mathrm{seq}\,(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*) - \{\emptyset\}$, with the notation $e_k = e^{-2i\pi\varepsilon_k}$, for $k \in [\![\, 1\,;\, r \,]\!]$, these are defined by:

$$\mathcal{H}e_+^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r}}(z) = \sum_{0<n_r<\cdots<n_1} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} . \tag{11}$$

$$\mathcal{H}e_-^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r}}(z) = \sum_{n_r<\cdots<n_1<0} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} . \tag{12}$$

The two previous result can easily be extended to these functions, because the "colors", *i.e.* the numerators ${e_1}^{n_1} \cdots {e_r}^{n_r}$, are complex numbers.

2. Elements of mould calculus

In this section we introduce briefly the notations of mould calculus. Many references can be found in many texts of Jean Ecalle. See for instance [20] or [21] ; other presentations of mould calculus are also available, see for instance [6], [15] or [53].

2.1. Notion of mould

If Ω is a set, $\Omega^\star$ will denote in the sequel the set of (finite) sequence of elements of Ω:

$$\Omega^\star = \{\emptyset\} \cup \bigcup_{r\in\mathbb{N}^*} \left\{(\omega_1;\cdots;\omega_r) \in \Omega^r\right\} .$$

A mould is a function defined over $\Omega^\star$, seen as a monoid, or sometimes over a subset of $\Omega^\star$, with values in an algebra $\mathbb{A}$. Concretely, this means that "a mould is a function with a variable number of variables" (Jean Ecalle, from the preface of [15], for instance). Thus, moulds depend on sequences $\underline{\mathbf{w}} = (w_1;\cdots;w_r)$ of any length r, where the w_i are elements of Ω.

Sometimes, it may be useful to see a mould as a collection of functions $(f_0, f_1, f_2, \cdots)$, where, for all nonnegative integers i, f_i is a function defined on Ω^i (and consequently, f_0 is a constant function) .

In all this article, we will use the mould notations:

1. Sequences will always be written in bold and underlined, with an upper indexation if necessary. We call length of $\underline{\mathbf{w}}$ and denote $l(\underline{\mathbf{w}})$ the number of elements of $\underline{\mathbf{w}}$ and the empty sequence (*i.e.* the sequence of length 0) is denoted by $\emptyset$. We will reserve the letter r to indicate the length of sequences.
2. For a given mould, we will prefer the notation $M^{\underline{\boldsymbol{\omega}}}$, that indicates the evaluation of the mould $M^\bullet$ on the sequence $\underline{\boldsymbol{\omega}}$ of seq(Ω), to the functional notation that would have been $M(\underline{\boldsymbol{\omega}})$.
3. We shall use the notation $\mathcal{M}^\bullet_{\mathbb{A}}(\Omega)$ to refer to the set of all moulds constructed over the alphabet Ω with values in the algebra $\mathbb{A}$.

2.2. Mould operations

Moulds can be, among other operations, added, multiplied by a scalar as well as multiplied, composed, and so on. In this article, only the multiplication needs to be defined: if $(A^\bullet; B^\bullet) \in (\mathcal{M}^\bullet_{\mathbb{A}}(\Omega))^2$, then, the mould multiplication $M^\bullet = A^\bullet \times B^\bullet$ is defined by

$$M^{\underline{\boldsymbol{\omega}}} = \sum_{\substack{(\underline{\boldsymbol{\omega}}^1;\,\underline{\boldsymbol{\omega}}^2)\in(\Omega^\star)^2\\ \underline{\boldsymbol{\omega}}^1\cdot\underline{\boldsymbol{\omega}}^2=\underline{\boldsymbol{\omega}}}} A^{\underline{\boldsymbol{\omega}}^1} B^{\underline{\boldsymbol{\omega}}^2} = \sum_{i=0}^{r} A^{\omega_1,\cdots,\omega_i} B^{\omega_{i+1},\cdots,\omega_r} . \tag{13}$$

Let us remark that the two deconcatenations $\emptyset \cdot \underline{\omega}$ and $\underline{\omega} \cdot \emptyset$ occur in the definition and refer respectively to the index $i = 0$ and $i = l(\underline{\omega})$.

Finally, $(\mathcal{M}^\bullet_{\mathbb{A}}(\Omega), +, \cdot, \times)$ is an associative, but noncommutative $\mathbb{A}$-algebra, with unit, whose invertible elements are easily characterised:

$$(\mathcal{M}^\bullet_{\mathbb{A}}(\Omega))^\times = \left\{M^\bullet \in \mathcal{M}^\bullet_{\mathbb{A}}(\Omega)\,;\, M^\emptyset \in \mathbb{A}^\times\right\} . \tag{14}$$

We will denote by $(M^\bullet)^{\times -1}$ the multiplicative inverse of a mould $M^\bullet$, when it exists.

As an example, we have the following

Lemma 2. *The multiplicative inverse* $\left(\mathcal{H}e_+^\bullet\right)^{\times -1}$ *and* $\left(\mathcal{H}e_-^\bullet\right)^{\times -1}$ *of the moulds* $\mathcal{H}e_+^\bullet$ *and* $\mathcal{H}e_-^\bullet$ *are given on nonempty sequences* $\underline{\mathbf{s}} = (s_1, \cdots, s_r)$ *by:*

$$\left(\mathcal{H}e_+^{s_1,\cdots,s_r}\right)^{\times -1}(z) = \sum_{0<n_1\leq\cdots\leq n_r} \frac{(-1)^r}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ , \ \text{if } \underline{\mathbf{s}} \in \mathcal{S}^\star \quad (15)$$

$$\left(\mathcal{H}e_-^{s_1,\cdots,s_r}\right)^{\times -1}(z) = \sum_{n_1\leq\cdots\leq n_r<0} \frac{(-1)^r}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ , \ \text{if } \overleftarrow{\underline{\mathbf{s}}} \in \mathcal{S}^\star \quad (16)$$

Proof. This is a direct consequence of the formula antipode for *QSym* (See corollary 2.3 of [39], p.973 – 974) □

2.3. Stuffle product

Let $(\Omega, \dot{+})$ be an alphabet with an additive semi-group structure. Let us first recall that the stuffle product (see [32]), denoted by $\uplus$, is recursively defined by:

$$\left\{ \begin{array}{rcl} \varepsilon \uplus u &=& u \uplus \varepsilon \;=\; u \ . \\ au \uplus bv &=& a\,(u \uplus bv) + b\,(ua \uplus v) + (a\dot{+}b)(u \uplus v) \ . \end{array} \right. \quad (17)$$

where ε is again the empty word.

Example 1. *In* $\mathbb{N}^\star$*, we have:* $12 \uplus 3 = 123 + 132 + 312 + 15 + 42$.

A visual representation of the stuffle product can be useful. Seeing a word as a deck of card, the stuffle of two words, a deck of blue cards and a deck of red cards, becomes the set of all the obtained results by inserting *magically* one deck of blue cards in a deck of red cards. By magically, we mean that some new cards may appear: these new ones are hybrid cards, that is, one of their sides is blue while the other is red. Such a hybrid card can only be obtained when two cards of different colors are situated side by side in a classical shuffle. In the previous example, the hybrid cards are 5, coming from $2+3$, and 4, from $3+1$.

This product is the product of the monomial basis of *QSym*. Consequently, it appears in many contexts related to it, such as the multizeta values, the Hurwitz multizeta functions and the multitangent functions (see [6]).

The multiset $sh\underline{e}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$, where $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are sequences in $\Omega^\star$, is defined to be the set of all monomials that appear in the non-commutative polynomial $\underline{\boldsymbol{\alpha}} \uplus \underline{\boldsymbol{\beta}}$, counted with their multiplicity.

2.4. Symmetr$\underline{e}$lity

When the alphabet Ω is an additive semi-group and $\mathbb{A}$ an algebra, we define a symmetr$\underline{e}$l mould $Me^\bullet$ to be a mould of $\mathcal{M}_{\mathbb{A}}^\bullet(\Omega)$ satisfiying for all $(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}) \in \left(\Omega^\star\right)^2$:

$$\left\{\begin{array}{l} Me^{\underline{\boldsymbol{\alpha}}} Me^{\underline{\boldsymbol{\beta}}} = \displaystyle\sum_{\underline{\boldsymbol{\gamma}}\in\Omega^\star} \langle \underline{\boldsymbol{\alpha}} \uplus \underline{\boldsymbol{\beta}} | \underline{\boldsymbol{\gamma}} \rangle Me^{\underline{\boldsymbol{\gamma}}} = \displaystyle\sum_{\underline{\boldsymbol{\gamma}}\in sh\underline{e}(\underline{\boldsymbol{\alpha}},\underline{\boldsymbol{\beta}})} Me^{\underline{\boldsymbol{\gamma}}} \ . \\ Me^{\emptyset} = 1 \ . \end{array}\right. \tag{18}$$

The symmetr$\underline{e}$lity imposes a strong rigidity since such properties impose a lot of relations, called the stuffle relations.

Example 2. *If $(x;y) \in \Omega^2$ and $Me^\bullet$ denote a symmetr$\underline{e}$l mould, then we have necessarily:*

$$\begin{aligned} Me^x Me^y &= Me^{x,y} + Me^{y,x} + Me^{x+y}. &(19)\\ Me^{x,y} Me^y &= Me^{y,x,y} + 2Me^{x,y,y} + Me^{x+y,y} + Me^{x,2y} \ . &(20) \end{aligned}$$

Nevertheless, there are some stability properties about symmetr$\underline{e}$lity.

Proposition 1. *Let us suppose that the target algebra $\mathbb{A}$ is commutative. Then, for any symmetr$\underline{e}$l moulds $Me^\bullet$ and $Ne^\bullet$ of $\mathcal{M}_{\mathbb{A}}^\bullet(\Omega)$, the product $Pe^\bullet = Me^\bullet \times Ne^\bullet$ is also a symmetr$\underline{e}$l mould of $\mathcal{M}_{\mathbb{A}}^\bullet(\Omega)$.*

Let us notice that this stability property does not hold when the target algebra $\mathbb{A}$ is not commutative.

Proof. From a mould $M^\bullet \in \mathcal{M}_{\mathbb{A}}^\bullet(\Omega)$, one can consider the noncommutative series

$$m = \sum_{\omega\in\Omega} M^\omega \omega \in \mathbb{A}\langle\langle\Omega\rangle\rangle \ . \tag{21}$$

Here, $\mathbb{A}\langle\langle\Omega\rangle\rangle$ is the algebra of all noncommutative series constructed over the alphabet Ω, whose product is the concatenation product. It turns out that $\mathbb{A}\langle\langle\Omega\rangle\rangle$ is also a Hopf algebra whose co-product is the dual of the stuffle product (see [50], chap. 1 for instance)

From this, it is easy to see that the mould $M^\bullet$ is symmetr$\underline{e}$l if, and only if, the series m is a group-like element of $\mathbb{A}\langle\langle\Omega\rangle\rangle$.

The conclusion now comes from the fact that the group-like elements of a Hopf algebra form a group. Details can be found, for instance, in [2], section 2.2. $\square$

The definition of symmetr$\underline{e}$lity may also apply to a mould defined only on a subset D of $\Omega^\star$ (in which case, we require D to be stable by the stuffle product).

Lemma 3. *The mould $\mathcal{H}e^{\bullet}_{+}$ and $\mathcal{H}e^{\bullet}_{-}$ are symmetr$\underline{\boldsymbol{e}}$l respectively on $\mathcal{S}^{\star}$ and $\overleftarrow{\mathcal{S}^{\star}} = \{\underline{\mathbf{s}} \in \mathbb{N}_1^{\star} \ ; \ \overleftarrow{\underline{\mathbf{s}}} \in \mathcal{S}^{\star}\}$.*

Proof. This is a direct consequence of Lemma 1 of [6], or of [32]. □

As a concluding remark on these reminders on symmetr$\underline{\boldsymbol{e}}$lity, we will always write in bold, italic and underlined the vowel that indicates not only a symmetry of the considered moulds, but also the nature of the products of sequences that will appear. Using this, it will become simpler to distinguish when a mould is symmetr$\underline{\boldsymbol{e}}$l or not.

2.5. Altern$\underline{\boldsymbol{e}}$lity

When the alphabet Ω is an additive semi-group and $\mathbb{A}$ an algebra, we define an altern$\underline{\boldsymbol{e}}$l mould $Men^{\bullet}$ to be a mould of $\mathcal{M}^{\bullet}_{\mathbb{A}}(\Omega)$ satisfying for all nonempty sequences $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ constructed over $\Omega^{\star}$:

$$\sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}}\,;\,\underline{\boldsymbol{\beta}})} Men^{\underline{\boldsymbol{\gamma}}} = 0 \ . \tag{22}$$

In the same way as for symmetr$\underline{\boldsymbol{e}}$lity, this sum is a shorthand: each element $\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}}\,;\,\underline{\boldsymbol{\beta}})$ have to be counted with its multiplicity.

The altern$\underline{\boldsymbol{e}}$lity, as well as the symmetr$\underline{\boldsymbol{e}}$lity, imposes a strong rigidity.

Example 3. *If $(x; y) \in \Omega^2$ and $Me^{\bullet}$ denote an altern$\underline{\boldsymbol{e}}$l mould, then we have necessarily:*

$$Me^{x,y} + Me^{y,x} + Me^{x+y} = 0 \ . \tag{23}$$

$$Me^{y,x,y} + 2Me^{x,y,y} + Me^{x+y,y} + Me^{x,2y} = 0 \ . \tag{24}$$

The definition of altern$\underline{\boldsymbol{e}}$lity may also apply to a mould defined only on a subset D of $\Omega^{\star}$ (in which case, we require D to be stable by stuffle).

As an example of altern$\underline{\boldsymbol{e}}$l mould, we have the following

Proposition 2. *Let D be a mould derivation,* i.e. *a derivation for the mould product.*
Then, if a symmetr$\underline{\boldsymbol{e}}$l mould $Me^{\bullet} \in \mathcal{M}^{\bullet}_{\mathbb{A}}(\Omega)$ satisfies $D(Me^{\bullet}) = Aen^{\bullet} \times Me^{\bullet}$, the mould $Aen^{\bullet}$ is naturally altern$\underline{\boldsymbol{e}}$l .

Proof. Note that, if t is a formal parameter, $\mathcal{M}^{\bullet}_{\mathbb{A}}(\Omega)$ naturally injects into $\mathcal{M}^{\bullet}_{\mathbb{A}[\![t]\!]}(\Omega)$. Moreover, $A_t = e^{tD}$ is a well-defined operator of $\mathbb{A}[\![t]\!]$ for the Krull topology. Consequently, $A_t(Me^{\bullet})$ is well-defined, for any mould $Me^{\bullet} \in \mathcal{M}^{\bullet}_{\mathbb{A}}(\Omega)$, and is valued in $\mathcal{M}^{\bullet}_{\mathbb{A}[\![t]\!]}(\Omega)$.

Since D is a mould derivation, $A_t = e^{tD}$ is a mould automorphism (for the mould product). Therefore, applying the automorphism A_t to both sides of the equalities given by the symmetr$\underline{e}$lity of $Me^\bullet$, we see that the mould $A_t(Me^\bullet)$ turns out to be a symmetr$\underline{e}$l mould.

Consequently, $Ae_t^\bullet = e^{tD}(Me^\bullet) \times (Me^\bullet)^{\times -1}$ is a symmetr$\underline{e}$l mould, according to Property 1.

Thus, for any sequences $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ of $\Omega^\star$, we have:

$$\frac{d\ Ae_t^{\underline{\mathbf{u}}}}{dt} \cdot Ae_t^{\underline{\mathbf{v}}} + Ae_t^{\underline{\mathbf{u}}} \cdot \frac{d\ Ae_t^{\underline{\mathbf{v}}}}{dt} = \sum_{\underline{\mathbf{w}} \in \mathrm{sh}\underline{e}(\underline{\mathbf{u}},\underline{\mathbf{v}})} \frac{d\ Ae_t^{\underline{\mathbf{w}}}}{dt} \tag{25}$$

Since $Aen^\bullet = \dfrac{d}{dt}(Ae_t^\bullet)\Big|_{t=0}$ and $Ae_0^\bullet = 1^\bullet$, we deduce from (25):

$$\sum_{\underline{\mathbf{w}} \in \mathrm{sh}\underline{e}(\underline{\mathbf{u}},\underline{\mathbf{v}})} Aen^{\underline{\mathbf{w}}} = Aen^{\underline{\mathbf{u}}}\ Ae_0^{\underline{\mathbf{v}}} + Ae_0^{\underline{\mathbf{u}}}\ Aen^{\underline{\mathbf{v}}} = Aen^{\underline{\mathbf{u}}}\ 1^{\underline{\mathbf{v}}} + 1^{\underline{\mathbf{u}}}\ Aen^{\underline{\mathbf{v}}}\ . \tag{26}$$

Finally, for any nonempty sequences $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ of $\Omega^\star$, we obtain

$$\sum_{\underline{\mathbf{w}} \in \mathrm{sh}\underline{e}(\underline{\mathbf{u}},\underline{\mathbf{v}})} Aen^{\underline{\mathbf{w}}} = 0\ , \tag{27}$$

which proves the altern$\underline{e}$lity of $Aen^\bullet$. □

2.6. Notations

To conclude this section, let us introduce two notations on sequences. Let $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$.

- If $\underline{\mathbf{s}}$ is a sequence of lenght r, we will use the condensed notation $s_{1..k} = s_1 + \cdots + s_k$ for all $k \in [\![1\,;\, r]\!]$.
- $||\underline{\mathbf{s}}||$ denotes the sum of all the elements of $\underline{\mathbf{s}}$.

3. A fundamental lemma

3.1. The operators τ , Δ_+ and Δ_-

We know that Hurwitz multizeta functions are a "translation" of multizeta values. Therefore, it is natural to examine how the shift operator acts on such functions.

For $p \in \mathbb{Z}$, let $\mathbb{Z}_{\leq p} = \{k \in \mathbb{Z}\,; k \leq p\}$. From now, let us consider operators $\tau : \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq -1}\big) \longrightarrow \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq -2}\big)$, $\Delta_+ : \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq -1}\big) \longrightarrow \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq -1}\big)$ and $\Delta_- : \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq -1}\big) \longrightarrow \mathcal{H}\big(\mathbb{C} - \mathbb{Z}_{\leq 0}\big)$ defined by :

$$\begin{array}{rcl} \tau(f)(z) & = & f(z+1)\ . \\ \Delta_+(f)(z) & = & f(z+1) - f(z)\ . \\ \Delta_-(f)(z) & = & f(z-1) - f(z)\ . \end{array} \tag{28}$$

3.2. The action of τ on Hurwitz multizeta functions

The next lemma is a fundamental one and will allow us to express the action of Δ_+ and Δ_- on a Hurwitz multizeta function.

Lemma 4. *Let $J^{\underline{\mathbf{s}}}$, for $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$, be the function defined by:*

$$J^{\underline{\mathbf{s}}}(z) = \begin{cases} \dfrac{1}{z^{s_1}} & , \textit{ if } l(\underline{\mathbf{s}}) = 1 \ . \\ 0 & , \textit{ otherwise } \ . \end{cases} \tag{29}$$

Then:

$$\tau(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet - \tau(\mathcal{H}e_+^\bullet \times J^\bullet) \ . \tag{30}$$

Proof. For all sequences $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$ with length r and $z \in \mathbb{C} - \mathbb{Z}_{\leq -1}^\star$, we successively have:

$$\begin{aligned} \mathcal{H}e_+^{\underline{\mathbf{s}}}(z+1) &= \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z+1)^{s_1}\cdots(n_r+z+1)^{s_r}} \\ &= \sum_{1<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \\ &= \left(\sum_{0<n_r<\cdots<n_1} - \sum_{1=n_r<\cdots<n_1}\right)\left(\frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\right) \\ &= \mathcal{H}e_+^{\underline{\mathbf{s}}}(z) - \frac{1}{(z+1)^{s_r}}\mathcal{H}e_+^{\underline{\mathbf{s}}^{<r}}(z+1) \ . \end{aligned}$$

We can rewrite this last equality with the following compact form:

$$\tau(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet - \tau\big(\mathcal{H}e_+^\bullet\big) \times \tau\big(J^\bullet\big) \ . \tag{31}$$

Because τ is an automorphism, this concludes the proof. $\square$

3.3. The action of Δ_+ on Hurwitz multizeta functions

Because it is a discrete analogue of the classical derivative, it is traditional to focus on the operator $\Delta_+ = \tau - Id$. We can remark that it is a $(\tau\,; Id)$-derivative as well as a $(Id\,;\tau)$-derivative[2]. We then have the following property explaining its action on a Hurwitz multizeta function:

[2]Let us remind that a (σ,τ)-derivation d defined on an algebra $\mathbb{A}$ is a linear map from $\mathbb{A}$ to $\mathbb{A}$ such that, forall $(a,b) \in \mathbb{A}^2$:

$$d(a\cdot b) = \sigma(a)\cdot d(b) + d(a)\cdot\tau(b) \ .$$

Proposition 3. *Let $K^{\underline{\mathbf{s}}}$, for $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$, be the function defined by:*

$$K^{\underline{\mathbf{s}}}(z) = \begin{cases} \dfrac{(-1)^{l(\underline{\mathbf{s}})}}{(z+1)^{||\underline{\mathbf{s}}||}} \ . & , \text{ if } l(\underline{\mathbf{s}}) \neq 0 \ . \\ 0 & , \text{ otherwise} \ . \end{cases} \tag{32}$$

Then:

$$\Delta_+(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet \times K^\bullet \ . \tag{33}$$

Proof. The previous lemma gives us: $\tau(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet - \tau\big(\mathcal{H}e_+^\bullet \times J^\bullet\big)$. Using it iteratively, we therefore obtain:

$$\begin{aligned} \tau(\mathcal{H}e_+^\bullet) &= \mathcal{H}e_+^\bullet - \mathcal{H}e_+^\bullet \times \tau\big(J^\bullet\big) + \tau\big(\mathcal{H}e_+^\bullet\big) \times \tau\big(J^\bullet\big)^2 \\ &\ \vdots \\ &= \mathcal{H}e_+^\bullet \times \left(\sum_{k \geq 0} (-1)^k \tau\big(J^\bullet\big)^k \right) \end{aligned}$$

So:

$$\Delta_+(\mathcal{H}e_+^{\underline{\mathbf{s}}}) = \mathcal{H}e_+^\bullet \times \left(\sum_{k \geq 1} (-1)^k \tau\big(J^\bullet\big)^k \right) \ . \tag{34}$$

We can notice that the right factor of this product is actually a locally finite sum, that is to say only a finite number of terms intervene in its evaluation on a sequence $\underline{\mathbf{s}}$. More precisely, if $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$ is of length $r = l(\underline{\mathbf{s}}) > 0$, we have:

$$\begin{aligned} \left(\sum_{k \geq 1} (-1)^k \tau\big(J^\bullet\big)^k \right)^{\underline{\mathbf{s}}} &= (-1)^r \left(\tau\big(J^\bullet\big)^r\right)^{\underline{\mathbf{s}}} = (-1)^r \tau\big(J^{s_1}\big) \cdots \tau\big(J^{s_r}\big) \\ &= \frac{(-1)^r}{(z+1)^{s_1 + \cdots + s_r}} = K^{\underline{\mathbf{s}}}(z) \ . \end{aligned} \tag{35}$$

Finally, if $\underline{\mathbf{s}} = \emptyset$, this sum equals $0 = K^{\underline{\mathbf{s}}}$, which ends the proof. □

Example 4. *We successively have:*

$$\begin{aligned} \tau(\mathcal{H}e_+^{3,2,1}) &= \mathcal{H}e_+^{3,2,1} - \Big(\tau\big(\mathcal{H}e_+^\bullet\big) \times \tau\big(J^\bullet\big)\Big)^{3,2,1} \\ &= \mathcal{H}e_+^{3,2,1} - \Big(\mathcal{H}e_+^\bullet - \tau\big(\mathcal{H}e_+^\bullet\big) \times \tau\big(J^\bullet\big)\Big)^{3,2} \cdot \tau\big(J^1\big) \\ &= \mathcal{H}e_+^{3,2,1} - \mathcal{H}e_+^{3,2} \cdot \tau\big(J^1\big) + \mathcal{H}e_+^3 \cdot \tau\big(J^2\big) \cdot \tau\big(J^1\big) \end{aligned}$$

Therefore: $\Delta_+(\mathcal{H}e_+^{3,2,1}) = \mathcal{H}e_+^{3,2} \cdot K^1 + \mathcal{H}e_+^3 \cdot K^{2,1}$.

Let us notice that the mould $K^\bullet$ could also be expressed by:

$$K^\bullet = -\tau(J^\bullet) \times \Big(1 + \tau(J^\bullet)\Big)^{\times -1} \ . \tag{36}$$

3.4. The action of Δ_- on Hurwitz multizeta functions

The Δ_+ derivative is complicated because the result involves several Hurwitz multizeta functions. We will prefer Δ_- to it, and this operator will become for all this article our discrete analogue of the classical derivative.

Recall that the mould $J^\bullet$ has been defined, for $\underline{\mathbf{s}} \in \mathbb{N}_1^\star$, by:

$$J^{\underline{\mathbf{s}}}(z) = \begin{cases} \dfrac{1}{z^{s_1}} & \text{, if } l(\underline{\mathbf{s}}) = 1 \ . \\ 0 & \text{, otherwise} \ . \end{cases} \tag{37}$$

Proposition 4. *We have:* $\Delta_-(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet \times J^\bullet$.

Proof. Lemma 4 gives us: $\tau(\mathcal{H}e_+^\bullet) = \mathcal{H}e_+^\bullet - \tau\big(\mathcal{H}e_+^\bullet \times J^\bullet\big)$. Composing this identity by τ^{-1}, we therefore obtain:

$$\mathcal{H}e_+^\bullet = \tau^{-1}\left(\mathcal{H}e_+^\bullet\right) - \mathcal{H}e_+^\bullet \times J^\bullet \ , \tag{38}$$

This gives the result. $\square$

Example 5. *Since we have* $\tau(\mathcal{H}e_+^{3,2,1}) = \mathcal{H}e_+^{3,2,1} - \big(\tau\big(\mathcal{H}e_+^\bullet\big) \times \tau\big(J^\bullet\big)\big)^{3,2,1}$, *we successively deduce:*

$$\begin{aligned} \tau^{-1}(\mathcal{H}e_+^{3,2,1}) &= \mathcal{H}e_+^{3,2,1} + \big(\mathcal{H}e_+^\bullet \times J^\bullet\big)^{3,2,1} \\ &= \mathcal{H}e_+^{3,2,1} + \mathcal{H}e_+^{3,2} \cdot J^1 \end{aligned}$$

Therefore: $\Delta_-(\mathcal{H}e_+^{3,2,1}) = \mathcal{H}e_+^{3,2} \cdot J^1$.

4. Resurgence theory and Hurwitz multizeta functions

This section is devoted to the study of the resurgent properties of Hurwitz multizeta functions. Reminders are available in Sections 4.1 and 4.2. A first example of the use of this theory is given in Section 4.3. The aim of this section is to prove Theorem 1 and Theorem 2.

References for the reminders are [19] in general, the book [13], the introduction of [4], the introductory articles [11], [16], [17], [38], [49], [51], [53] as well as the new introductory course focused on this subject [54].

4.1. A few reminders on the Borel transform and Borel resummation process

4.1.1. Borel transform

The formal Borel transform $\mathcal{B}$ is defined by:

$$\begin{array}{rccc} \mathcal{B}: & z^{-1}\mathbb{C}[\![z^{-1}]\!] & \longrightarrow & \mathbb{C}[\![\zeta]\!] \\ & \displaystyle\sum_{n\geq 0} \frac{c_n}{z^{n+1}} & \longmapsto & \displaystyle\sum_{n\geq 0} \frac{c_n}{n!}\zeta^n \ . \end{array} \tag{39}$$

Let us first remind the terminology and the traditional notations. A formal power series φ, near infinity, will systematically be denoted with a tilde ($\widetilde{\ }$), while the notation for its formal Borel transform will have a hat ($\widehat{\ }$): $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$. The variable near infinity will also systematically be denoted by z while the variable of its formal Borel transform will be ζ. We say we are working in the "formal model" when we are dealing with formal power series, *i.e.* with the z variable; when we are dealing with Borel transforms, *i.e.* with the ζ variable, we say we are working in the "convolutive model" (we will see why in the next proposition).

Moreover, let us denote by $\circ$, $\cdot$ and ∂_z the composition, the multiplication and the derivative with respect to z. Finally, we define $l : \mathbb{C} \longrightarrow \mathbb{C}$ to be the 1-unit translation (*i.e.* $l(z) = z + 1$) and $\star$ to be the convolution of formal power series defined by:

$$\Big(\sum_{n=0}^{+\infty} \frac{a_n}{n!}\zeta^n\Big) \star \Big(\sum_{n=0}^{+\infty} \frac{b_n}{n!}\zeta^n\Big) = \sum_{n=0}^{+\infty} \frac{1}{(n+1)!}\Big(\sum_{p=0}^{n} a_p b_{n-p}\Big)\zeta^{n+1} \ , \tag{40}$$

which extends the classical integral definition to formal power series.

Using these notations, we can remind the classical properties of the formal Borel transform (see. [38]):

Proposition 5. *Let* $(\widetilde{\varphi}\,;\widetilde{\psi}) \in \left(z^{-1}\mathbb{C}[\![z^{-1}]\!]\right)^2$.
Then :
1. $\mathcal{B}(\partial_z \widetilde{\varphi})(\zeta) = -\zeta\widehat{\varphi}(\zeta)$.
2. $\mathcal{B}(\widetilde{\varphi} \circ l)(\zeta) = e^{-\zeta}\widehat{\varphi}(\zeta)$ *and* $\mathcal{B}(\widetilde{\varphi} \circ l^{-1})(\zeta) = e^{+\zeta}\widehat{\varphi}(\zeta)$.
3. $\mathcal{B}(\widetilde{\varphi} \cdot \widetilde{\psi})(\zeta) = \left(\widehat{\varphi} \star \widehat{\psi}\right)(\zeta)$.

Moreover, it is also easy to characterize the series of $z^{-1}\mathbb{C}[\![z^{-1}]\!]$ whose image under the Borel transform are analytical germs at the origin. Their set is denoted by $z^{-1}\mathbb{C}[\![z^{-1}]\!]_1$ and these series called 1-Gevrey formal series.

Definition 1. *A formal series near infinity* $\widetilde{\varphi}(z) = \sum_{n \geq 0} \frac{c_n}{z^{n+1}}$ *is an element of* $z^{-1}\mathbb{C}[\![z^{-1}]\!]_1$, i.e. *has a Borel transform which is an analytical germ at the origin, if, and only if, there exist two positive constants* (C_0, C_1) *such that:*

$$\forall n \in \mathbb{N} \ , \ |c_n| \leq C_0 C_1^n n! \ . \tag{41}$$

We can conclude these reminders by the following

Proposition 6. $\mathcal{B}\left(z^{-1}\mathbb{C}[\![z^{-1}]\!]_1\right) = \mathbb{C}\{\zeta\}$.

In other terms, $\mathcal{B}$ realizes an isomorphism between $z^{-1}\mathbb{C}[\![z^{-1}]\!]_1 \subset z^{-1}\mathbb{C}[\![z^{-1}]\!]$ and $\mathbb{C}\{\zeta\}$.

4.1.2. Laplace transform

The Laplace transform $\mathcal{L}^\theta$ in the direction $\theta \in \mathbb{R}$ is the linear operator defined by:

$$\mathcal{L}^\theta(\varphi)(z) = \int_0^{e^{i\theta}\infty} \varphi(\zeta) e^{-z\zeta}\, d\zeta\ . \tag{42}$$

This is well-defined over the set of analytical functions φ on an open set of $\mathbb{C}$ containing $\{\zeta \in \mathbb{C}\ ;\ \arg \zeta = \theta\}$ that have at most exponential growth in the direction θ. This last condition means that there exist two positive constants C and τ such that for all $r > 0$:

$$\left|\varphi(re^{i\theta})\right| \leq Ce^{\tau r}\ . \tag{43}$$

Since $\dfrac{1}{z^{n+1}} = \mathcal{L}^\theta\left(\dfrac{\zeta^n}{n!}\right)(z)$, for $z \in \mathbb{C}$ such that $\Re e\,(ze^{i\theta}) > 0$, the Borel transform can be seen as a formal inverse of the Laplace transform, where formally means here that we formally permute the symbols sum and integral. This is actually licit for entire functions of exponential type in every direction.

Dealing with an analytical function φ defined over an open set containing $\{\zeta \in \mathbb{C}\ ;\ \arg \zeta = \theta\}$ which has at most an exponential growth in the direction θ (with an associated constant τ) allows us to use the theorem about the holomorphy of a parameter-dependent integral. This proves that $\mathcal{L}^\theta(\varphi)$ is a holomorphic function in the half-plane $\left\{z \in \mathbb{C}\ ;\ \Re e\,\left(e^{i\theta}z\right) > \tau\right\}$.

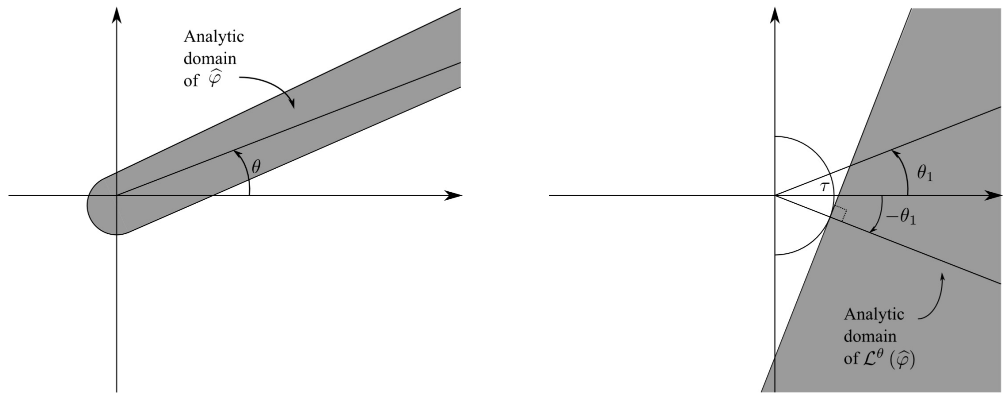


Figure 1: Image of the analycity domain under the Laplace transform.

4.1.3. Borel summation process

The Borel transform and Laplace transform interact to define what is now called the Borel summation process:

Definition 2. *For a given* $\theta \in \mathbb{R}$*, we say that* $\widetilde{\varphi} \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$ *is Borel-summable in the direction* θ*, and we denote* $\widetilde{\varphi} \in \mathcal{S}_{\mathcal{B},\theta}$*, when the following conditions are satisfied:*

1. $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$ *can be analytically extended on a neighborhood* Ω *of* $e^{i\theta}\mathbb{R}_+$.
2. *there exist two positive constants* C *and* τ *such that for all* $\zeta \in \Omega$,

$$|\widehat{\varphi}(\zeta)| \leq Ce^{\tau|\zeta|} \ . \tag{44}$$

In this case, the corresponding Borel sum, denoted by $S^{\theta}(\widetilde{\varphi})$*, is defined by:*

$$S^{\theta}(\widetilde{\varphi}) = \mathcal{L}^{\theta}\left(\mathcal{B}(\widetilde{\varphi})\right) \ . \tag{45}$$

Thus, such a sum is automatically an analytic function on the half-plane $\mathcal{P}^{\theta}(\tau) = \{z \in \mathbb{C} \ ; \ \Re e\,(e^{i\theta}z) > \tau\}$.

This process is satisfactory, in the following sense:

1. If $\widetilde{\varphi} \in \mathbb{C}\left\{z^{-1}\right\}$ is well-defined on a neighborhood Ω of infinity, then, the Borel sums $S^{\theta}(\widetilde{\varphi})$ coincide with $\widetilde{\varphi}$ on Ω, for all directions $\theta \in \mathbb{R}$.
2. $S^{\theta} \ : \ \mathcal{S}_{\mathcal{B},\theta} \longrightarrow \bigcup_{\tau>0} \mathcal{H}\left(\mathcal{P}^{\theta}(\tau)\right)$ is an injective homomorphism which commutes with the derivation[3].
3. If $\widetilde{\varphi} \in \mathcal{S}_{\mathcal{B},\theta}$, then $\widetilde{\varphi}$ is the asymptotic expansion, near infinity, of $S^{\theta}(\widetilde{\varphi})$.

To summarize, we have the diagram on Figure 2 .

4.1.4. Sectorial resummation

A formal power series $\widetilde{\varphi}$ is said to be uniformly Borel-summable in the interval of direction $]\theta_1;\theta_2[$ when the following conditions are satisfied:

1. $\widehat{\varphi}$ can be analytically extended to a neighborhood of $\Omega = \{z \in \mathbb{C} \ ; \ \theta_1 < arg\ z < \theta_2\}$.
2. there exist two positive constants C and τ such that for all $\zeta \in \Omega$,

$$|\widehat{\varphi}(\zeta)| \leq Ce^{\tau|\zeta|} \ . \tag{46}$$

For such a formal power series, when we apply the Borel summation process, it is possible to vary the direction of the summation between θ_1 and θ_2. A natural question is then to understand the links between the different Borel sums. The residue theorem gives the answer: they are mutual analytical extensions.

Therefore, we can define a Borel sum, denoted by $S^{[\theta_1;\theta_2]}$, on the angular sector

$$\left\{z \in \mathbb{C} \ ; \ -\theta_2 - \frac{\pi}{2} < arg\ z < -\theta_1 + \frac{\pi}{2}\right\} \tag{47}$$

[3] $\mathcal{H}(\mathcal{U})$ denotes classicaly the space of holomorphic functions on the open subset $\mathcal{U}$ of $\mathbb{C}$.

Formal model:

$$\widetilde{\varphi}(z) = \sum_{n=0}^{+\infty} \frac{c_n}{z^{n+1}} \in z^{-1}\mathbb{C}[\![z^{-1}]\!] \cap \mathcal{S}_{\mathcal{B},\theta} \quad .$$

(formal power series, Borel summable in the direction θ) .

$\xrightarrow{\mathcal{B}}$

Convolutive model:

$$\widehat{\varphi}(\zeta) = \sum_{n=0}^{+\infty} \frac{c_n}{n!}\zeta^n \quad .$$

(function that can be analytically exten- -ded on a neighborhood of $e^{i\theta}\mathbb{R}_+$ and that has at most exponential growth) .

$\xrightarrow{\mathcal{L}^\theta}$

Geometric model:

$$S^\theta(\widetilde{\varphi})(z) = \mathcal{L}^\theta\widehat{\varphi}(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}(\zeta)e^{-\zeta z}d\zeta \quad .$$

(analytical function defined on the half-plane $\{z \in \mathbb{C} \ ; \ \Re e\,(e^{i\theta}z) > 0\}$) .

asymptotic expansion near infinity.

Figure 2: Borel summation process in the direction θ .

by glueing the Borel sums obtained when θ varies between θ_1 and θ_2 (see Figure 3), because two Borel sums, corresponding to angles satisfying (47) coincide on the intersection of their definition domains (thanks to contour integrals of holomorphic functions).

This is true as long as $\widehat{\varphi}$ has no singularities in the sector Ω . Otherwise, the residue theorem shows that the difference between $\mathcal{L}^{\theta^-}(\widehat{\varphi})$ and $\mathcal{L}^{\theta^+}(\widehat{\varphi})$, the Borel sums just before and just after a singularity in the direction θ, do not coincide any more: this is the so-called *Stokes phenomenon*.

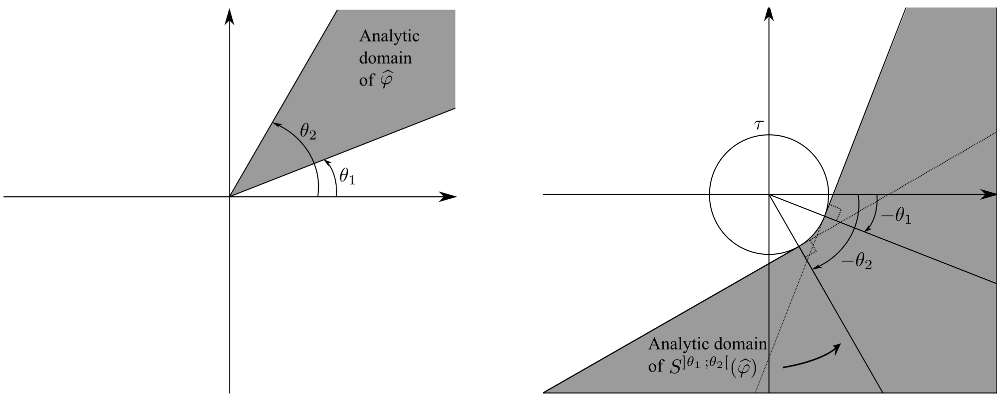


Figure 3: Sectorial resummation.

4.2. Reminders on resurgence theory

As we have just seen, we need to have a precise knowledge of the different singularities of $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$ to compare the Borel sectorial sums of the divergent series $\widetilde{\varphi}$, which is supposed to be uniformly Borel-summable (in an interval) .

Since we will only deal with simple resurgent functions in the sequel, we only remind the definition of this class of functions. For more information, we refer the reader to Ecalle's text (the three volumes of [19]), or to the introductory article/book [51] and [13].

4.2.1. Simple singularities

We say that a function $\widehat{\varphi}$, defined and holomorphic on an open set D, has a simple singularity at $\omega \in \mathbb{C}$ adherent to D when $\widehat{\varphi}$ can be expanded near ω as the sum of a logarithmic singularity, a polar singularity and a regular part. Thus, there exist $C \in \mathbb{C}$ and two germs of holomorphic functions $\widehat{\Phi}$ and reg such that, in a neighborhood U of ω, the following equality is satisfied:

$$\widehat{\varphi}(\zeta) \underset{(\zeta \longrightarrow \omega)}{=} \frac{C}{2i\pi(\zeta-\omega)} + \frac{1}{2i\pi}\widehat{\Phi}(\zeta-\omega)\log(\zeta-\omega) + reg(\zeta-\omega) \ . \tag{48}$$

The number C is called the residue of $\widehat{\varphi}$ and $\widehat{\Phi}$ is called the variation of $\widehat{\varphi}$. These objects can be computed from $\widehat{\varphi}$, independently from the choice of the branch of the logarithm, by:

$$C = 2i\pi \lim_{\substack{\zeta \longrightarrow \omega \\ \zeta \in U}} (\zeta - \omega)\widehat{\varphi}(\zeta) \quad , \quad \widehat{\Phi}(\zeta) = \widehat{\phi}(\omega + \zeta) - \widehat{\phi}\left(\omega + \zeta e^{-2i\pi}\right) \ , \tag{49}$$

where $\widehat{\phi}\left(\omega + \zeta e^{-2i\pi}\right)$ means the evaluation at the point $\omega + \zeta$ of the analytic continuation of $\widehat{\phi}$ following the path $t \longmapsto \omega + \zeta e^{-2i\pi t}$, $t \in [0;1]$.

(48) is usually denoted in a simpler form:

$$\mathrm{sing}_\omega \widehat{\varphi} = C\delta + \widehat{\Phi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \ . \tag{50}$$

Here, the operator $\mathrm{sing}_\omega \widehat{\varphi}$ denotes the asymptotic expansion near the point ω of $\widehat{\varphi}$, while δ denotes a Dirac. Let us remind that the algebra $(\mathbb{C}\{\zeta\}, \star)$ has no unit. We add a formal one by an extension of the Borel transform:

$$\mathcal{B} : \mathbb{C}[\![z^{-1}]\!]_1 \stackrel{\sim}{\longrightarrow} \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \ , \tag{51}$$

where

$$\delta = \mathcal{B}(1) \ . \tag{52}$$

4.2.2. Endlessly continuable germs

We say that a holomorphic germ φ at the origin is an endlessly continuable germ when, for any finite broken line L, there exists a finite set $\Omega_L \subset L$ of singularities such that φ has an analytic continuation along all the possible paths obtained by following L and getting around each point of Ω_L turning left or right.

4.2.3. S-resurgent functions

We define here the notion of simple resurgent function, which is a particular case of a more general notion (see [11], [13], [16], [19], [51]).

In the convolutive model, S-resurgent functions are defined to be endlessly continuable holomorphic germs at the origin, with simple singularities.

In the formal model, S-resurgent functions are formal power series in $z^{-1}\mathbb{C}[\![z^{-1}]\!]$ such that their Borel transform is an S-resurgent function in the convolutive model. Let us remark that, necessarily, an S-resurgent function in the formal model is a 1-Gevrey series.

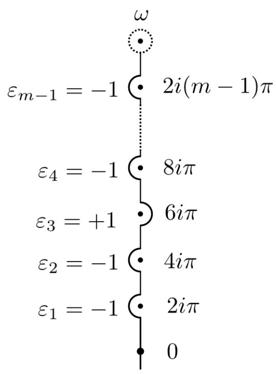
We will denote by $\widetilde{\mathrm{RES}}^{\mathrm{simple}}$ and $\widehat{\mathrm{RES}}^{\mathrm{simple}}$ the set of simple resurgent functions in the formal and convolutive models.

The important point is the stability of $\widetilde{\mathrm{RES}}^{\mathrm{simple}}$ (resp. $\widehat{\mathrm{RES}}^{\mathrm{simple}}$) under the ordinary product (resp. convolution product) of formal power series (see [52]).

4.2.4. Alien derivations

From now on, we will restrict to a particular case of S-resurgent functions, those dealing with Borel transform singularities located in $\Omega = 2i\pi\mathbb{Z}^*$.

The alien derivatives are linear operators acting on the resurgent functions that "measure" the singularities near each $\omega \in \Omega$. Let us emphasize that they indeed define derivations (for the usual product in the formal model or for the convolution product in the convolutive model), since they come from the logarithm of the Stokes automorphism.



To make it explicit in the convolutive model, let us consider $\omega = 2im\pi \in \Omega$. Given $\underline{\boldsymbol{\varepsilon}} = (\varepsilon_{\pm 1}; \cdots ; \varepsilon_{\pm(m-1)}) \in \{+1; -1\}^{|m|-1}$, we define the path $\gamma(\underline{\boldsymbol{\varepsilon}})$ constructed by following $]0; \omega[$ and getting around the intermediate singularities $2ik\pi$, $\pm k \in [\![\, 1\, ;\, |m| - 1\,]\!]$ on half-circles, the k-th being oriented on the left side of the imaginary axis when $\varepsilon_k = -1$ and oriented on the right side of the same axis when $\varepsilon_k = +1$.

Thus, the alien derivative $\boldsymbol{\Delta}_\omega$ acts on $\widehat{\varphi} \in \widehat{\mathrm{RES}}^{\mathrm{simple}}$ by

$$\boldsymbol{\Delta}_\omega(\widehat{\varphi}) = \sum_{\underline{\boldsymbol{\varepsilon}} = (\varepsilon_{\pm 1}; \cdots ; \varepsilon_{\pm(m-1)}) \in \{+1;-1\}^{|m|-1}} \frac{p(\underline{\boldsymbol{\varepsilon}})!\, q(\underline{\boldsymbol{\varepsilon}})!}{m!} \mathrm{sing}_\omega(\mathrm{cont}_{\gamma(\underline{\boldsymbol{\varepsilon}})} \widehat{\varphi}) \ .$$

In this formula, $p(\underline{\boldsymbol{\varepsilon}})$ and $q(\underline{\boldsymbol{\varepsilon}})$ stand respectively for the number of signs $+$ and $-$ in $\underline{\boldsymbol{\varepsilon}}$. Moreover, $\mathrm{cont}_{\gamma(\underline{\boldsymbol{\varepsilon}})}$ is the analytic continuation of $\widehat{\varphi}$ along the path $\gamma(\underline{\boldsymbol{\varepsilon}})$.

In the formal model, we define the linear operator

$$\boldsymbol{\Delta}_\omega : \widetilde{\mathrm{RES}}^{\mathrm{simple}} \longmapsto \widetilde{\mathrm{RES}}^{\mathrm{simple}} \tag{53}$$

by the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{RES}}^{\mathrm{simple}} \arrow[r, "\boldsymbol{\Delta}_\omega"] \arrow[d, "\mathcal{B}"'] & \widetilde{\mathrm{RES}}^{\mathrm{simple}} \arrow[d, "\mathcal{B}"] \\
\widehat{\mathrm{RES}}^{\mathrm{simple}} \arrow[r, "\boldsymbol{\Delta}_\omega"] & \widehat{\mathrm{RES}}^{\mathrm{simple}}
\end{tikzcd}$$

Proposition 7. *If $\widetilde{f}$ and $\widetilde{g}$ are two simple resurgent functions, then $\widetilde{h}(z) = f(z + g(z))$ is also a resurgent function, and we have:*

$$\boldsymbol{\Delta}_\omega \widetilde{h}(z) = e^{-\omega \widetilde{g}(z)} \left(\boldsymbol{\Delta}_\omega \widetilde{f} \right) (z + \widetilde{g}(z)) + (\partial \widetilde{f})(z + \widetilde{g}(z)) \cdot \boldsymbol{\Delta}_\omega \widetilde{g}(z) \ . \tag{54}$$

Proof. See, for instance [19], Vol. 1, Section 2e, or [54], Section 30. $\square$

4.3. On a generic 1*-order difference equation*

Examples of the use of Borel-Laplace resummation can be found in the literature. For example, we refer the reader to the third volume of [19] (p. 243 to 246) to see a resurgent approach of the function $\log \Gamma$, to [16] for the Euler equation $\varphi'(z) + \varphi(z) = z^{-1}$ or to [17] and [49] for a resurgent study of the Airy function.

We recall here the use of the Laplace transform for a simple 1-order difference equation (see [4] and [51]). This equation will be the prototype of the equations we will deal with in the next section.

Let us fix a holomorphic germ $a \in z^{-2}\mathbb{C}\left\{z^{-1}\right\}$ and study the 1-order difference equation:

$$\widetilde{\varphi}(z-1) - \widetilde{\varphi}(z) = a(z) \ . \tag{55}$$

We will study it first when the unknown $\widetilde{\varphi}$ is a formal power series at infinity, then construct solutions which are holomorphic functions (expressed as a Laplace transform) which turn out to have explicit expressions depending on the germ a .

4.3.1. Preliminary upper bounds

Since $a(z) = \displaystyle\sum_{n \geq 1} \frac{a_n}{z^{n+1}} \in z^{-2}\mathbb{C}\{z^{-1}\}$, there exist two positive constants C_0 and C_1 such that

$$|a_n| \leq C_0 {C_1}^n \ , \ \text{for all } n \in \mathbb{N}^* \ . \tag{56}$$

Consequently, we have:

$$\left|\widehat{a}(\zeta)\right| \leq C_0 \left(e^{C_1|\zeta|} - 1\right) \leq C_0|\zeta| e^{C_1|\zeta|} \text{ for all } \zeta \in \mathbb{C} \ . \tag{57}$$

Moreover, we have the following

Lemma 5. *If $\zeta = re^{i\theta}$, with $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$, we have:*

$$\left|\frac{\zeta}{e^{\zeta}-1}\right| \leq \frac{e^{|\zeta|}}{|\cos\theta|} \ . \tag{58}$$

Proof. If $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$,

$$\begin{aligned}\left|\frac{\zeta}{e^{\zeta}-1}\right| &\leq |\zeta| \cdot |e^{-\zeta}| \cdot \left|\frac{1}{1-e^{-\zeta}}\right| \\ &\leq re^{-r\cos\theta}\frac{1}{1-e^{-r\cos\theta}} \\ &\leq \frac{1}{\cos\theta}\frac{r\cos\theta}{e^{r\cos\theta}-1} \leq \frac{1}{\cos\theta} \\ &\leq \frac{e^{|\zeta|}}{\cos\theta} \ .\end{aligned}$$

If $\theta \in \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$,
$$\begin{aligned}\left|\frac{\zeta}{e^{\zeta}-1}\right| &\leq |\zeta|\cdot\left|\frac{1}{1-e^{\zeta}}\right| \leq \frac{r}{1-e^{r\cos\theta}} \\ &\leq \frac{1}{|\cos\theta|}e^{-r\cos\theta}\frac{-r\cos\theta}{e^{-r\cos\theta}-1} \leq \frac{e^{r}}{|\cos\theta|} \\ &\leq \frac{e^{|\zeta|}}{|\cos\theta|} \ .\end{aligned}$$

4.3.2. The resurgent character of the formal solution

Using the Borel transform, if $\widetilde{\varphi}$ is a solution of (55), we naturally obtain:

1. $e^{\zeta}\widehat{\varphi}(\zeta) - \widehat{\varphi}(\zeta) = \widehat{a}(\zeta)$, which implies $\widehat{\varphi}(\zeta) = \dfrac{\widehat{a}(\zeta)}{e^{\zeta}-1}$.
2. $\widehat{\varphi}$ defines a meromorphic function on $\mathbb{C}$ of exponential type in all directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$, with poles that can only be located in $2i\pi\mathbb{Z}^*$, since $\widehat{a}$ is an entire function vanishing at the origin satisfying:
$$|\widehat{\varphi}(\zeta)| = \left|\frac{\widehat{a}(\zeta)}{e^{\zeta}-1}\right| \leq \frac{C_0}{|\cos\theta|}e^{(C_1+1)|\zeta|} \ , \text{ for all } \zeta \in \mathbb{C} - i\mathbb{R} \ , \tag{59}$$
according to (57) and Lemma 5.

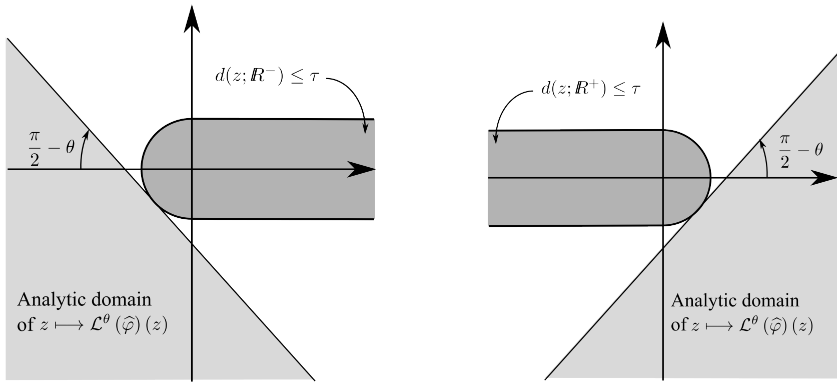
Thus, Equation (55) has a unique solution in $z^{-1}\mathbb{C}[\![z^{-1}]\!]$
$$\widetilde{\varphi} = \mathcal{B}^{-1}\left(\frac{\widehat{a}(\zeta)}{e^{\zeta}-1}\right) \ , \tag{60}$$
which is an S-resurgent function and satisfies:
$$\boldsymbol{\Delta}_{\omega}\widehat{\varphi} = \widehat{a}(\omega)\ \delta \tag{61}$$
for all $\omega \in 2i\pi\mathbb{Z}^*$. This explains the term of resurgent functions, since we can conceptually write (even if $\widehat{\varphi}(\omega)$ is infinite) $\boldsymbol{\Delta}_{\omega}\widehat{\varphi} = (e^{\omega}-1)\widehat{\varphi}(\omega)\ \delta$, and thus $\widehat{\varphi}$ "reappears" in the singularity ω .

4.3.3. Sectorial resummations

Moreover, $\widetilde{\varphi}$ is Borel-summable in all directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$.

The sectorial resummation principle gives us two analytical functions, φ_e and φ_w, which, by construction, are two resurgent functions in the geometric model and defined respectively on the east half-plane and west half-plane by:
$$\begin{cases} \varphi_e = \mathcal{L}^{\theta}\left(\dfrac{\widehat{a}(\zeta)}{e^{\zeta}-1}\right) = \mathcal{L}^{\theta}(\widehat{\varphi}),\ \theta \in \left]-\dfrac{\pi}{2};\dfrac{\pi}{2}\right[\\ \varphi_w = \mathcal{L}^{\theta}\left(\dfrac{\widehat{a}(\zeta)}{e^{\zeta}-1}\right) = \mathcal{L}^{\theta}(\widehat{\varphi}),\ \theta \in \left]\dfrac{\pi}{2};\dfrac{3\pi}{2}\right[\ . \end{cases} \tag{62}$$

Denoting by $d(z; F)$ the distance of $z \in \mathbb{C}$ to the set F and taking into account the properties of the Laplace transform, there exists $\tau > 0$ such that these two solutions are analytical on the domains $\mathcal{D}_e(\tau)$ and $\mathcal{D}_w(\tau)$ (see Figure 4) defined by $\mathcal{D}_e(\tau) = \mathbb{C} - \{z \in \mathbb{C}\ ;\ d(z; \mathbb{R}_-) \leq \tau\}$ and $\mathcal{D}_w(\tau) = \mathbb{C} - \{z \in \mathbb{C}\ ;\ d(z; \mathbb{R}_+) \leq \tau\}$, according to Equation (59).



Holomorphic domain of the sectorial resummation φ_w .

Holomorphic domain of the sectorial resummation φ_e .

Figure 4: Illustration of the complementary set of $\mathcal{D}_e(\tau)$ and $\mathcal{D}_w(\tau)$.

4.3.4. Characterization of φ_e and φ_w

It is not difficult to see that $\varphi_{e|\mathbb{R}_{>\tau}}$ is a solution of (55) which is defined, by construction, on $\mathbb{R}_{>\tau} = \{x \in \mathbb{R}\ ;\ x > \tau\}$ and admits 0 as limit near $+\infty$ (since φ_e is a Laplace transform). Actually, such a solution turns out to be unique: if ψ is another solution of (55) defined on $\mathbb{R}_{>\tau'}$ with 0 as limit near $+\infty$, $\psi - \varphi_e$ becomes a 1-periodic function defined on $\mathbb{R}_{>max(\tau,\tau')}$ which also has 0 as limit near $+\infty$; this necessarily imposes the equality $\psi = \varphi_e$ in $\mathbb{R}_{>max(\tau,\tau')}$.

Likewise, $\varphi_{w|\mathbb{R}^-_{<-\tau}}$, where $\mathbb{R}^-_{<-\tau} = \{x \in \mathbb{R}\ ;\ x < -\tau\}$, is the unique solution of (55) defined on a set $\mathbb{R}^-_{<-\tau}$ and admiting 0 as limit near $-\infty$.

On the other side, from Equation (56), we deduce that

$$|a(z)| \leq \frac{C_0 C_1}{|z|(|z| - C_1)}\ ,\quad \text{if } z \in \mathbb{C} - D_e(C_1)\ . \tag{63}$$

Moreover, it is clear, if $z \in \mathbb{C} - D_e(C_1)$, where C_1 is defined by Equation (56), that we have $z + k \in \mathbb{C} - D_e(C_1)$ for all $k \in \mathbb{N}^*$. In particular, $|z + k| > C_1$. Therefore,

$$|a(z+k)| \leq \frac{C_0 C_1}{|z+k|(|z+k| - C_1)}\ ,\quad \text{if } z \in \mathbb{C} - D_e(C_1) \text{ and } k \in \mathbb{N}^*\ . \tag{64}$$

Moreover, if K is a compact subset of $\mathbb{C} - D_e(C_1)$, and if $a > 0$ is such that $K \subset D(0, a)$, we obtain:

$$|a(z+k)| \leq \frac{C_0 C_1}{\left(\sqrt{k(k-2a)+C_1^2} - C_1\right)^2} \ , \tag{65}$$

if $z \in K \subset\subset \mathbb{C} - D_e(C_1)$ and $k \in \mathbb{N}^*_{>2a}$, according to $|z+k| \geq \sqrt{k^2 - 2ak + C_1^2}$ for all $z \in \mathbb{C} - D_e(C_1)$ and $k \in \mathbb{N}^*$.

Consequently, $z \mapsto \sum_{k>0} a(z+k)$ is a normally convergent series on every compact subset of $\mathbb{C} - D_e(C_1)$ of holomorphic functions on $\mathbb{C} - D_e(C_1)$. Thus, it defines a holomorphic function over $\mathbb{C} - D_e(C_1)$.

Using simillar arguments with $\mathbb{C} - D_w(C_1)$, we see that $z \mapsto -\sum_{k \leq 0} a(z+k)$ is also a holomorphic function of $\mathbb{C} - D_w(C_1)$.

Moreover, it is clear that $z \mapsto \sum_{k>0} a(z+k)$ and $z \mapsto -\sum_{k \leq 0} a(z+k)$ are two solutions of (55). The limit of these at $+\infty$ and $-\infty$ respectively is 0. Consequently, they coincide with φ_e and φ_w on a set which type is $\mathbb{R}_{>\tau}$ and $\mathbb{R}_{<-\tau}$ for a certain constant $\tau > 0$. According to the analytic continuation principle, we therefore have proven the following

Lemma 6. *Let $a \in z^{-2}\mathbb{C}\{z^{-1}\}$.*
1. *The difference equation $\varphi(z-1) - \varphi(z) = a(z)$ admits a unique solution defined on a set $\mathbb{R}_{>\tau}$, for a certain constant $\tau > 0$, with 0 as limit near $+\infty$, given by:*

$$\varphi_e(z) \quad = \quad \mathcal{L}^0\left(\frac{\widehat{a}(\zeta)}{e^\zeta - 1}\right) \quad = \quad \sum_{k>0} a(z+k) \ . \tag{66}$$

It turns out that φ_e is holomorphic on $\mathbb{C} - D_e(\tau)$.
2. *The difference equation $\varphi(z-1) - \varphi(z) = a(z)$ admits a unique solution defined on a set $\mathbb{R}_{<-\tau}$, for a certain constant $\tau > 0$, with 0 as limit near $-\infty$, given by:*

$$\varphi_w(z) \quad = \quad \mathcal{L}^\pi\left(\frac{\widehat{a}(\zeta)}{e^\zeta - 1}\right) \quad = \quad -\sum_{k \leq 0} a(z+k) \ . \tag{67}$$

It turns out that φ_w is holomorphic on $\mathbb{C} - D_w(\tau)$.

4.3.5. Stokes phenomenon

Since $\mathcal{D}_e \cap \mathcal{D}_w$ has two connected components, which are $\{z \in \mathbb{C} \, ; \, \Im m\, z > \tau\}$ and $\{z \in \mathbb{C} \, ; \, \Im m\, z < -\tau\}$, we can evaluate the difference $\varphi_e - \varphi_w$ in each of these connected components. Then, the residue theorem gives us for z such that $\Im m\, z < -\tau$:

$$\begin{aligned} \int_\gamma \widehat{\varphi}(\zeta) e^{-z\zeta} \, d\zeta &= \sum_{k=1}^{m} 2i\pi \mathrm{Res} \left(z \longmapsto \frac{\widehat{a}(\zeta) e^{-\omega z}}{e^{\zeta} - 1} \; ; \; 2ik\pi \right) \quad (68) \\ &= \sum_{k=1}^{m} 2i\pi \widehat{a}(2ik\pi) e^{-2ik\pi z} \; , \end{aligned}$$

where γ is the path described on Figure 5.

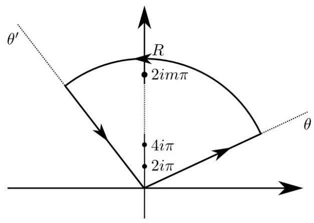


Figure 5: An integration path.

Besides, it is easy to see that the limit of the previous integral on the path restricted to the arc of circle is 0 as R approaches $+\infty$. The series $\sum_{k>0} 2i\pi \widehat{a}(2ik\pi) e^{-2ik\pi z}$ is a convergent one when $\Im m\, z < -\tau$. Thus, when R approaches $+\infty$, we obtain:

$$\forall z \in \mathbb{C} \; , \; \Im m\, z < -\tau \; , \; \varphi_e(z) - \varphi_w(z) = \sum_{\omega \in 2i\pi \mathbb{N}^*} 2i\pi \widehat{a}(\omega) e^{-\omega z} \; , \quad (69)$$

which expresses the Stokes phenomenon in the south part of the complex plane.

In the same way, but with a symmetric contour relative to the real axis, we obtain:

$$\forall z \in \mathbb{C} \; , \; \Im m\, z > \tau \; , \; \varphi_e(z) - \varphi_w(z) = \sum_{\omega \in -2i\pi \mathbb{N}^*} 2i\pi \widehat{a}(\omega) e^{-\omega z} \; , \quad (70)$$

which expresses the Stokes phenomenon in the north part of the complex plane.

4.3.6. Conclusion of the solution of the generic 1*-order difference equation.*

The difference equation

$$\widetilde{\varphi}(z-1) - \widetilde{\varphi}(z) = a(z) \; ,$$

where $a \in z^{-2}\mathbb{C}\left\{z^{-1}\right\}$ is a holomorphic germ near infinity, has:

- a unique formal solution $\widetilde{\varphi}$ in $z^{-1}\mathbb{C}[\![z^{-1}]\!]$ which turns out to be S-resurgent.
- a unique solution $\widehat{\varphi}$ in the convolutive model which is also S-resurgent.
- two analytical solutions φ_e and φ_w, defined respectively as the sectorial resummations of $\widehat{\varphi}$ in the east and west directions which have an asymptotic expansion near infinity (in those respective directions) which is nothing but $\widetilde{\varphi}$.

4.4. On the resurgence of Hurwitz multizeta functions

In this section, we will use recursively the resurgent study of the generic 1-order difference equation made in the previous section, in order to generalize it to a resurgent study of the 1-order mould difference equation

$$\Delta_-(M^\bullet) = M^\bullet \times J^\bullet \ . \tag{71}$$

Let us notice that any solution of this mould equation, with perhap another condition (such as a given limit near infinity or the absence of constant term), will turn out to be a symmetr$\underline{e}$l mould. Consequently, to remind us of this property, we are going to add systematically the letter $\underline{e}$ to the name of the different types of solutions we will consider.

This resurgent study of (71) will lead us to define four moulds, related to Hurwitz multizeta functions, in the same way that we have found four solutions of the generic 1-order difference equation:

- $\rightsquigarrow$ $\widetilde{\mathcal{H}e}^\bullet$ will denote a mould of formal solutions of (71).
- $\rightsquigarrow$ $\widehat{\mathcal{H}e}^\bullet$ will denote the Borel transform of any formal solution $\widetilde{\mathcal{H}e}^\bullet$ of (71):

$$\widehat{\mathcal{H}e}^{\underline{\mathbf{s}}} = \mathcal{B}\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{s}}}\right) \ . \tag{72}$$

- $\rightsquigarrow$ $\mathcal{H}e_e^\bullet$ and $\mathcal{H}e_w^\bullet$, defined respectively, if possible, as the sectorial resummations of $\widehat{\mathcal{H}e}^\bullet$ in the east and west directions.

In order to always have $\Delta_-(M^{\underline{\mathbf{s}}}) \in z^{-2}\mathbb{C}[\![z^{-1}]\!]$, we will define these moulds on the subset $\mathcal{S}^\star$ of $\mathbb{N}_1^\star$ of sequences $\underline{\mathbf{s}}$ of positive integers such that $s_1 \geq 2$.

In order to be consistent with the previously introduced notations, let us mention that all these four notations might have a subscript '+' that we have omitted throughout all the rest of this section for simplicity. Nevertheless, we still use the subscript '+' or '-' to specify which Hurwitz multizeta function we are currently dealing with.

These notations and restrictions being introduced, we now are going to prove the following

Theorem 1. 1. *There exists a unique mould* $\widetilde{\mathcal{H}e}^{\bullet}$*, defined over the subset* $\mathcal{S}^\star$ *of* $\mathbb{N}_1^\star$*, of formal series near infinity which are solutions of the mould difference equation* $\Delta_-(M^\bullet) = M^\bullet \times J^\bullet$ *. These functions turn out to be:*

- *formal* S*-resurgent functions.*
- *Borel-summable formal series in all directions* $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ *and* $\theta \in \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$.

2. *There exist two unique moulds* $\mathcal{H}e_e^\bullet$ *and* $\mathcal{H}e_w^\bullet$ *of analytical functions defined respectively on the east half complex plane and the west half complex plane (* i.e. *on domains such as those shown on Figure 4), that admit* 0 *as a limit near infinity, but in the real axis: they are the moulds of the sectorial sums of* $\widetilde{\mathcal{H}e}^{\bullet}$ *in the geometric model and turn out to be moulds valued in the algebra of resurgent functions.*
3. *The Stokes phenomenon obtained from the sectorial sums can be expressed by:*
$$\mathcal{H}e_e^\bullet \times (\mathcal{H}e_w^\bullet)^{\times -1} = \mathcal{T}e^\bullet \ , \tag{73}$$
where $\mathcal{T}e^\bullet$ *denotes the mould of multitangent functions.*

4.4.1. Reminders on the multitangent functions

The mould $\mathcal{T}e^\bullet$ denotes the mould of multitangent functions and is defined for all sequences $\underline{\mathbf{s}}$ of positive integers satisfying $s_1 \geq 2$ and $s_r \geq 2$ (if $\underline{\mathbf{s}}$ is of length r) and for all $z \in \mathbb{C} - \mathbb{Z}$ by:

$$\mathcal{T}e^{s_1,\cdots,s_r}(z) = \sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ . \tag{74}$$

These multitangent functions have been extensively studied in [6] from a combinatorial, algebraic and analytic point of view. It results from this study that the multitangent functions are a nice functional generalization of the multizeta values.

The multitangents satisfy not only the stuffle relations (defined in Section 2.4) but also some quite mysterious nontrivial linear relations (see [6]) which are analogue in a certain sense to the other relations satisfied by the multizeta values (the shuffle relations as well as the double shuffle relations, to be precise; see [56] for an introduction).

For instance, one has:

$$4\mathcal{T}e^{3,1,3} - 2\mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,1,2,2} = 0 \ , \tag{75}$$

which is equivalent to

$$\zeta(2,1,1) = 4\zeta(3,1) \tag{76}$$

and

$$\zeta(2)\zeta(2,1) = \zeta(2,1,2) + 6\zeta(3,1,1) + 3\zeta(2,2,1)\ . \tag{77}$$

Note that (77) is exactly a shuffle relation for the multizeta values and (76) is a consequence of the three families of relations (the shuffle relations, the stuffle relations and the double shuffle relations) which are conjectured to span all the algebraic relations between multizeta values (see [20] and [56]).

There is no doubt that, in comparison to the algebra of multitangent functions or multizeta values, the Hurwitz multizeta functions span a simple algebra since they satisfy only algebraic relations coming from the stuffle relations, which is nevertheless an interesting one because it is isomorphic to a famous algebra in combinatorics, the algebra of quasisymmetric functions *QSym*, as will be seen in Section 5.

Moreover, the mould of multitangent functions has a nice mould factorization in terms of Hurwitz multizeta functions $\mathcal{H}e_+^\bullet$ and $\mathcal{H}e_-^\bullet$ which will enlighten the Stokes phenomena of Hurwitz multizeta functions.

Lemma 7. *We have :* $\mathcal{T}e^\bullet = \mathcal{H}e_+^\bullet \times (1^\bullet + J^\bullet) \times \mathcal{H}e_-^\bullet$.

PROOF. See [6], p. 55. □

4.4.2. On the case of length 0 *and* 1

Because we want the moulds $\widetilde{\mathcal{H}e}^\bullet$, $\widehat{\mathcal{H}e}^\bullet$, $\mathcal{H}e_e^\bullet$ and $\mathcal{H}e_w^\bullet$ to be symmetr$\underline{e}$l moulds, we necessarily have to define their values on the empty sequence to be the unit of the product used for their symmetr$\underline{e}$lity properties, *i.e.* the ordinary product for $\widetilde{\mathcal{H}e}^\bullet$, $\mathcal{H}e_e^\bullet$ and $\mathcal{H}e_w^\bullet$, but the convolution product for $\widehat{\mathcal{H}e}^\bullet$:

$$\widetilde{\mathcal{H}e}^\emptyset = \mathcal{H}e_e^\emptyset = \mathcal{H}e_w^\emptyset = 1 \qquad , \qquad \widehat{\mathcal{H}e}^\emptyset = \delta\ . \tag{78}$$

The case of length 1 has already been done in Section 4.3. Let us remind what we have obtained. The difference equation

$$\Delta_- \widetilde{\mathcal{H}e}^s(z) = \frac{1}{z^s}\ ,\ s \geq 2\ , \tag{79}$$

has a unique formal solution in $z^{-2}\mathbb{C}[\![z^{-1}]\!]$ which is S-resurgent:

$$\widetilde{\mathcal{H}e}^s = \mathcal{B}^{-1}(\widehat{\mathcal{H}e}^s)\ ,\ \text{where } \widehat{\mathcal{H}e}^s(\zeta) = \frac{1}{(s-1)!}\frac{\zeta^{s-1}}{e^\zeta - 1} \in \zeta\mathbb{C}\{\zeta\}\ . \tag{80}$$

It satisfies for all $n \in \mathbb{Z}^*$:

$$\mathbf{\Delta}_{2in\pi}\widehat{\mathcal{H}e}^s = \text{Res}\left(\widehat{\mathcal{H}e}^s, 2in\pi\right)\delta = \frac{(2in\pi)^{s-1}}{(s-1)!}\delta = \widehat{J}^s(2in\pi)\delta\ . \tag{81}$$

Moreover, $\widetilde{\mathcal{H}e}^{s}$ is Borel-summable in all the directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ or $\theta \in \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$ and, according to (66) and (67):

- Equation (79) has a unique solution, denoted by $\mathcal{H}e^s_e$, defined over $\mathbb{R}^+$ such that $\mathcal{H}e^s_e(z) \underset{\substack{z\longrightarrow+\infty \\ z\in\mathbb{R}^+}}{\longrightarrow} 0$:

$$\mathcal{H}e^s_e = \mathcal{L}^0\left(\widehat{\mathcal{H}e}^{s}\right) = \mathcal{H}e^s_+ \ . \tag{82}$$

- Equation (79) has a unique solution, denoted by $\mathcal{H}e^s_w$, defined over $\mathbb{R}^-$ such that $\mathcal{H}e^s_w(z) \underset{\substack{z\longrightarrow-\infty \\ z\in\mathbb{R}^-}}{\longrightarrow} 0$:

$$\mathcal{H}e^s_w = \mathcal{L}^\pi\left(\widehat{\mathcal{H}e}^{s}\right) = -(J^s + \mathcal{H}e^s_-) \ . \tag{83}$$

Finally, the Stokes phenomena are given in two different ways. The first one uses the expression of $\mathcal{H}e^s_e$ and $\mathcal{H}e^s_w$ in terms of $\mathcal{H}e^s_+$ and $\mathcal{H}e^s_-$ and the trifactorization of multitangent functions. The second one uses the residue theorem, as seen in Section 4.3.

$$\begin{aligned}
\mathcal{H}e^s_e(z) - \mathcal{H}e^s_w(z) &= \mathcal{H}e^s_+(z) + J^s(z) + \mathcal{H}e^s_-(z) = \mathcal{T}e^s(z) \ . && (84)\\
\mathcal{H}e^s_e(z) - \mathcal{H}e^s_w(z) &= \mathrm{sg}(\Im m\, z)\frac{(2i\pi)^s}{(s-1)!}\sum_{k>0} k^{s-1}e^{-2ik\pi z} \ . && (85)
\end{aligned}$$

Let us remind that all these quantities are well-defined since $s \geq 2$.

4.4.3. The case of length 2

Let us now consider the difference equation

$$\Delta_-(\widetilde{\mathcal{H}e}^{s_1,s_2}) = \widetilde{\mathcal{H}e}^{s_1}\cdot J^{s_2} \ , \tag{86}$$

with $s_1 \geq 2$ and $s_2 \geq 1$.

Its formal solution is simple, according to

Lemma 8. *There exists a unique formal series near infinity* $\widetilde{\mathcal{H}e}^{s_1,s_2}$*, which is a solution of the difference equation* (86)*:*

$$\widetilde{\mathcal{H}e}^{s_1,s_2} = \mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right) \ , \tag{87}$$

where

$$\widehat{\mathcal{H}e}^{s_1,s_2}(\zeta) = \frac{1}{e^\zeta - 1}\left(\widehat{\mathcal{H}e}^{s_1} \star \widehat{J}^{s_2}\right)(\zeta) \ . \tag{88}$$

Proof. Any formal solution $\widetilde{\mathcal{H}e}^{s_1,s_2}$ of (86) has a Borel transform $\widehat{\mathcal{H}e}^{s_1,s_2}$ which satisfies:

$$(e^{\zeta}-1)\widehat{\mathcal{H}e}^{s_1,s_2}(\zeta)=\left(\widehat{\mathcal{H}e}^{s_1}\star\widehat{J}^{s_2}\right)(\zeta)\ . \tag{89}$$

Consequently, $\widetilde{\mathcal{H}e}^{s_1,s_2}$ is necessarily defined by (87) and (88), which proves unicity.

Conversely, defining $\widetilde{\mathcal{H}e}^{s_1,s_2}$ by (87) and (88), and using successively the second point of Property 5 rewritten as

$$\mathcal{B}^{-1}\left(\widehat{\varphi}\right)\circ l^{-1}=\mathcal{B}^{-1}\left(exp\cdot\widehat{\varphi}\right) \tag{90}$$

and then its third point, we successively have:

$$\begin{aligned}
\Delta_{-}\left(\widetilde{\mathcal{H}e}^{s_1,s_2}\right)(z) &= \mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right)\circ l^{-1}(z)-\mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right)(z)\\
&= \mathcal{B}^{-1}\left(exp\cdot\widehat{\mathcal{H}e}^{s_1,s_2}\right)(z)-\mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right)(z)\\
&= \mathcal{B}^{-1}\left(\frac{e^{\zeta}}{e^{\zeta}-1}\cdot\left(\widehat{\mathcal{H}e}^{s_1}\star\widehat{J}^{s_2}\right)\right)(z)\\
&\quad -\mathcal{B}^{-1}\left(\frac{1}{e^{\zeta}-1}\cdot\left(\widehat{\mathcal{H}e}^{s_1}\star\widehat{J}^{s_2}\right)\right)(z)\\
&= \mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1}\star\widehat{J}\right)(z)\\
&= \widetilde{\mathcal{H}e}^{s_1}\cdot J^{s_2}\ ,
\end{aligned}$$

which proves the existence of a formal solution near infinity of (86) and concludes the proof of the lemma. □

Note that:

Lemma 9. *$\widehat{\mathcal{H}e}^{s_1,s_2}$ is a S-resurgent function, as well as $\widetilde{\mathcal{H}e}^{s_1,s_2}$.*

Proof. As recalled in Section 4.2, the important point concerns the stability of $\widetilde{\mathrm{RES}}^{\mathrm{simple}}$ (resp. $\widehat{\mathrm{RES}}^{\mathrm{simple}}$) under the ordinary product (resp. convolution product) of formal power series (see [52]).

Since $\widehat{\mathcal{H}e}^{s_1}$, $\zeta\longmapsto\widehat{J}^{s_2}(\zeta)$ and $\zeta\longmapsto\dfrac{1}{e^{\zeta}-1}$ are S-resurgent functions, $\widehat{\mathcal{H}e}^{s_1,s_2}$ is a S-resurgent function in the convolutive model. Consequently, $\widetilde{\mathcal{H}e}^{s_1,s_2}$ is a S-resurgent function in the formal model. □

We are now able to look at the Borel-summability of $\widetilde{\mathcal{H}e}^{s_1,s_2}$.

Lemma 10. *$\zeta\mathbb{C}\{\zeta\}\star\mathbb{C}\{\zeta\}\subset\zeta^2\mathbb{C}\{\zeta\}$, where $\star$ denotes the convolution product.*

PROOF. It is enough to prove, for all non negative integers p and q, that $\zeta^{p+1} \star \zeta^q \in \zeta^2\mathbb{C}\{\zeta\}$, which is straightforward. $\square$

As a consequence of (80), $\zeta \longmapsto \widehat{\mathcal{H}e}^{s_1} \star \widehat{J}^{s_2}(\zeta)$ is a meromorphic function over $\mathbb{C}$ vanishing at the origin with poles located in $2i\pi\mathbb{Z}$. Therefore, $\widehat{\mathcal{H}e}^{s_1,s_2}$ defines a meromorphic function on $\mathbb{C}$ with poles which can only be localised in $2i\pi\mathbb{Z}^*$. Let us mention here that this fact will be reobtained in Section 4.4.6 as a consequence of the computation of the alien derivatives done in Theorem 2. This proves its Borel summability in all directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$ if it is of exponential type in all directions. But, this is a consequence of the exponential type of $\zeta \longrightarrow \left(e^\zeta - 1\right)^{-1}$ and the fact that if f and g are of exponential type 0, it is also the case for $f \star g$.

So, we have proved

Lemma 11. *$\widetilde{\mathcal{H}e}^{s_1,s_2}$ is a Borel summable formal series in all directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ and $\theta \in \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$ and gives rise to two analytical functions $\mathcal{H}e_e^{s_1,s_2}$ and $\mathcal{H}e_w^{s_1,s_2}$ which are solutions of (86) and defined by:*

$$\mathcal{H}e_e^{s_1,s_2}(z) = \mathcal{L}^0\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right) \quad \text{and} \quad \mathcal{H}e_w^{s_1,s_2}(z) = \mathcal{L}^\pi\left(\widehat{\mathcal{H}e}^{s_1,s_2}\right), \tag{91}$$

on domains as shown in Figure 4.

Using the characterization of Lemma 6, we see that:

Lemma 12. 1. *$\mathcal{H}e_e^{s_1,s_2}$ is the unique function defined over a neighbourhood of $\mathbb{R}^+$ satisfying:*

$$\begin{cases} \Delta_- \mathcal{H}e_e^{s_1,s_2} = \mathcal{H}e_e^{s_1} \cdot J^{s_2} \\ \mathcal{H}e_e^{s_1,s_2}(z) \underset{\substack{z \longrightarrow +\infty \\ z\in\mathbb{R}^+}}{\longrightarrow} 0 \end{cases} \tag{92}$$

and is given by

$$\begin{aligned} \mathcal{H}e_e^{s_1,s_2}(z) &= \sum_{n_2>0} \mathcal{H}e_e^{s_1}(n_2+z)J^{s_2}(n_2+z) \\ &= \sum_{n_1>n_2>0} \frac{1}{(n_1+z)^{s_1}(n_2+z)^{s_2}} = \mathcal{H}e_+^{s_1,s_2}(z) \end{aligned} \tag{93}$$

2. *$\mathcal{H}e_w^{s_1,s_2}$ is the unique function defined over a neighbourhood of $\mathbb{R}^-$ satisfying:*

$$\begin{cases} \Delta_- \mathcal{H}e_w^{s_1,s_2} = \mathcal{H}e_w^{s_1} \cdot J^{s_2} \\ \mathcal{H}e_w^{s_1,s_2}(z) \underset{\substack{z \longrightarrow -\infty \\ z\in\mathbb{R}^-}}{\longrightarrow} 0 \end{cases} \tag{94}$$

and is given by

$$\begin{aligned}\mathcal{H}e_w^{s_1,s_2}(z) &= -\sum_{n_2\leq 0}\mathcal{H}e_w^{s_1}(n_2+z)J^{s_2}(n_2+z)\\ &= +\sum_{n_1\leq n_2\leq 0}\frac{1}{(n_1+z)^{s_1}(n_2+z)^{s_2}}\end{aligned} \tag{95}$$

4.4.4. On the general case

Because the general case is proven exactly in the same way as in length 2, we only give here the recursion formulas which are useful and conclude the proof of parts 1 and 2 of Theorem 1.

Let us suppose that, for a positive integer r, we have constructed $\widetilde{\mathcal{H}e}^{s_1,\cdots,s_r}$, $\widehat{\mathcal{H}e}^{s_1,\cdots,s_r}$, $\mathcal{H}e_e^{s_1,\cdots,s_r}$ and $\mathcal{H}e_w^{s_1,\cdots,s_r}$ for all $(s_1,\cdots,s_r)\in(\mathbb{N}^*)^r$ such that $s_1\geq 2$.

Then, the resurgent study of the difference equation

$$\Delta_-(\widetilde{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}) = \widetilde{\mathcal{H}e}^{s_1,\cdots,s_r}\cdot J^{s_{r+1}}\ , \tag{96}$$

with $s_1\geq 2$ and $(s_2,\cdots,s_{r+1})\in(\mathbb{N}^*)^r$, is given by:

$$\widehat{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}(\zeta) = \frac{1}{e^\zeta-1}\left(\widehat{\mathcal{H}e}^{s_1,\cdots,s_r}\star\widehat{J}^{s_{r+1}}\right)(\zeta)\ . \tag{97}$$

$$\widetilde{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}(z) = \mathcal{B}^{-1}\left(\widehat{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}\right)(z)\ . \tag{98}$$

$$\mathcal{H}e_e^{s_1,\cdots,s_{r+1}}(z) = \mathcal{L}^0\left(\widehat{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}\right)(z) \tag{99}$$

$$\mathcal{H}e_w^{s_1,\cdots,s_{r+1}}(z) = \mathcal{L}^\pi\left(\widehat{\mathcal{H}e}^{s_1,\cdots,s_{r+1}}\right)(z)\ . \tag{100}$$

These equations define four moulds over $\mathcal{S}^\star$: $\widetilde{\mathcal{H}e}^\bullet$, $\widehat{\mathcal{H}e}^\bullet$, $\mathcal{H}e_w^\bullet$ and $\mathcal{H}e_e^\bullet$. As we have explained at the begining of Section 4.4.2, we wanted these moulds to be symmetr$\underline{e}$l .

Proposition 8. *The mould $\widetilde{\mathcal{H}e}^\bullet$ is a symmetr$\underline{e}$l mould, for the usual product of formal power series.*

Proof. We have to prove that, for all sequences $(\underline{\mathbf{u}},\underline{\mathbf{v}})\in(\mathcal{S}^\star)^2$

$$\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}}\cdot\widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}} = \sum_{\underline{\mathbf{w}}\in sh\underline{e}(\underline{\mathbf{u}},\underline{\mathbf{v}})}\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}}\ . \tag{101}$$

Let us show this proposition by an induction process on $l(\underline{\mathbf{u}})+l(\underline{\mathbf{v}})$. Equation (101) is true for $\underline{\mathbf{u}}=\underline{\mathbf{v}}=\emptyset$, which allows to begin the induction process, but also when $\underline{\mathbf{u}}=\emptyset$ or $\underline{\mathbf{v}}=\emptyset$.

Therefore, let us suppose that Equation (101) is true for all sequences $(\underline{\mathbf{u}},\underline{\mathbf{v}})\in(\mathcal{S}^\star)^2$ such that $l(\underline{\mathbf{u}})+l(\underline{\mathbf{v}})\leq N$, with $N\in\mathbb{N}^*$. From now on, we fix two particular sequences $(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \in (\mathcal{S}^\star)^2$ such that $l(\underline{\mathbf{u}}) + l(\underline{\mathbf{v}}) = N - 1$ and two particular letters a and b (*i.e.* $a,\ b \in \mathbb{N}^*$). We will show that Equation (101) is valid for $(\underline{\mathbf{u}} \cdot a, \underline{\mathbf{v}} \cdot b)$.

1. Using the fact that Δ_- is a (Id, τ^{-1})-derivation and the recursive definition (17) of the stuffle product, we successively have:

$$\begin{aligned}
\Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}\cdot a} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}\cdot b}\right) &= \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}\cdot a}\right) \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}\cdot b} + \tau^{-1}\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}\cdot a}\right) \cdot \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}\cdot b}\right) \\
&= \widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}} \cdot J^a \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}\cdot b} + \widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}} \cdot J^a \cdot J^b + \widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}\cdot a} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}} \cdot J^b \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}}\cdot b)} \widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}} \cdot J^a + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}} \cdot J^{a+b} \\
&\quad + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}} \cdot J^b \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}}\cdot b)} \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}\cdot a}\right) + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}})} \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}\cdot (a+b)}\right) \\
&\quad + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}})} \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}\cdot b}\right) \\
&= \left(\sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}})\cdot b} + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}})\cdot (a+b)} + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}, \underline{\mathbf{v}}\cdot b)\cdot a}\right) \left(\Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}}\right)\right) \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}}\cdot b)} \Delta_-\left(\widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}}\right)
\end{aligned}$$

Therefore, for all sequences $(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \in (\mathcal{S}^\star)^2$ such that $l(\underline{\mathbf{u}}) + l(\underline{\mathbf{v}}) = N - 1$ and two letters a and b, there exists a correction $corr^{\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}}\cdot b} \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, which is 1-periodic and satisfies:

$$\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}}\cdot a} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}}\cdot b} = \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}}\cdot b)} \widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}} + corr^{\underline{\mathbf{u}}\cdot a, \underline{\mathbf{v}}\cdot b} \ . \tag{102}$$

2. We now extend the definition of the correction $corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}}$ to sequences $(\underline{\mathbf{a}}, \underline{\mathbf{b}}) \in (\mathcal{S}^\star)^2$ such that $l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) \leq N$ by $corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} = 0$, so that

$$\widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{b}}} = \sum_{\underline{\mathbf{c}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{b}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} + corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \ , \tag{103}$$

for all sequences $(\underline{\mathbf{a}}, \underline{\mathbf{b}}) \in (\mathcal{S}^\star)^2$ such that $l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) \leq N+1$. Consequently, for any sequences $(\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}) \in (\mathcal{S}^\star)^3$ satisfying $l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) + l(\underline{\mathbf{c}}) \leq N+1$, we have:

$$\begin{aligned}
\widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} &= \widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot \left(\sum_{\underline{\mathbf{d}} \in sh\underline{e}(\underline{\mathbf{b}}, \underline{\mathbf{c}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{d}}} + corr^{\underline{\mathbf{b}}, \underline{\mathbf{c}}} \right) \\
&= \sum_{\underline{\mathbf{d}} \in sh\underline{e}(\underline{\mathbf{b}}, \underline{\mathbf{c}})} \sum_{\underline{\mathbf{e}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{d}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{d}}} + \widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot corr^{\underline{\mathbf{b}}, \underline{\mathbf{c}}} \\
&= \sum_{\underline{\mathbf{e}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{e}}} + \widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot corr^{\underline{\mathbf{b}}, \underline{\mathbf{c}}}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} &= \left(\sum_{\underline{\mathbf{d}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{b}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{d}}} + corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \right) \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} \\
&= \sum_{\underline{\mathbf{d}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{b}})} \sum_{\underline{\mathbf{e}} \in sh\underline{e}(\underline{\mathbf{c}}, \underline{\mathbf{d}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{e}}} + corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} \\
&= \sum_{\underline{\mathbf{e}} \in sh\underline{e}(\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}})} \widetilde{\mathcal{H}e}^{\underline{\mathbf{e}}} + corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}}
\end{aligned}$$

Consequently, $l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) + l(\underline{\mathbf{c}}) \leq N+1 \Longrightarrow \widetilde{\mathcal{H}e}^{\underline{\mathbf{a}}} \cdot corr^{\underline{\mathbf{b}}, \underline{\mathbf{c}}} = corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}}$. Better, if $l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) + l(\underline{\mathbf{c}}) \leq N+1$ and $\underline{\mathbf{a}} \neq \emptyset$, we necessarily have $l(\underline{\mathbf{b}}) + l(\underline{\mathbf{c}}) \leq N$, so $corr^{\underline{\mathbf{b}}, \underline{\mathbf{c}}} = 0$, which implies for all sequences $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}} \in \mathcal{S}^\star$ that:

$$\begin{cases} l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) + l(\underline{\mathbf{c}}) \leq N+1 \\ \underline{\mathbf{a}} \neq \emptyset \end{cases} \Longrightarrow corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{c}}} = 0 \Longrightarrow corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} = 0 \; .$$

Moreover, it is clear that $corr^{\underline{\mathbf{a}}, \emptyset} = corr^{\emptyset, \underline{\mathbf{b}}} = 0$ if $l(\underline{\mathbf{a}}) = l(\underline{\mathbf{b}}) = N+1$. This allow us to sum up the values of the correction $corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}}$:

$$l(\underline{\mathbf{a}}) + l(\underline{\mathbf{b}}) \leq N+1 \Longrightarrow corr^{\underline{\mathbf{a}}, \underline{\mathbf{b}}} = 0 \; . \tag{104}$$

Finally, we can update Equation (102):

$$\widetilde{\mathcal{H}e}^{\underline{\mathbf{u}} \cdot a} \cdot \widetilde{\mathcal{H}e}^{\underline{\mathbf{v}} \cdot b} = \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}} \cdot a, \underline{\mathbf{v}} \cdot b)} \widetilde{\mathcal{H}e}^{\underline{\mathbf{w}}} \; , \tag{105}$$

which concludes the proof of the proposition, by an induction process. □

We therefore obtain the following

Corollary 1. 1. *The mould $\widehat{\mathcal{H}e}^\bullet$ is a symmetr$\underline{e}$l mould, for the convolutive product of formal power series.*
2. *The moulds $\mathcal{H}e_w^\bullet$ and $\mathcal{H}e_e^\bullet$ are symmetr$\underline{e}$l moulds, for the usual product of holomorphic functions.*

PROOF. The first point is straighforward, applying the Borel transform to the symmetr$\underline{e}$lity relations of $\widetilde{\mathcal{H}e}^\bullet$.

The second point is a direct application of the first point and the classical formula

$$\mathcal{L}^\theta(f)\cdot\mathcal{L}^\theta(g)=\mathcal{L}^\theta(f\star g)\ . \tag{106}$$

□

4.4.5. The Stokes phenomenon

Now, we have to focus on the Stokes phenomenon. Note that we have not eluded its study in length 2 because it cannot be expressed as a difference of the two Borel resummations on the east and west side. Nevertheless, it will be reinforced.

From the characterization given in Section 4.3.4 of φ_e and φ_w, it follows that:

$$\mathcal{H}e_e^{s_1,\cdots s_r}(z)=\sum_{0<n_r<\cdots<n_1<+\infty}\frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\ . \tag{107}$$

$$\mathcal{H}e_w^{s_1,\cdots s_r}(z)=\sum_{-\infty<n_1\leq\cdots\leq n_r\leq 0}\frac{(-1)^r}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\ . \tag{108}$$

According to Lemma 2, we consequently obtain:

Lemma 13. $\mathcal{H}e_w^\bullet=\Big((1^\bullet+J^\bullet)\times\mathcal{H}e_-^\bullet\Big)^{\times -1}$.

PROOF. In the sum defining $\mathcal{H}e_w^\bullet$, we can distinguish whether the index n_1 is equal to 0 or not. This gives successively:

$$\begin{aligned}
\mathcal{H}e_w^{s_1,\cdots s_r}(z) &= \sum_{-\infty<n_1\leq\cdots\leq n_r<0}\frac{(-1)^r}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\\
&\quad+\sum_{-\infty<n_1\leq\cdots\leq n_{r-1}\leq n_r=0}\frac{(-1)^r}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\\
&= \left(\mathcal{H}e_-^{s_1,\cdots,s_r}\right)^{\times -1}(z)-\frac{1}{z^{s_r}}\sum_{-\infty<n_1\leq\cdots\leq n_{r-1}<0}\frac{(-1)^{r-1}}{(n_1+z)^{s_1}\cdots(n_{r-1}+z)^{s_{r-1}}}\\
&\quad+\frac{1}{z^{s_{r-1}+s_r}}\sum_{-\infty<n_1\leq\cdots\leq n_{r-2}\leq n_{r-1}=n_r=0}\frac{(-1)^{r-2}}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}\\
&=\\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
&= \\
&= \left(\mathcal{H}e_-^{s_1,\cdots,s_r}\right)^{\times-1}(z) - \frac{1}{z^{s_r}}\left(\mathcal{H}e_-^{s_1,\cdots,s_{r-1}}\right)^{\times-1}(z) + \\
&\quad \frac{1}{z^{s_{r-1}+s_r}}\left(\mathcal{H}e_-^{s_1,\cdots,s_{r-2}}\right)^{\times-1}(z) + \cdots + \frac{(-1)^r}{z^{s_1+\cdots+s_r}}\ .
\end{aligned}
$$

Consequently, we have:

$$
\begin{aligned}
\mathcal{H}e_w^\bullet &= \left(\mathcal{H}e_-^\bullet\right)^{\times-1} \times \sum_{n\geq 0}(-J^\bullet)^{\times n} \qquad (109)\\
&= \left(\mathcal{H}e_-^\bullet\right)^{\times-1} \times (1^\bullet + J^\bullet)^{\times-1}\\
&= \Big((1^\bullet + J^\bullet) \times \mathcal{H}e_-^\bullet\Big)^{\times-1}\ .
\end{aligned}
$$

□

Therefore, the Stokes phenomena are expressed multiplicatively by:

$$
\mathcal{H}e_e^\bullet \times (\mathcal{H}e_w^\bullet)^{\times-1} = \mathcal{H}e_+^\bullet \times (1^\bullet + J^\bullet) \times \mathcal{H}e_-^\bullet = \mathcal{T}e^\bullet\ , \qquad (110)
$$

which concludes the proof of Theorem 1.

Note that Theorem 1 also has an analogue for the difference equation using Δ_+ satisfied by the Hurwitz multizeta functions. The east Borel resummation would have been the inverse of the mould $\mathcal{H}e_+^\bullet \times (1^\bullet + J^\bullet)$, while the west Borel resummation would have been the mould $\mathcal{H}e_-^\bullet$. Thus, the Stokes phenomenon would have been expressed as the inverse of the mould of multitangent functions.

4.4.6. On the alien derivations of $\widetilde{\mathcal{H}e}^\bullet$

Let us come back in this section to the computation of the alien derivatives of $\widetilde{\mathcal{H}e}^\bullet$ and $\widehat{\mathcal{H}e}^\bullet$. In an explicit way, this computation is actually difficult. But, from the mould difference equation $\Delta_-\widetilde{\mathcal{H}e}^\bullet = \widetilde{\mathcal{H}e}^\bullet \times J^\bullet$, it is possible to predict some formulas.

We will see that $\widetilde{\mathcal{H}e}^\bullet$ turns out to be a resurgent monomial. Such a function, according to [21], Section 3, is a resurgent function which behaves nicely under product (which is the case for $\widetilde{\mathcal{H}e}^\bullet$ because of the symmetr$\underline{e}$lity), under the ordinary derivation (which is also the case according to (5)) as well as under the alien derivatives. Thus, "the resurgent monomials are to the alien derivatives what the ordinary monomials are to the natural derivative" (see [21], p. 104).

Theorem 2. *For each $\omega \in \mathbb{C}$, there exists a scalar-valued altern$\underline{e}$l mould $Hen^\bullet_\omega$ defined over $\mathcal{S}^*$ such that:*

$$\boldsymbol{\Delta}_\omega \widetilde{\mathcal{H}e}^\bullet = Hen^\bullet_\omega \times \widetilde{\mathcal{H}e}^\bullet \ . \tag{111}$$

Moreover, $Hen^\bullet_\omega = 0^\bullet$ if $\omega \notin 2i\pi\mathbb{Z}^*$.

Let us remind that such a theorem directly goes back to the origin of the name of the resurgent function and generalizes the remark made relatively to Equation (61). Actually, computing an alien derivative of a resurgent function gives rise to a singular phenomenon: the resurgent function resurges in the variation of the singularities of its analytic continuation.

PROOF. Let us fix $\omega \in \mathbb{C}$.

We can always define a mould $\widetilde{Hen}^\bullet_\omega$ by $\boldsymbol{\Delta}_\omega \widetilde{\mathcal{H}e}^\bullet = \widetilde{Hen}^\bullet_\omega \times \widetilde{\mathcal{H}e}^\bullet$ where $\widetilde{Hen}^\bullet_\omega$ is a priori valued in $\mathbb{C}[\![z^{-1}]\!]$.

According to Property 7, we have:

$$\boldsymbol{\Delta}_\omega \left(\widetilde{\varphi}(z-1)\right) = e^\omega \left(\boldsymbol{\Delta}_\omega \widetilde{\varphi}\right)(z-1) \text{ , for all } \widetilde{\varphi} \in \widetilde{\mathrm{RES}}^{\mathrm{simple}} \ . \tag{112}$$

We can now compute in two different ways $\boldsymbol{\Delta}_\omega \left(\Delta_- \widetilde{\mathcal{H}e}^\bullet\right)$:

$$\begin{aligned}
\boldsymbol{\Delta}_\omega \left(\Delta_- \widetilde{\mathcal{H}e}^\bullet\right)(z) &= e^\omega \ \boldsymbol{\Delta}_\omega\big(\widetilde{\mathcal{H}e}^\bullet\big)(z-1) - \boldsymbol{\Delta}_\omega\big(\widetilde{\mathcal{H}e}^\bullet\big)(z) \\
&= (e^\omega - 1)\, \widetilde{Hen}^\bullet_\omega(z-1) \times \widetilde{\mathcal{H}e}^\bullet(z-1) + \Delta_- \left(\widetilde{Hen}^\bullet_\omega \times \widetilde{\mathcal{H}e}^\bullet\right)(z) \\
&= (e^\omega - 1)\, \widetilde{Hen}^\bullet_\omega(z-1) \times \widetilde{\mathcal{H}e}^\bullet(z-1) \\
&\quad + \Delta_- \left(\widetilde{Hen}^\bullet_\omega\right)(z) \times \widetilde{\mathcal{H}e}^\bullet(z-1) + \widetilde{Hen}^\bullet_\omega(z) \times \Delta_-\big(\widetilde{\mathcal{H}e}^\bullet\big)(z) \ .
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Delta}_\omega \left(\Delta_- \widetilde{\mathcal{H}e}^\bullet\right)(z) &= \boldsymbol{\Delta}_\omega\big(\widetilde{\mathcal{H}e}^\bullet\big)(z) \times J^\bullet(z) + \widetilde{\mathcal{H}e}^\bullet(z) \times \boldsymbol{\Delta}_\omega\left(J^\bullet\right)(z) \\
&= \widetilde{Hen}^\bullet_\omega(z) \times \widetilde{\mathcal{H}e}^\bullet(z) \times J^\bullet(z) + \widetilde{\mathcal{H}e}^\bullet(z) \times \mathcal{B}^{-1}\big(\boldsymbol{\Delta}_\omega \widehat{J}^\bullet\big)(z) \\
&= \widetilde{Hen}^\bullet_\omega(z) \times \Delta_-\big(\widetilde{\mathcal{H}e}^\bullet\big)(z) + \widetilde{\mathcal{H}e}^\bullet(z) \times \mathcal{B}^{-1}(0)(z) \\
&= \widetilde{Hen}^\bullet_\omega(z) \times \Delta_-\big(\widetilde{\mathcal{H}e}^\bullet\big)(z) \ .
\end{aligned}$$

Consequently, we obtain:

$$\widetilde{Hen}^\bullet_\omega(z+1) = e^\omega \widetilde{Hen}^\bullet_\omega(z) \ . \tag{113}$$

Using the fact that the Borel transform induces an isomorphism from $\mathbb{C}[\![z^{-1}]\!]$ to $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$, we can denote $\mathcal{B}(\widetilde{Hen}^\bullet_\omega)$ by

$$\mathcal{B}(\widetilde{Hen}^\bullet_\omega)(\zeta) = Hen^\bullet_\omega \delta + \widehat{Hen}^\bullet_\omega(\zeta) \ . \tag{114}$$

Now, we can interpret (113) in the convolutive model, by taking its Borel transform:

$$Hen^{\bullet}_{\omega}\delta + e^{-\zeta}\widehat{H}en^{\bullet}_{\omega}(\zeta) = e^{\omega}Hen^{\bullet}_{\omega}\delta + e^{\omega}\widehat{H}en^{\bullet}_{\omega}(\zeta) \ , \tag{115}$$

which is equivalent to:

$$\begin{cases} (e^{\omega}-1)Hen^{\bullet}_{\omega} = 0^{\bullet} \ . \\ (e^{\omega}-e^{-\zeta})\widehat{H}en^{\bullet}_{\omega} = 0^{\bullet} \ . \end{cases} \tag{116}$$

Consequently, we have necessarily $\widehat{H}en^{\bullet}_{\omega} = 0^{\bullet}$ for all $\omega \in \mathbb{C}$, and hence

$$\mathcal{B}(\widetilde{H}en^{\bullet}_{\omega})(\zeta) = Hen^{\bullet}_{\omega}\delta \ , \ \ i.e. \ \widetilde{H}en^{\bullet}_{\omega} = Hen^{\bullet}_{\omega} \in \mathbb{C} \ . \tag{117}$$

Moreover, from (116), we deduce that $Hen^{\bullet}_{\omega} = 0^{\bullet}$ if $\omega \notin 2i\pi\mathbb{Z}$.

To conclude the proof, we now have to prove that the mould $Hen^{\bullet}_{\omega}$ is altern$\underline{e}$l , but this is a consequence of Property 2. □

Equation (111), expressed in the convolutive model, gives for all $n \in \mathbb{Z}^*$:

$$\mathbf{\Delta}_{2in\pi}\widehat{\mathcal{H}e}^{\emptyset} = 0 \implies Hen^{\emptyset}_{2in\pi} = 0 \ . \tag{118}$$

$$\mathbf{\Delta}_{2in\pi}\widehat{\mathcal{H}e}^{s} = \widehat{J}(2in\pi)\ \delta \implies Hen^{s}_{2in\pi} = \widehat{J}(2in\pi) \ . \tag{119}$$

Nevertheless, there is, a priori, no easy way to compute the numbers $Hen^{s_1,\cdots,s_r}_{2in\pi}$ if $r \geq 2$ because it appears to be the residue at $2in\pi$ of a complicated combination of terms $\widehat{He}^{\boldsymbol{\omega}}$:

$$\mathbf{\Delta}_{2in\pi}\widehat{\mathcal{H}e}^{s_1,\cdots,s_r}(\zeta) = Hen^{s_1,\cdots,s_r}_{2in\pi}\ \delta + \sum_{i=1}^{r-1} Hen^{s_1,\cdots,s_i}_{2in\pi}\widehat{\mathcal{H}e}^{s_{i+1},\cdots,s_r}(\zeta) \ . \tag{120}$$

4.5. Application to the asymptotic expansion of Hurwitz multizeta functions

As an application of the resurgent character of Hurwitz multiple zeta functions, and more precisely of Figure 4.1.3, we shall give the asymptotic expansion of Hurwitz multizeta functions near infinity.

As a first calculation, we have

Lemma 14. *Let k and r be two positive integers and $s \geq 2$. If $A(\zeta) = \sum\limits_{n_1,\cdots,n_r \geq 0} a_{n_1,\cdots,n_r}\dfrac{\zeta^{n_1+\cdots+n_r+k}}{(n_1+\cdots+n_r+k)!}$ and $B(\zeta) = \dfrac{\zeta^{s-1}}{(s-1)!}$ are two formal power series at the origin, then:*

$$\frac{(A \star B)(\zeta)}{e^{\zeta}-1} = \sum_{n_1,\cdots,n_{r+1}\geq 0} \binom{||\underline{\mathbf{n}}||+k+s-1}{n_{r+1}-1} a_{n_1,\cdots,n_r}\frac{b_{n_{r+1}}}{n_{r+1}}\frac{\zeta^{||\underline{\mathbf{n}}||+k+s-1}}{(||\underline{\mathbf{n}}||+k+s-1)!} \ ,$$

where $||\underline{\mathbf{n}}|| = n_1 + \cdots + n_{r+1}$*,* b_k *denotes the* k*-th Bernoulli number and the summand is interpreted to be equal to*

$$a_{n_1,\cdots,n_r}\frac{\zeta^{||\underline{\mathbf{n}}||+k+s-1}}{(||\underline{\mathbf{n}}||+k+s)!}\ ,$$

when $n_{r+1} = 0$*.*

Proof. This is a direct computation since:

$$(A \star B)(\zeta) = \sum_{n_1,\cdots,n_r \geq 0} a_{n_1,\cdots,n_r}\frac{\zeta^{n_1+\cdots+n_r+k+s}}{(n_1+\cdots+n_r+k+s)!}\ . \tag{121}$$

Therefore, we obtain:

$$\frac{(A \star B)(\zeta)}{e^\zeta - 1} = \sum_{n_1,\cdots,n_{r+1} \geq 0} a_{n_1,\cdots,n_r}\frac{b_{n_{r+1}}}{n_{r+1}!}\frac{\zeta^{n_1+\cdots+n_{r+1}+k+s-1}}{(n_1+\cdots+n_r+k+s)!}\ . \tag{122}$$

□

From

$$\widehat{He}^{s_1}(\zeta) = \sum_{n_1 \geq 0}\binom{n_1+s_1-2}{s_1-1}\frac{b_{n_1}}{n_1}\frac{\zeta^{n_1+s_1-2}}{(n_1+s_1-2)!} \tag{123}$$

for all integers $s_1 \geq 2$, we deduce from Lemma 14 that

$$\widehat{He}^{s_1,s_2}(\zeta) = \sum_{n_1,n_2 \geq 0}\binom{n_1+s_1-2}{n_1-1}\binom{n_{12}+s_{12}-3}{n_2-1}\frac{b_{n_1}}{n_1}\frac{b_{n_2}}{n_2}\frac{\zeta^{n_{12}+s_{12}-3}}{(n_{12}+s_{12}-3)!} \tag{124}$$

and more generally

$$\widehat{He}^{s_1,\cdots,s_r}(\zeta) = \sum_{n_1,\cdots,n_r \geq 0}\left(\prod_{k=1}^{r}\binom{n_{1\cdots k}+s_{1\cdots k}-k-1}{n_k-1}\frac{b_{n_k}}{n_k}\right)\frac{\zeta^{n_{1\cdots r}+s_{1\cdots r}-r-1}}{(n_{1\cdots r}+s_{1\cdots r}-r-1)!}\ , \tag{125}$$

where $n_{1\cdots k}$ denotes $n_1 + \cdots + n_k$ and $s_{1\cdots k} = s_1 + \cdots s_k$.

Now, it is easy to deduce from it

Proposition 9. *For any sequence* $(s_1, \cdots, s_r) \in \mathcal{S}^\star$*,* $\mathcal{H}e_+^{s_1,\cdots,s_r}$ *has an asymptotic expansion near infinity given by*

$$\widetilde{He}^{s_1,\cdots,s_r}(z) = \sum_{n_1,\cdots,n_r \geq 0}\left(\prod_{k=1}^{r}\binom{n_{1\cdots k}+s_{1\cdots k}-k-1}{n_k-1}\frac{b_{n_k}}{n_k}\right)\frac{1}{z^{n_{1\cdots r}+s_{1\cdots r}-r}}\ . \tag{126}$$

Let us notice that another proof of this asymptotic expansion could be obtained from a recursive use of the Euler-Maclaurin formula.

Example 6.

$$\widetilde{He}^{2}(z) = \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots \tag{127}$$

$$\widetilde{He}^{3}(z) = \frac{1}{2z^2} - \frac{1}{z^3} + \frac{1}{4z^4} + \cdots \tag{128}$$

$$\widetilde{He}^{2,1}(z) = \frac{1}{z} - \frac{3}{2z^2} + \frac{11}{9z^3} + \cdots \tag{129}$$

5. On the algebraic structure of $\mathcal{H}mzv_{cv}$

In this section, we are going to study the algebraic structure of $\mathcal{H}mzv_{cv}$, which is the algebra spanned by the convergent Hurwitz multizeta functions. The first question is the linear independence over $\mathbb{C}$ of the Hurwitz multiple zeta values. This has already been established in [33], p. 381.

As an easy consequence of Example 6, we can show that:

Lemma 15. *The functions* 1, $\mathcal{H}e_+^2$, $\mathcal{H}e_+^{2,1}$ *and* $\mathcal{H}e_+^3$ *are* $\mathbb{C}$*-linearly independant.*

This method can be applied for any particular case, but will not be so easy in a general way. So, we will go back to Lemma 4 to use another method.

5.1. Linear independence of Hurwitz multizeta functions on the rational fraction fields

Despite the fact we are going to use the $\mathbb{C}$-linear independence in order to study the algebraic structure of $\mathcal{H}mzv_{cv}$, we will show more than this:

Theorem 3. *The family* $\left(\mathcal{H}e_+^{\mathbf{s}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}^\star}$ *is* $\mathbb{C}(z)$*-linearly independent.*

Let us notice that this result has not been pointed out in [33], even if the method is actually similar.

5.1.1. Preliminary examples

As a first example of the method developped in this section, let us consider three rational functions F_2, $F_{2,1}$ and G valued in $\mathbb{C}$ such that

$$F_2 \mathcal{H}e_+^2 + F_{2,1}\mathcal{H}e_+^{2,1} = G \ . \tag{130}$$

If $F_{2,1} \neq 0$, we can assume that $F_{2,1} = 1$. Then, applying the operator Δ_- to this relation, we obtain:

$$\widetilde{F}_2 \mathcal{H}e_+^2 = \widetilde{G} \ , \tag{131}$$

where

$$\begin{cases} \widetilde{F}_2(z) = \Delta_-(F_2)(z) + \dfrac{1}{z} = \Delta_-(F_2 + \mathcal{H}e^1_+)(z) \\ \widetilde{G}(z) = \Delta_-(G)(z) - \dfrac{1}{z^2} \cdot F_2(z-1) \end{cases} \tag{132}$$

and

$$\mathcal{H}e^1_+(z) = \sum_{n>0} \left(\frac{1}{n+z} - \frac{1}{n} \right) . \tag{133}$$

If $\widetilde{F}_2 \neq 0$, Equation (131) would imply that $\mathcal{H}e^2_+$ is a rational fraction, and therefore has a finite number of poles. But, each $k \in \mathbb{N}^*$ is a pole of $\mathcal{H}e^2_+$. Consequently, Equation (131) imposes $\widetilde{F}_2 = \widetilde{G} = 0$.

Consequently, $F_2 + \mathcal{H}e^1_+$ is a 1-periodic function. Note that the rational fraction F_2 has a finite number of poles, so there exist $N \in \mathbb{N}^*$ such that N and $-N$ are not a pole of F_2, but N is a pole of $\mathcal{H}e^1_+$. Consequently, $-N$ is a pole of $F_2 + \mathcal{H}e^1_+$. Using the 1-periodicity, we therefore obtain that each integer is a pole of $F_2 + \mathcal{H}e^1_+$. This is, in particular, the case of the positive integer N, which show us a contradiction.

Therefore, $F_{2,1} = 0$, which also implies, as we have just seen from Equation (131), that $F_2 = G = 0$.

Thus, we can state the following

Lemma 16. *The functions* 1, $\mathcal{H}e^2_+$ *and* $\mathcal{H}e^{2,1}_+$ *are* $\mathbb{C}(z)$*-linearly independent.*

The following example will explain how an induction process can be used to prove Theorem 3.

So, as a second example, let us consider this time rational functions F_2, $F_{2,1}$, F_3 and G valued in $\mathbb{C}$ such that

$$F_2\mathcal{H}e^2_+ + F_3\mathcal{H}e^3_+ + F_{2,1}\mathcal{H}e^{2,1}_+ = G . \tag{134}$$

We will apply exactly the same steps as in the previous example, that is:

1. assume that $F_3 \neq 0$.
   (a) assume more precisely that $F_3 = 1$.
   (b) apply the operator Δ_- to Equation (134) to obtain

$$\widetilde{F}_2(z)\mathcal{H}e^2_+ + \widetilde{F}_{2,1}\mathcal{H}e^{2,1}_+ = \widetilde{G}(z) . \tag{135}$$

where

$$\begin{cases} \widetilde{F}_2(z) = \Delta_-(F_2)(z) + \dfrac{1}{z} \cdot F_{2,1}(z-1) \\ \widetilde{F}_{2,1}(z) = \Delta_-(F_{2,1})(z) \\ \widetilde{G}(z) = \Delta_-(G)(z) - \dfrac{1}{z^3} - \dfrac{1}{z^2} \cdot F_2(z-1) \end{cases} \tag{136}$$

(c) use Lemma 16 to conclude that:

$$\begin{cases} \Delta_-(F_2)(z) + \dfrac{1}{z} \cdot F_{2,1}(z-1) = 0 \ . \\ \Delta_-(F_{2,1})(z) = 0 \ . \\ \Delta_-(G)(z) - \dfrac{1}{z^3} - \dfrac{1}{z^2} \cdot F_2(z-1) = 0 \ . \end{cases} \tag{137}$$

(d) partially solve System (137):

$$\begin{cases} F_{2,1} \text{ is a constant } f_{2,1} \in \mathbb{C} \ . \\ f_{2,1} \cdot \mathcal{H}e_+^1 + F_2 = 1 - \text{periodic function} \ . \end{cases} \tag{138}$$

(e) obtain a contradiction in the same way as in the previous example, and thus $F_3 = 0$.

2. rewrite Equation (134) as (130), and use Lemma 16 to conclude that

$$F_2 = F_{2,1} = F_3 = G = 0 \ . \tag{139}$$

Finally, Lemmas 15 and 16 can be improved to:

Lemma 17. *The functions* 1, $\mathcal{H}e_+^2$, $\mathcal{H}e_+^{2,1}$ *and* $\mathcal{H}e_+^3$ *are* $\mathbb{C}(z)$*-linearly independent.*

5.1.2. Proof of Theorem 3

Using the algorithm described in the second example of the previous section, we will prove this theorem by an induction process based on the degree of Hurwitz multizeta functions. But, before beginning the proof, let us introduce some notations, order relations and propositions.

1. For all $d \in \mathbb{N}$, let $\mathcal{S}^\star_{\leq d}$ and $\mathcal{S}^\star_d$ be the sets defined by

$$\begin{aligned} \mathcal{S}^\star_{\leq d} &= \{\underline{\mathbf{s}} \in \mathcal{S}^\star \ ; \ d^\circ \underline{\mathbf{s}} \leq d\} \ , \\ \mathcal{S}^\star_d &= \{\underline{\mathbf{s}} \in \mathcal{S}^\star \ ; \ d^\circ \underline{\mathbf{s}} = d\} \ , \end{aligned} \tag{140}$$

where the degree of a sequence $\underline{\mathbf{s}}$ is defined to be the difference between the weight and the length of the sequence $\underline{\mathbf{s}}$: $d^\circ \underline{\mathbf{s}} = ||\underline{\mathbf{s}}|| - l(\underline{\mathbf{s}})$.

2. We can order the sequences of $\mathcal{S}^\star_{d+1}$ by numbering first the sequences of length 1, then those of length 2, etc. For the following proof, let us remark that it will not be necessary to precise the numbering within a given length. So, we will consider the sets $\mathcal{S}^\star_{d+1} = \{\underline{\mathbf{s}}^n \ ; \ n \in \mathbb{N}^*\}$ and, for $n \in \mathbb{N}$, $S_n^{(d+1)} = \{\underline{\mathbf{s}}^i \ ; \ 1 \leq i \leq n\}$. Of course, the sequences $\underline{\mathbf{s}}^n$ depend on d, but we omit it for simplicity.

3. We will finally consider the following properties:

$$\begin{aligned} &\mathcal{D}(d) : && \text{“the family } \left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}^\star_{\leq d}} \text{ is } \mathbb{C}(z)\text{-free.”} \\ &\mathcal{P}(d,n) : && \text{“the family } \left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}^\star_{\leq d}} \bigcup \left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in S_n^{(d+1)}} \text{ is } \mathbb{C}(z)\text{-free.”} \end{aligned} \tag{141}$$

Proving the theorem amounts to proving that the property $\mathcal{D}(d)$ is true for all $d \in \mathbb{N}$. Since $\mathcal{S}^\star_{\leq 0} = \emptyset$, the property $\mathcal{D}(0)$ is obviously true. Consequently, we have to prove the heredity of $\mathcal{D}(n)$.

This will also be done by an induction process, showing that we can add one by one the sequences of $\mathcal{S}^\star_{d+1}$ in the statement given by $\mathcal{D}(n)$ to obtain the property $\mathcal{D}(d+1)$: the heredity of $\mathcal{P}(d,n)$ will justify this. Since $\mathcal{P}(d,0) = \mathcal{D}(d)$, the initialisation of the property $\mathcal{P}(d,n)$ is obvious.

In other terms, we will prove the property $\mathcal{P}(d,n)$ by induction on $(d,n) \in \mathbb{N}^2$, where $\mathbb{N}^2$ is ordered by the lexicographic order.

Then, the proof of the theorem boils down to showing the following implication:

$$\forall (d,n) \in \mathbb{N}^2 \ , \ \mathcal{P}(d,n) \Longrightarrow \mathcal{P}(d,n+1) \ ,$$

which is what we will now do.

Proof. Assume Property $\mathcal{P}(d,n)$ for a pair $(d,n) \in \mathbb{N}^2$ and let us show that $\mathcal{P}(d,n+1)$ is, therefore, true .

In order to do this, let us consider the relation:

$$\sum_{\substack{\underline{\mathbf{s}} \in \mathcal{S}^\star_{\leq d} \cup S^{(d+1)}_{n+1} \\ \underline{\mathbf{s}} \neq \emptyset}} F_{\underline{\mathbf{s}}} \mathcal{H}e^{\underline{\mathbf{s}}}_{+} = F \ , \tag{142}$$

where F and $F_{\underline{\mathbf{s}}}$, $\underline{\mathbf{s}} \in \mathcal{S}^\star_{\leq d} \cup S^{(d+1)}_{n+1}$, are rational fractions.

We will show that $F_{\underline{\mathbf{s}}^{n+1}} = 0$ by exhibiting a contradiction. Consequently, the property $\mathcal{P}(d,n)$ will imply that all others rational fractions are zero ; therefore, the property $\mathcal{P}(d,n+1)$ will be proven. So, let us now assume that $F_{\underline{\mathbf{s}}^{n+1}} \neq 0$.

Up to dividing (142) by $F_{\underline{\mathbf{s}}^{n+1}}$, we can assume without loss of generality that $F_{\underline{\mathbf{s}}^{n+1}} = 1$.

Let us set

$$\underline{\mathbf{s}}^{n+1} = \underline{\mathbf{u}} \cdot p \text{ with } \begin{cases} p \geq 1 \ . \\ \underline{\mathbf{u}} \in \mathcal{S}^\star_{\leq d+2-p} \ . \end{cases} \tag{143}$$

Let us remark that it is possible to have $\underline{\mathbf{u}} = \emptyset$ when $n = 0$. Moreover, we will use the notation $F_{\underline{\mathbf{s}}}$ even if the sequences $\underline{\mathbf{s}}$ do not appear a priori in Relation (142) ; in this case, we will set $F_{\underline{\mathbf{s}}} = 0$.

<u>**Step** 1 **:**</u> Application of Δ_- on Relation (142) .

Since Δ_- is a $(1\,;\tau^{-1})$-derivative, if we apply it to Relation (142), we obtain:

$$\Delta_-\left(\mathcal{H}e_+^{\underline{\mathbf{s}}^{n+1}}\right)+\sum_{\substack{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}\\ \underline{\mathbf{s}}\neq\emptyset}}\left(\Delta_-\left(F_{\underline{\mathbf{s}}}\right)\cdot\mathcal{H}e_+^{\underline{\mathbf{s}}}+\tau^{-1}\left(F_{\underline{\mathbf{s}}}\right)\cdot\Delta_-\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)\right)=\Delta_-(F)\ . \tag{144}$$

Using a change of index, we obtain:

$$\begin{aligned}\sum_{\substack{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}\\ \underline{\mathbf{s}}\neq\emptyset}}\tau^{-1}\left(F_{\underline{\mathbf{s}}}\right)\cdot\Delta_-\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right) &= \sum_{\substack{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}\\ \underline{\mathbf{s}}\neq\emptyset}}\tau^{-1}\left(F_{\underline{\mathbf{s}}}\right)\cdot\mathcal{H}e_+^{\underline{\mathbf{s}}^{<r}}\cdot J^{s_r}\\ &= \sum_{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}}\ \sum_{\substack{k\in\mathbb{N}^*\\ \underline{\mathbf{s}}\cdot k\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}}}\tau^{-1}\left(F_{\underline{\mathbf{s}}\cdot k}\right)\cdot J^k\cdot\mathcal{H}e_+^{\underline{\mathbf{s}}}\ .\end{aligned} \tag{145}$$

So, we have:

$$\begin{aligned}\Delta_-(F) &= J^p\cdot\mathcal{H}e_+^{\underline{\mathbf{u}}}+\sum_{\substack{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}\\ \underline{\mathbf{s}}\neq\emptyset}}\Delta_-\left(F_{\underline{\mathbf{s}}}\right)\cdot\mathcal{H}e_+^{\underline{\mathbf{s}}}\\ &\quad+\sum_{\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}}\ \sum_{\substack{k\in\mathbb{N}^*\\ \underline{\mathbf{s}}\cdot k\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}}}\tau^{-1}\left(F_{\underline{\mathbf{s}}\cdot k}\right)\cdot J^k\cdot\mathcal{H}e_+^{\underline{\mathbf{s}}}\ .\end{aligned} \tag{146}$$

The Hurwitz multizeta $\mathcal{H}e_+^{\underline{\mathbf{s}}}$ that appear in this relation all satisfy $\underline{\mathbf{s}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}$. Indeed, let us recall that we have numbered the sequences of the set $\mathcal{S}^\star_{d+1}$ by the length. Since $\Delta_-\left(\mathcal{H}e_+^{\underline{\mathbf{s}}^{n+1}}\right)=\Delta_-\left(\mathcal{H}e_+^{\underline{\mathbf{u}}\cdot p}\right)=\mathcal{H}e_+^{\underline{\mathbf{u}}}\cdot J^p$, we therefore have $\underline{\mathbf{u}}\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}$.

We obtain the following system by distinguishing, in the third term of the last equation, whether the summation sequence $\underline{\mathbf{s}}$ is empty or not, and then by an application of the induction hypothesis $\mathcal{P}(d,n)$:

$$\begin{cases}\forall\underline{\mathbf{s}}\in\left(\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}\right)-\{\emptyset\}\ ,\ \Delta_-(F_{\underline{\mathbf{s}}})+\displaystyle\sum_{\substack{k\in\mathbb{N}^*\\ \underline{\mathbf{s}}\cdot k\in\mathcal{S}^\star_{\leq d}\cup S_n^{(d+1)}}}\tau^{-1}(F_{\underline{\mathbf{s}}\cdot k})\cdot J^k+\delta_{\underline{\mathbf{u}},\underline{\mathbf{s}}}J^p=0\ .\\ \Delta_-(F)=\displaystyle\sum_{k=2}^{d+1}\tau^{-1}(F_k)J^k+(1-\delta_{n,0})\tau^{-1}(F_{d+2})J^{d+2}\ .\end{cases} \tag{147}$$

<u>**Step 2:**</u> A lemma which gives some partial solutions to System (147) .

The contradiction will be highlighted by the following lemma. So, we will look for a relation as in it, which will be found by a partial solving of the previous system and will be done in Lemma 19.

Lemma 18. *Let F be a rational fraction and f a 1-periodic function. If, for an n-tuple $(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, with $n \in \mathbb{N}$, the equality*

$$F + \sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i = f \tag{148}$$

is valid, then we necessarily have: $\begin{cases} \lambda_1 = \cdots = \lambda_n = 0 \ . \\ F \text{ and } f \text{ are constant functions} \ . \end{cases}$

Proof. Let us assume that $\sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i \neq 0$. Then, this function has its poles located at negative integers.
Besides, there exists $N \in \mathbb{N}$ such that N and $-N$ are not poles of F . Thus, $-N$ is a pole of f, which therefore admits all integers as poles, according to its 1-periodicity. Since N is a pole of f, N must be a pole of either F, or of $\sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i \neq 0$, which highlights a contradiction.

Then, we have:

1. $\sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i = 0$, which produces $\lambda_1 = \cdots = \lambda_n = 0$. In fact, when applying Δ_-, we obtain: $\sum_{i=1}^{n} \lambda_i J^i = 0$, which gives $\lambda_1 = \cdots = \lambda_n = 0$ according to the partial fraction expansion uniqueness.
2. $F = f$, which proves that F and f are constant functions.

$\square$

According to this lemma, we will now be able to solve partially System (147). Let us remind that we have noted $\underline{\mathbf{s}}^{n+1} = \underline{\mathbf{u}} \cdot p$, where $\begin{cases} p \geq 1 \ . \\ \underline{\mathbf{u}} \in \mathcal{S}^\star_{\leq d+2-p} \ . \end{cases}$

Lemma 19. *Let r be a positive integer and $p \geq 2$. Let us also consider two r-tuples, $(n_1; \cdots; n_r) \in \mathbb{N}^r$ and $(k_1; \cdots; k_r) \in (\mathbb{N}^*)^r$ such that $\sum_{i=1}^{r} (k_i - 1) \leq p - 2$.*

Then, $F_{\underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdot \ldots \cdot k_r \cdot 1^{[n_r]}} = \begin{cases} 0 & , \text{ if } n_r > 0 \ . \\ cste & , \text{ if } n_r = 0 \ . \end{cases}$

PROOF. Let us denote $\underline{\mathbf{v}} = \underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdots \cdot k_r$ and $\delta(\underline{\mathbf{v}}) = p - 2 - \sum_{i=1}^{r} (k_i - 1)$.
Let us show this lemma by induction on $\delta(\underline{\mathbf{v}})$ (even if $\delta(\underline{\mathbf{v}})$ can only take a finite number of values).

• If $\delta(\underline{\mathbf{v}}) = 0$, System (147) applied to $\underline{\mathbf{v}} \cdot 1^{[n]}$, $n \in \mathbb{N}$, give us:

$$\forall n \in \mathbb{N} \ , \ \Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n]}} \right) + \tau^{-1} \left(F_{\underline{\mathbf{v}} \cdot 1^{[n+1]}} \right) J^1 = 0 \ . \tag{149}$$

In fact:

$$\begin{aligned} \underline{\mathbf{v}} \cdot 1^{[n]} \cdot k \in \mathcal{S}^{\star}_{\leq d} \cup S_n & \iff d^\circ (\underline{\mathbf{v}} \cdot 1^{[n]} \cdot k) \leq d \\ & \iff d^\circ \underline{\mathbf{u}} - \delta(\underline{\mathbf{v}}) + p + k - 3 \leq d \\ & \iff k \leq \delta(\underline{\mathbf{v}}) + 1 \\ & \iff k = 1 \end{aligned} \tag{150}$$

We know that there exists $n_0 \in \mathbb{N}$ such that $F_{\underline{\mathbf{v}} \cdot 1^{[n_0]}} = 0$, that is to say that the sequence $\underline{\mathbf{v}} \cdot 1^{[n_0]}$ does not appear in (142). Then, (149) gives $\Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \right) = 0$. Thus, $F_{\underline{\mathbf{v}} \cdot 1^{[n_0-1]}}$ is a 1-periodic function, and therefore a constant function according to the previous lemma, which will now be denoted by $f_{\underline{\mathbf{v}} \cdot 1^{[n_0-1]}}$. (149) gives us again[4]: $\Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n_0-2]}} + f_{\underline{\mathbf{v}} \cdot 1^{[n_0-1]}} \mathcal{H}e^1_+ \right) = 0$, so $F_{\underline{\mathbf{v}} \cdot 1^{[n_0-2]}} + f_{\underline{\mathbf{v}} \cdot 1^{[n-1]}} \mathcal{H}e^1_+$ is a 1-periodic function. Again, the previous lemma imposes to $F_{\underline{\mathbf{v}} \cdot 1^{[n_0-2]}}$ being a constant function as well as $F_{\underline{\mathbf{v}} \cdot 1^{[n_0-1]}} = f_{\underline{\mathbf{v}} \cdot 1^{[n_0-1]}}$ being the null function.

We obtain, by repeating the same reasoning:

$$F_{\underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdots \cdot k_r \cdot 1^{[n_r]}} = \begin{cases} 0 & \text{, if } n_r > 0 \ . \\ \text{cste} & \text{, if } n_r = 0 \ . \end{cases} \tag{151}$$

• Let us assume the lemma proved for all sequence $\underline{\mathbf{v}}$ such that $\delta(\underline{\mathbf{v}}) \leq k$ and let us show it for sequences such that $\delta(\underline{\mathbf{v}}) = k + 1$.
System (147) applied this time to $\underline{\mathbf{v}} \cdot 1^{[n]}$, $n \in \mathbb{N}$, gives:

$$\Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n]}} \right) + \tau^{-1} \left(F_{\underline{\mathbf{v}} \cdot 1^{[n+1]}} \right) J^1 + \sum_{l=2}^{\delta(\underline{\mathbf{v}})+1} \tau^{-1} \left(F_{\underline{\mathbf{v}} \cdot 1^{[n]} \cdot l} \right) J^l = 0 \ . \tag{152}$$

According to the induction hypothesis, we have that for all $l \in [\![2 \, ; \delta(\underline{\mathbf{v}}) + 1]\!]$, $F_{\underline{\mathbf{v}} \cdot 1^{[n]} \cdot l}$ is a constant function which will be, as usual, denoted by $f_{\underline{\mathbf{v}} \cdot 1^{[n]} \cdot l}$.

[4]Let us remark that we have $\Delta_-(\mathcal{H}e^1_+) = J^1$, but this does not come from Property 4 because we have, in it, ruled out the case of divergent Hurwitz multizeta functions. Nevertheless, this equality is true.

Consequently, the last equation can be written as:

$$\forall n \in \mathbb{N}\ ,\ \Delta_-\left(F_{\underline{\mathbf{v}}\cdot 1^{[n]}}\right) + \tau^{-1}\left(F_{\underline{\mathbf{v}}\cdot 1^{[n+1]}}\right) J^1 + \sum_{l=2}^{\delta(\underline{\mathbf{v}})+1} f_{\underline{\mathbf{v}}\cdot 1^{[n]}\cdot l} J^l = 0\ . \quad (153)$$

We will now apply the same reasoning as in the case $\delta(\underline{\mathbf{v}}) = 0$. The only modification we have to do is the application of the previous lemma to the relation

$$F_{\underline{\mathbf{v}}\cdot 1^{[n-1]}} + \sum_{l=1}^{\delta(\underline{\mathbf{v}})+1} f_{\underline{\mathbf{v}}\cdot 1^{[n-1]}\cdot l} \mathcal{H}e_+^l = 0 \quad (154)$$

instead of $F_{\underline{\mathbf{v}}\cdot 1^{[n-1]}} + F_{\underline{\mathbf{v}}\cdot 1^{[n]}} \mathcal{H}e_+^1 = 0$.

So, we have again proved that:

$$F_{\underline{\mathbf{u}}\cdot k_1\cdot 1^{[n_1]}\cdots\cdot k_r\cdot 1^{[n_r]}} = \begin{cases} 0 & , \text{si } n_r > 0\ . \\ \text{cste} & , \text{si } n_r = 0\ . \end{cases} \quad (155)$$

• The lemma is thus proved for any value of $\delta(\underline{\mathbf{v}})$. □

Step 3 : Revealing of the contradiction.

Equation (147), applied to $\underline{\mathbf{u}}$, give us:

$$\Delta_-(F_{\underline{\mathbf{u}}}) + \sum_{k=1}^{p-1} \tau^{-1}(F_{\underline{\mathbf{u}}\cdot k}) J^k + J^p = 0\ . \quad (156)$$

The previous lemma gives us, in particular, that $F_{\underline{\mathbf{u}}\cdot 1}\ ,\ \cdots\ ,\ F_{\underline{\mathbf{u}}\cdot p-1}$ are some rational fractions which are constant functions and now denoted by $f_{\underline{\mathbf{u}}\cdot 1}\ ,\ \cdots\ ,\ f_{\underline{\mathbf{u}}\cdot p-1}$. This can be rewritten:

$$\Delta_-\left(F_{\underline{\mathbf{u}}} + \sum_{k=1}^{p-1} f_{\underline{\mathbf{u}}\cdot k} \mathcal{H}e_+^k + \mathcal{H}e_+^p\right) = 0\ . \quad (157)$$

Thus, $F_{\underline{\mathbf{u}}} + \sum_{k=1}^{p-1} f_{\underline{\mathbf{u}}\cdot k} \mathcal{H}e_+^k + \mathcal{H}e_+^p$ defines a 1-periodic function. The coefficients being not all zero in this relation, this enlights a contradiction of Lemma 18 .

Thus, $F_{\underline{\mathbf{s}}^{n+1}} = 0$ and (142) can be rewritten:

$$F = \sum_{\underline{\mathbf{s}} \in \mathcal{S}^\star_{\leq d} \cup S_n} F_{\underline{\mathbf{s}}} \mathcal{H}e_+^{\underline{\mathbf{s}}}\ . \quad (158)$$

It only remains to use the induction hypothesis $\mathcal{P}(d,n)$ to obtain that all other rational fractions are null. Therefore, we have shown that:

$$\mathcal{P}(d,n) \Longrightarrow \mathcal{P}(d,n+1) \ .$$

This completes the proof of the theorem. $\square$

5.2. Algebraic relations in $\mathcal{H}mzv_{cv}$

In this section, we give some corollaries of Theorem 3. We will begin with some reminders on the algebra *QSym* of quasi-symmetric functions, and then we will give an explicit isomorphism between this algebra and $\mathcal{H}mzf_{cv}$. Thus, we will be able to raise up any relation between convergent Hurwitz multizeta functions in *QSym*. Consequently, we will give an answer, for the algebra of convergent Hurwitz multizeta functions, to each question we ask and would be able to answer for the algebra of multizeta values.

5.2.1. Reminders on QSym

Let us consider an infinite commutative alphabet $X = \{x_1, x_2, x_3, \cdots\}$. Thus, we can consider (commutative) formal power series, valued in a ring R, with indeterminates x_1, x_2, $\cdots$. In particular, we are interested in series such that the coefficient of the monomial $x_1^{s_1} \cdots x_r^{s_r}$ is equal to the coefficient of the monomial $x_{n_1}^{s_1} \cdots x_{n_r}^{s_r}$ for any strictly increasing sequence $0 < n_1 < n_2 < \cdots < n_r$ of positive integers indexing the variables (and for any positive integer sequence of exponents s_1, $\cdots$, s_r).

Such a series is called a quasi-symmetric function(see [28], or [3], [30], [31] and [39] for a more recent presentation).

From the definition, it is clear that the set of quasi-symmetric functions is an R-vector space denoted $QSym_R(X)$, or more simply $QSym_R$, as well as *QSym* if there is no possible ambiguity.

This vector space has a natural basis, the so-called monomial basis. Let us notice that *QSym* has other interesting bases (see [3], for the fundamental basis, but also [47] for a new basis related to multizetas values) . A composition I of length r being given, we denote by

$$M_I(X) = \sum_{0<n_1<\cdots<n_r} x_{n_1}^{i_1} \cdots x_{n_r}^{i_r} \ , \tag{159}$$

the monomial basis. These series are quasi-symmetric functions. Moreover, according to the definition, it turns out that they clearly span *QSym*. On the other hand, if we order the indeterminates by

$$x_1 > x_2 > x_3 > \cdots \ , \tag{160}$$

and extend this order to words to be the lexicographic order, then the leading term of $M_I(X)$ is $x_1^{i_1} \cdots x_r^{i_r}$. Thus, the M_I's are also linearly independent and consequently are a basis of *QSym*.

Since the monomials M_I are iterated sums, the M_I's multiply themselves using the stuffle product (see [32], for instance). Therefore, the product of two quasi-symmetric functions is a quasi-symmetric function, and *QSym* is actually an R-algebra.

Let us conclude these reminders by noticing that we have given the classical definition of quasi-symmetric function: the sequence of positive integers indexing the variables is a strictly increasing sequence. Nevertheless, the multizetas' convention, which is the most used today, and consequently the Hurwitz multizetas' convention, is the opposite one: the sequence of positive integers indexing the variables is a strictly decreasing sequence.

If the both conventions were identically the same, the Hurwitz multizeta functions should be seen as a particular evaluation of the monomials via the specialisation $x_n \longmapsto \dfrac{1}{n+z}$, according to the condition that the result is a convergent series. In order to use jointly both of these conventions, we can consider the alphabet $X = \{x_{-1}; x_{-2}; x_{-3}; \cdots\}$ and the specialisation $x_{-n} \longmapsto \dfrac{1}{n+z}$. However, to make links between *QSym* and multizeta values, we only need to have an infinite totally ordered alphabet.

5.2.2. Corollaries to Theorem 3

A totally ordered and infinite alphabet $A = \{a_1, a_2, \cdots\}$ being given, a word $\omega = w_1 w_2 \cdots w_l$ over A is said to be a Lyndon word if it is strictly smaller than any of its non-empty proper right factors for the lexicographic order:

$$\omega < w_i w_{i+1} \cdots w_l \text{ for all } i \in [\![\, 2\, ;\, l \,]\!] \ . \tag{161}$$

To have more information about Lyndon words, we refer the reader to [37] for historical references as well as to [50].

The definition of Lyndon words allows us to deduce from Theorem 3 a ring basis of $\mathcal{H}mzf_{cv}$:

Corollary 2. *Let us denote by $QSym_{cv}$ the subalgebra of QSym spanned by the monomials M_I with a composition $I = (i_1, \cdots, i_r)$ such that $i_1 \geq 2$. Then:*

$$\mathcal{H}mzf_{cv} \simeq QSym_{cv} \simeq \mathbb{Q}\big[\mathcal{L}yn(\mathbb{N}^*) - \{1\}\big] \ , \tag{162}$$

where $\mathcal{L}yn(y_1; y_2; \cdots)$ denotes the set of Lyndon words over the alphabet $Y = \{y_1; y_2; \cdots\}$.

Proof. 1. Let us define $\mathfrak{h} : QSym_{cv} \longrightarrow \mathcal{H}mzf_{cv}$ on the monomial basis by $\mathfrak{h}(M_{s_1,\cdots,s_r}) = \mathcal{H}e_+^{s_1,\cdots,s_r}$.
$\mathfrak{h}$ is a algebra morphism, since the monomial basis of *QSym* and the Hurwitz multizeta functions are multiplied by the same rule, the stuffle product.

Moreover, $\mathfrak{h}$ is a surjective map, by construction. Finally, $\mathfrak{h}$ is injective, according to Theorem 3. Consequently, $\mathfrak{h}$ is an algebra isomorphism.

2. A stuffle contains all the terms of the ordinary shuffle together with contractions obtained by adding two consecutive parts coming from different terms. So, by a triangular change of variable, the quasi-shuffle product defines an algebraic structure, which is actually isomorphic to an ordinary shuffle algebra over the same set (see [32]).

According to the Radford theorem (see [48]), a shuffle algebra over the alphabet A is a polynomial algebra in the Lyndon words over A as generators. Thus, a stuffle algebra over the alphabet A is also a polynomial algebra spanned by an algebraically independent familly indexed by Lyndon words over A (see [32]). Consequently, if we define a subalgebra of a stuffle algebra, we just have to exclude some relevant Lyndon words:

$$QSym_{cv} \simeq \mathbb{Q}\big\langle \mathcal{L}yn(y_1; y_2; \cdots) - \{y_1\}\big\rangle \tag{163}$$

$\square$

Since it is well-known that $QSym$ is a graded algebra, $QSym_{cv}$ is also a graded algebra whose n^{th} homogeneous component is of dimension 2^{n-2} (and respectively 1 and 0 if $n = 0$ and $n = 1$). Consequently, the isomorphism introduced in Corollary 2 first implies:

Corollary 3. *Let us denote the algebra of convergent Hurwitz multizeta functions of weight n by $\mathcal{H}mzf_{cv,n}$, that is the subalgebra of $\mathcal{H}mzf_{cv}$ spanned by the Hurwitz multizetas $\mathcal{H}e_+^{s_1,\cdots,s_r}$ such that $s_1 + \cdots + s_r = n$ and $s_1 \geq 2$. Then:*

1. *$\mathcal{H}mzf_{cv}$ is graded by the weight:*

$$\begin{cases} \mathcal{H}mzf_{cv} = \displaystyle\bigoplus_{n\in\mathbb{N}} \mathcal{H}mzf_{cv,n} \ . \\ \forall (p,q) \in \mathbb{N}^2 \ , \ \mathcal{H}mzf_{cv,p} \cdot \mathcal{H}mzf_{cv,q} \subset \mathcal{H}mzf_{cv,p+q} \ . \end{cases} \tag{164}$$

2. $\begin{cases} dim\ \mathcal{H}mzf_{cv,0} = 1 \ , \ dim\ \mathcal{H}mzf_{cv,1} = 0. \\ dim\ \mathcal{H}mzf_{cv,n} = 2^{n-2} \text{ for all } n \geq 2. \end{cases}$

Finally, the isomorphism $\mathfrak{h}$ allows us to completely describe the algebraic relations between convergent Hurwitz multizeta functions since these relations can be lifted to the formal level:

Corollary 4. *Each algebraic relation in $\mathcal{H}mzf_{cv}$ comes from the expansion of the stuffle products.*

PROOF. Let us consider the algebra $A = \mathbb{Q}\langle \Omega \rangle$, where

$$\Omega = \bigcup_{n \in \mathbb{N}} \{y_I ; I \text{ composition of n}\} \ , \tag{165}$$

with the concatenation product. Let us recall that the stuffle product is recursively defined by (17) for any alphabet which has a semi-group structure. Thus, this definition is a valid one for the alphabet $\mathbb{N}_1$ and give us, by bilinearity, a stuffle over A:

$$y_I \uplus y_J = \sum_{K \in \mathbb{N}_1^\star} \langle I \uplus J | K \rangle y_K \tag{166}$$

Here, it is natural to use the same symbol $\uplus$ to indicate the stuffle over A and over the compositions since it is morally the same product.

Let $\mathcal{I}$ be the ideal of A spanned by the elements $y_I \cdot y_J - y_I \uplus y_J$, where I and J are two compositions and $\cdot$ denote the concatenation product. Thus, $A/\mathcal{I}$ is an algebra, which is naturally graded since $\mathcal{I}$ is an homogeneous ideal and A is graded by the weight (with $dy_I = n$, if I is a composition of n).

The homogeneous component of weight n of $A/\mathcal{I}$ is of dimension 2^{n-1}, since each element of $A/\mathcal{I}$ is generated by the y_I, I being a composition of n, these elements are linearly independent (according to the definition of A) and are 2^{n-1}.

Finally, let us consider the linear map $\varphi : A/\mathcal{Y} \longrightarrow \mathit{QSym}$ defined by $\varphi(y_I) = M_I$. It is an algebra morphism:

$$\begin{aligned} \varphi(y_I \cdot y_J) &= \varphi(y_I \uplus y_J) &&= \varphi\Big(\sum_{K \in \mathbb{N}_1^\star} \langle I \uplus J | K \rangle y_K \Big) \\ &= \sum_{K \in \mathbb{N}_1^\star} \langle I \uplus J | K \rangle M_K &&= M_I M_J = \varphi(y_I)\varphi(y_J) \ . \end{aligned} \tag{167}$$

Moreover, φ, restricted to the homogeneous component of weight n of $A/\mathcal{I}$, is valued in QSym_n, is surjective by definition and injective according to the dimensions. Consequently, φ is an isomorphism between $A/\mathcal{Y}$ and QSym.

If we have an algebraic relation between convergent Hurwitz multizeta functions, like

$$\sum_{i=1}^{n} \lambda_i \left(\mathcal{H}e_+^{\underline{\mathbf{s}}_1^i} \right)^{\alpha_1^i} \cdots \left(\mathcal{H}e_+^{\underline{\mathbf{s}}_{n_i}^i} \right)^{\alpha_{n_i}^i} = 0 \tag{168}$$

where, for all $i \in [\![1 \, ; n]\!]$, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{N}^*$ and $\underline{\mathbf{s}}_j^i \in \mathcal{S}^\star$, we can lift it to the formal level, from $\mathcal{H}mzf_{cv}$ to QSym_{cv}, and then to $A/\mathcal{I}$:

$$\sum_{i=1}^{n} \lambda_i \left(y_{\underline{\mathbf{s}}_1^i} \right)^{\alpha_1^i} \cdots \left(y_{\underline{\mathbf{s}}_{n_i}^i} \right)^{\alpha_{n_i}^i} = 0 \text{ in } A/\mathcal{I} \ . \tag{169}$$

Consequently:

$$\sum_{i=1}^{n} \lambda_i \left(y_{\underline{\mathbf{s}}_1^i}\right)^{\alpha_1^i} \cdots \left(y_{\underline{\mathbf{s}}_{n_i}^i}\right)^{\alpha_{n_i}^i} \in \mathcal{I} \ . \tag{170}$$

Therefore, all the algebraic relations between convergent Hurwitz multizeta functions, like (168), is directly a consequence of the stuffle relations. □

6. Extension to the divergent Hurwitz multizeta functions

First of all, let us remind the following Lemma from [6] which gives us an algebraic way to extend the definition of a symmetr$\underline{e}$l mould:

Lemma 20. *Let $\mathcal{S}e^{\bullet}$ be a symmetr$\underline{e}$l mould over the alphabet $\Omega = \mathbb{N}^*$, with values in a commuative algebra $\mathbb{A}$, well-defined for sequences in $S^{\star} = \{\underline{\mathbf{s}} \in \mathbb{N}_1^{\star} \ ; \ s_1 \geq 2\}$*

1. *For all $\theta \in \mathbb{A}$, there exists a unique symmetr$\underline{e}$l extension of $\mathcal{S}e^{\bullet}$ to $\mathbb{N}_1^{\star}$, denoted by $\mathcal{S}e_{\theta}^{\bullet}$, such that $\mathcal{S}e_{\theta}^{1} = \theta$.*
2. *For all $\gamma \in \mathbb{A}$, let $\mathcal{N}e_{\gamma}^{\bullet}$ be the symmetr$\underline{e}$l mould defined on sequences of $\mathbb{N}_1^{\star}$ by: $\mathcal{N}e_{\gamma}^{\underline{\mathbf{s}}} = \begin{cases} \dfrac{\gamma^r}{r!} & , \text{if } \underline{\mathbf{s}} = 1^{[r]} \ . \\ 0 & , \text{otherwise.} \end{cases}$*
   Then, for all $(\theta_1 \, ; \, \theta_2) \in \mathbb{A}^2$, we have:

$$\mathcal{S}e_{\theta_1}^{\bullet} = \mathcal{N}e_{\theta_1-\theta_2}^{\bullet} \times \mathcal{S}e_{\theta_2}^{\bullet} \ . \tag{171}$$

6.1. Algebraic structure of $\mathcal{H}mzf$

According to Lemma 20, to extend the definition of the Hurwitz multizeta functions to the divergent case, *i.e.* when $s_1 = 1$, we just need to choose which function will be $\mathcal{H}e_{+}^{1}$. Let us notice that, for any choice, the algebra $\mathcal{H}mzv$ spanned by all the Hurwitz multizeta functions (the convergent and divergent ones) is then described by

$$\mathcal{H}mzf = \mathcal{H}mzf_{cv}\left[\mathcal{H}e_{+}^{1}\right] \ . \tag{172}$$

Thus, Theorem 3 can be extended to all Hurwitz multizeta functions:

Theorem 4. *The family $\left(\mathcal{H}e_{+}^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\mathbb{N}_1^{\star}}$ is $\mathbb{C}(z)$-linearly independent.*

Its corollary can also be extended easily to obtain:

Corollary 5. *$\mathcal{H}mzf \simeq QSym \simeq \mathbb{Q}\left[\mathcal{L}yn(\mathbb{N}^*)\right]$, where $\mathcal{L}yn(y_1; y_2; \cdots)$ denotes the set of Lyndon words over the alphabet $Y = \{y_1; y_2; \cdots\}$.*

Corollary 6. *Let us denote the algebra of Hurwitz multizeta functions of weight n by $\mathcal{H}mzf_n$.*
Then:

1. *$\mathcal{H}mzf$ is graded by the weight: $\mathcal{H}mzf = \bigoplus_{n\in\mathbb{N}} \mathcal{H}mzf_n$.*
2. $\begin{cases} \textit{dim } \mathcal{H}mzf_0 = 1 \ , \\ \textit{dim } \mathcal{H}mzf_n = 2^{n-1} \textit{ for all } n \geq 1. \end{cases}$

Corollary 7. *Each algebraic relation in $\mathcal{H}mzv$ comes from the expansion of the stuffle products.*

6.2. About a possible resurgent character of divergent Hurwitz multizeta functions

Since the properties of convergent Hurwitz multizeta are to be extended, we want to define the divergent Hurwitz multizeta $\mathcal{H}e^1_+$ in order that it satisfies:

1. $\mathcal{H}e^1_+$ is defined from a Borel-Laplace summation of a certain $\widetilde{f} \in -\ln z + \mathbb{C}[\![z^{-1}]\!]$ satisfying the difference equation
$$\Delta_-\widetilde{f}(z) = \frac{1}{z} \ . \tag{173}$$
2. $\mathcal{H}e^1_+$ is asymptotically equal, near infinity, to $-\ln z + \sum_{n>0} \frac{b_n}{nz^n}$, where b_n is the nth Bernoulli number.

If there exists such a function $\mathcal{H}e^1_+$, this one is unique. Indeed, the first point gives us that $\Delta_-\widetilde{f}'(z) = \frac{-1}{z^2}$ with $\widetilde{f}' \in \mathbb{C}[\![z^{-1}]\!]$. Thus, we necessarily have $\widetilde{f}$ fixed, up to a constant since $\widetilde{f}' = -\widetilde{He}^2$. Then, the second point prove the unicity of the constant. Moreover, provided such a function exists, this last differential property can be generalized to all divergent Hurwitz multizeta functions such that Equation (5) is valid for any divergent Hurwitz multizeta functions.

On the other hand, it is clear that (173) has a unique solution $\widetilde{f}(z) = -\ln z + \widetilde{g}(z) \in -\ln z + \mathbb{C}[\![z^{-1}]\!]$ caracterized by:

$$\Delta_-\widetilde{g}(z) = \frac{1}{z} + \ln\left(1 - \frac{1}{z}\right) \in z^{-2}\mathbb{C}[\![z^{-1}]\!] \ . \tag{174}$$

Then, the resurgent treatment of the generic 1-order difference equation can be used to produce a simple resurgent function defined in the geometric model for all $z \in \mathbb{C}$, $\Re e\, z > 0$, by

$$\begin{aligned} g_e(z) &= \sum_{n>0} \left[\frac{1}{n+z} + \ln\left(1 - \frac{1}{n+z}\right) \right] \\ &= \ln z + \gamma + \sum_{n>0} \left(\frac{1}{n+z} - \frac{1}{n} \right) \\ &= \ln z - \frac{1}{z} - \frac{\Gamma'(z)}{\Gamma(z)} \ , \end{aligned} \tag{175}$$

where $\gamma \simeq 0.5772156649$ denotes the Euler - Mascheroni constant and Γ the Gamma function which satisfies the well-known identity:

$$\Gamma(z+1) = z\Gamma(z) \tag{176}$$

Consequently, the function f defined over $\Re e\, z > 0$ by

$$f(z) = -\frac{1}{z} - \frac{\Gamma'(z)}{\Gamma(z)} = \gamma + \sum_{n>0} \left(\frac{1}{n+z} - \frac{1}{n} \right) \tag{177}$$

satisfies as required (173) and thus the first point.

Finally, it is well-known that the derivative of the logarithm of the Γ function has an asymptotic expansion near infinity given by

$$\frac{\Gamma'(z)}{\Gamma(z)} \asymp \ln z - \frac{1}{2z} - \sum_{n>0} \frac{b_{2n}}{2nz^{2n}} \ . \tag{178}$$

which shows that f also satisfies the second point. Thus, we shall choose

$$\mathcal{H}e^1_+(z) = \gamma + \sum_{n>0} \left(\frac{1}{n+z} - \frac{1}{n} \right) \tag{179}$$

which allows us to extend multizeta values to the divergent case by setting $\mathcal{Z}e^1 = \gamma$, which is exactly the choice one would have, according to [55]. But for simplicity, we prefer to choose $\mathcal{Z}e^1 = 0$ and use, if necessary, the "change of constant formula" (171). Therefore, we set:

$$\mathcal{H}e^1_+(z) = \sum_{n>0} \left(\frac{1}{n+z} - \frac{1}{n} \right) \tag{180}$$

Let us notice that this regularization is actually set out in [55] for the divergent multizeta values from a different point of view, namely this of special functions. It is nice to see that the resurgent point of view agrees with this one.

From the previous resurgent treatment of (173), we now know that $\mathcal{H}e^1_+$ is equal to a simple resurgent function minus the principal branch of the complex logarithm. Therefore, Theorem 1 is extended as:

Theorem 5. *Any Hurwitz multizeta function is a polynomial in* $-\ln$ *with coefficients in the algebra of simple resurgent functions whose singularities are over* $2\pi i\mathbb{Z}^*$.

References

[1] F. V. Atkinson : *The mean-value of the Riemann zeta function*, Acta Math., **81**, 1949, p. 353-376.

[2] E. Vieillard-Baron : *From resurgent functions to real resummation through combinatorial Hopf algebras*, Ph.D. Thesis, Université de Bourgogne, 228p., 2014.

[3] F. Bergeron : *Algebraic Combinatorics and Coinvariant Spaces*, CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON, 2009.

[4] O. Bouillot : *Invariants analytiques des difféomorphismes et multizêtas*, Ph.D. Thesis, Paris-Sud 11 University, 290 p., 2011.

[5] O. Bouillot, J. Ecalle : *Invariants of identity-tangent diffeomorphisms: explicit formulae and effective computation*, 42p., 2012, http://arxiv.org/abs/1404.1042

[6] O. Bouillot : *The Multitangent Functions Algebra*, Journal of Algebra, **91p.**, n ° 410 (2014), p.148-238.

[7] O. Bouillot : *The Algebra of Hurwitz Multizeta Functions*, C. R. Acad. Sci. Paris, Ser. I, 352, n ° 11 (2014), p. 865-869.

[8] O. Bouillot : *Multiple Bernoulli Polynomials*, in preparation.

[9] F. Brown : *Mixed Tate motives over* $\mathbb{Z}$, Annals of Math., volume 175 (2012), *n* ° 2, p. 949-976.

[10] F. Brown : *On the decomposition of motivic multiple zeta values*, *in* Galois-Teichmller Theory and Arithmetic Geometry, H. Nakamura et. al. (eds.), Adv. Studies in Pure Math. 68, Math. Soc. Japan, Tokyo, (2012), p. 31-58.

[11] B. Candelpergher : *Une introduction à la résurgence*, Gazette des Mathématiciens, (Soc. math. France), *n* ° 42 (1989), p. 36-64.

[12] P. Cartier : *P. Cartier, Fonctions polylogarithmes, nombres polyzeta et groupes pro-unipotents*, Séminaire N. Bourbaki, (2000-2001), exp. 885, p. 137-173.

[13] B. Candelpergher, J.-C. Nosmas, F. Pham : *Approche de la résurgence*, Actualités Math., Hermann, Paris, 1993.

[14] Y. Choie, K. Matsumoto : *Functional equations for double series of Euler-Hurwitz-Barnes type with coefficients*, http://arxiv.org/abs/1403.1940

[15] J. Cresson : *Calcul moulien*, Annales de la faculté des sciences de Toulouse, Sér. 6 (2009), 18, n 2, p. 307-395.

[16] E. Delabaere : *Introduction to the Ecalle theory*, Computer Algebra and Differential Equations, number 193, London Math. Soc., Lecture Note Series, Cambridge University Press, 1994, p. 59-102.

[17] E. Delabaere : *Effective resummation methods for an implicit resurgent function*, Prépu. Math. Angers 220.

[18] K. Ebrahimi-Fard, L. Guo : *Quasi-shuffles, Mixable Shuffles and Hopf Algebras*, J. Algebraic Combinatorics 24, (2006), p. 83-101.

[19] J. Ecalle : *Les fonctions résurgentes*, Publications mathématiques d'Orsay [vol. 1 : 81-05, vol. 2 : 81-06, vol. 3 : 85-05], 1981, 1985.

[20] J. Ecalle : *ARI/GARI, la dimorphie et l'arithmétique des multizêtas, un premier bilan*, Journal de Théorie des Nombres de Bordeaux, **15** (2003), n$^\circ$2, p. 411-478.

[21] J. Ecalle : *Singularités non abordables par la géométrie*, Annales de l'institut Fourier, 42 (1992), $n^\circ 1-2$, p. 73-164.

[22] J. Ecalle : *Recent Advances in the Analysis of Divergence and Singularities*, Proceedings of the July 2002 Montreal Seminar on Bifurcations, Normal forms and Finiteness Problems in Differential Equations, C. Rousseau,Yu. Ilyashenko Publ., 2004, Kluwer Acad. Publ., p. 87-187.

[23] J. Ecalle : *The Flexion Structure and Dimorphy : Flexion Units, Singulators, Generators, and the Enumeration of Multizeta Irreducibles*, in "Asymptotic in dynamics, geometry and PDEs ; generalized Borel summation", Eds. O. Costin, F. Fauvet, F. Menous, D. Sauzin, Publications of the Scuola Normale Superiore, Pisa, 2011, Volume 12, p. 201-218.

[24] M. Eie, W-C. Liaw : *Double Euler sums on Hurwitz zeta functions* Rocky Mountain J. Math., **39** (2009), p. 1869-1883.

[25] J.-Y. Enjalbert, H. N. Minh : *Combinatorial study of colored Hurwitz polyztas*, Discrete Mathematics, 312, 24, (2013), p. 3489-3497.

[26] L. Euler : *Meditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol., **20** (1775), p. 140-186, Reprinted in "Opera Omia", Ser.1, Vol. 15, B. G. Teubner, Berlin, 1927, p. 217-267.

[27] H. Furusho, Y. Komori, K. Matsumoto, H. Tsumura : *Desingularization of multiple zeta-functions of generalized Hurwitz-Lerch type*, preprint 1404.4758[NT].

[28] I. Gessel : *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983), p. 289-317, Contemp. Math., 34, Amer. Math. Soc., Providence, RI, 1984.

[29] L. Guo, B. Zhang : *Renormalization of multiple zeta values*, J. Algebra, **319** (2008), $n^\circ 9$, p. 3770-3809.

[30] M. Hazewinkel : *Quasi-symmetric Functions*, in 12th Annual International Conference on Formal Power Series and Algebraic Combinatorics, Moscow, 2000, p. 30-44.

[31] F. Hivert : *Local Action of the Symmetric Group and Generalizations of Quasi-symmetric Functions*, in 17th Annual International Conference on Formal Power Series and Algebraic Combinatorics, Taormina, 2005.

[32] M. Hoffman : *Quasi-shuffle products*, J. Algebraic Combinatorics, n° **11**, 2000, p. 49-68.

[33] S. Joyner : *An algebraic version of the Knizhnik-Zamolodchikov equation*, Ramanujan J., 28, $n^\circ 3$, (2012), p. 361-384.

[34] K. Kamano : *The multiple Hurwitz zeta function and a generalization of Lerch's formula* Tokyo J. Math., **29**, 2006, p. 61-73.

[35] J. P. Kelliher, R. Masri : *Analytic continuation of multiple Hurwitz zeta functions*, Math. Proc. Camb. Phil. Soc., **145**, 2008, p. 605-617.

[36] S. Leurent, D. Volin : *Multiple zeta functions and double wrapping in planar $N = 4$ SYM*, Nuclear Phys. B **875**, 2013, p. 757-789.

[37] R. C. Lyndon : *On Burnside's problem*, Trans. Amer. Math. Soc., **77** (1954), p. 202-215.

[38] B. Malgrange : *Resommation des séries divergentes*, Exp. Math. **13** (1995), p. 163-222.

[39] C. Malvenuto, C. Reutenauer : *Duality Between Quasi-symmetric Functions and the Solomon Descent Algebra*, Journal of Algebra, n° **177** (1995), p. 967-982.

[40] D. Manchon, S. Paycha : *Nested sums of symbols and renormalised multiple zeta values*, Int. Math. Res. Notices, **24**, (2010), p. 4628-4697.

[41] K. Matsumoto : *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I*, J. Number Theory, **101** (2003), p. 223-243.

[42] H. N. Minh, G. Jacob, N. E. Oussous, M. petitot : *De l'algèbre des ζ de Riemann multivariées à l'algèbre des ζ de Hurwitz multivariées*, journ. electr. sem. Loth. comb, $B44e$, (2001).

[43] M. R. Murty, K.Sinha : *Multiple Hurwitz zeta function*, *in* Multiple Dirichlet series, automorphic forms and analytic number theory, Proc. Symp. in Pure Math., 75, (2006), p. 135-156.

[44] T. Nakamura : *Zeroes and the universality for the Euler-Zagier-Hurwitz type of multiple zeta values* Bull. London Math. Soc., **41** (2009), p. 691-700.

[45] T. Nakamura : *Real zeroes of Hurwitz-Lerch zeta and Hurwitz-Lerch type of Euler-Zagier double zeta functions*, http://arxiv.org/abs/1405.1504

[46] T. Nakamura : *Hurwitz-Lerch zeta and Hurwitz-Lerch type of Euler-Zagier double zeta distributions*, http://arxiv.org/abs/1405.1799

[47] J. C. Novelli, J. Y. Thibon : *Binary Shuffle Bases for Quasi-symmetric Functions*, ArXiV 1305.5032v1

[48] D. E. Radford : *A natural ring basis for the shuffle algebra and an application to group schemes*, Journal of Algebra, n° **58** (1979), p. 432-454.

[49] J.P. Ramis : *Séries divergentes et théories asymptotiques*, Panoramas et synthèses, 0, Supplément au Bulletin de la S.M.F., 121, 1993.

[50] C. Reutenauer : *Free Lie Algebras*, London Math. Soc. Monographs, News series **7**, Oxford Sciences Publications, Oxford, 1993.

[51] D. Sauzin : *Resurgent functions and splitting problems*, RIMS Kokyuroku, n° **1493** (2005), p. 48-117.

[52] D. Sauzin : *On the stability under convolution of resurgent functions*, Funkcial. Ekvac. **56** (2013), n° 3, p. 397-413.

[53] D. Sauzin : *Mould Expansion for the Saddle-node and Resurgence Monomials*, in "Renormalization and Galois theories", A. Connes, F. Fauvet, J. P. Ramis. Eds., IRMA Lectures in Mathematics and Theoretical Physics, 15, European Mathematical Society, Zürich, 2009, pp. 83-163.

[54] D. Sauzin : *Introduction to 1 summability and the resurgence theory*, available on http://hal.archives-ouvertes.fr/hal-00860032.

[55] M. Waldschmidt : *Valeurs zêta multiples. Une introduction*, Journal de Théorie des Nombres de Bordeaux, **12** (2000), p. 581-592.

[56] W. Zudilin : *Algebraic relations for multiple zeta values*, Russian Mathematical Surveys, **58**, 2003, p. 1-29.