About Links Between Multizeta Values and Multitangent Functions.

Workshop Periods and Motives, A New Perspective on Renormalization.

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Outline

A few words on moulds - First notations.

Concrete definition : A **mould** is a function with a varying number of variables.

Mathematical definition : A mould is a function defined on a monoid.

Typical example : *The Multizetas Values !*

	Functional notations	Mould notations
Evaluation	$f(x)$	$M^{\underline{s}}$
Name	f	$M^{\bullet}$

Why do we need some "new" notations ?

For analytical reasons, moulds might be contracted with dual objects, called **co-moulds** :

$$\sum_{\underline{\boldsymbol{\omega}} \in \Omega^\star} M^{\underline{\boldsymbol{\omega}}} B_{\underline{\boldsymbol{\omega}}} \ .$$

Important remark :

All mould's definitions come from such an interpretation !

In particular, this is the case for
- the mould algebra's structure.
- the mould's symmetries.

Let $M^\bullet$ and $N^\bullet$ defined on an alphabet Ω (valued in an algebra) and λ a scalar.

- **Addition:**

$$S^\bullet = M^\bullet + N^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \ , \ S^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}} + N^{\underline{\boldsymbol{\omega}}} \ .$$

- **Scalar multiplication:**

$$(\lambda M)^\bullet = \lambda \cdot M^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \ , \ (\lambda M)^{\underline{\boldsymbol{\omega}}} = \lambda \, M^{\underline{\boldsymbol{\omega}}} \ .$$

- **Mould multiplication:**

$$P^\bullet = M^\bullet \times N^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \ , \ P^{\underline{\boldsymbol{\omega}}} = \sum_{\substack{(\underline{\boldsymbol{\omega}}^1 \, ; \, \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \\ \underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}}^1 \cdot \underline{\boldsymbol{\omega}}^2}} M^{\underline{\boldsymbol{\omega}}^1} N^{\underline{\boldsymbol{\omega}}^2} \ .$$

Example :

$$P^{\omega_1,\omega_2,\omega_3} = M^{\omega_1,\omega_2,\omega_3} N^{\emptyset} + M^{\omega_1,\omega_2} N^{\omega_3} + M^{\omega_1} N^{\omega_2,\omega_3} + M^{\emptyset} N^{\omega_1,\omega_2,\omega_3} \ .$$

With these three operations, the set of moulds is an unitary associative (non commutative) algebra.

- **Symmetrality $\longleftrightarrow$ shuffle product** :

If $\Omega = \{x_i \,;\, i \in I\}$:

$$\begin{cases} \varepsilon \sqcup\!\sqcup\, \omega = \omega \sqcup\!\sqcup\, \varepsilon = \omega \ . \\ (x_i\omega^1) \sqcup\!\sqcup\, (x_j\omega^2) = x_i\left(\omega^1 \sqcup\!\sqcup\, (x_j\omega^2)\right) + x_j\left((x_i\omega^1) \sqcup\!\sqcup\, \omega^2\right) \ . \end{cases}$$

Let $sh\underline{a}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) = \{$words appear in $\omega^1 \sqcup\!\sqcup\, \omega^2$ (with multiplicity)$\}$.
A mould $Ma^\bullet$ is symmetr$\underline{a}$l when :

$$\forall(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \ Ma^{\underline{\boldsymbol{\omega}}^1} Ma^{\underline{\boldsymbol{\omega}}^2} = \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{a}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2)} Ma^{\underline{\boldsymbol{\omega}}} \ .$$

Example : $\quad \mathcal{W}a^{\omega_1, \cdots, \omega_r} = (-1)^{\sharp(\underline{\boldsymbol{\omega}})} \displaystyle\int_{0<t_r<\cdots<t_1<1} \frac{dt_1 \cdots dt_r}{(\omega_1 - t_1) \cdots (\omega_r - t_r)}$,

where $\begin{cases} \Omega = \{0 \,;\, 1\} \\ \sharp(\underline{\boldsymbol{\omega}}) = \sharp\{i \,;\, \omega_i = 0\} \\ \omega_1 = 0 \text{ and } \omega_r = 1 \end{cases}$, is symmetr$\underline{a}$l.

- **Symmetrelity $\longleftrightarrow$ quasi-shuffle product (ou stuffle product)** :

If $\Omega = \{y_i \, ; \, i \in \mathbb{N}\}$:

$$\begin{cases} \varepsilon \star \omega = \omega \star \varepsilon = \omega \ . \\ (y_i \omega_1) \star (y_j \omega_2) = y_i \big(\omega_1 \star (y_j \omega_2)\big) + y_j \big((y_i \omega_1) \star \omega_2\big) + y_{i+j} (\omega_1 \star \omega_2) \ . \end{cases}$$

Let $sh\underline{e}(\underline{\boldsymbol{\omega}}^1 \, ; \underline{\boldsymbol{\omega}}^2) = \{\text{words appear in } \omega^1 \star \omega^2 \text{ (with multiplicity)}\}$.
A mould $Me^\bullet$ is symmetr$\underline{e}$l when :

$$\forall (\underline{\boldsymbol{\omega}}^1 \, ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \ Me^{\underline{\boldsymbol{\omega}}^1} Me^{\underline{\boldsymbol{\omega}}^2} = \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{e}(\underline{\boldsymbol{\omega}}^1 \, ; \underline{\boldsymbol{\omega}}^2)} Me^{\underline{\boldsymbol{\omega}}} \ .$$

Example : $$\mathcal{Z}e^{s_1, \cdots, s_r} = \sum_{1 \leq n_r < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \ , \text{ where } \begin{cases} \Omega = \mathbb{N}^* \\ s_1 \geq 2 \end{cases}$$

is symmetr$\underline{e}$l.

- **Symmetrility**

If $\Omega = \{y_i \,;\, i \in \mathbb{N}\}$:

$$\begin{cases} \varepsilon \,\widetilde{\star}\, \omega = \omega \,\widetilde{\star}\, \varepsilon = \omega \ . \\ (y_i\omega_1)\,\widetilde{\star}\,(y_j\omega_2) = y_i\big(\omega_1 \,\widetilde{\star}\,(y_j\omega_2)\big) + y_j\big((y_i\omega_1)\,\widetilde{\star}\,\omega_2\big) + (y_i \circledast y_j)(\omega_1 \,\widetilde{\star}\, \omega_2) \ . \end{cases}$$

Here, $\circledast$ is a abstract contraction which replace addition in symmetr<u>**e**</u>lity.
Let $sh\underline{\mathbf{i}}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) = \{$words appear in $\omega^1 \,\widetilde{\star}\, \omega^2$ (with multiplicity)$\}$
A mould $Mi^\bullet$ is symmetr<u>i</u>l when :

$$\forall(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \ Mi^{\underline{\boldsymbol{\omega}}^1} Mi^{\underline{\boldsymbol{\omega}}^2} = \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{i}}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2)} Mi^{\underline{\boldsymbol{\omega}}} \ ,$$

with the following recursive evaluation rule:

$$Mi^{\underline{\mathbf{v}} \cdot (x \circledast y) \cdot \underline{\mathbf{w}}} = \begin{cases} \dfrac{Mi^{\underline{\mathbf{v}} \cdot x \cdot \underline{\mathbf{w}}} - Mi^{\underline{\mathbf{v}} \cdot y \cdot \underline{\mathbf{w}}}}{x - y} \ , & \text{if} \quad x \neq y \ . \\ \dfrac{\partial Mi^{\underline{\mathbf{v}} \cdot x \cdot \underline{\mathbf{w}}}}{\partial x} \ , & \text{if} \quad x = y \ . \end{cases}$$

Introduction to Multitangent Functions: Definition.

Definition:

Let $\mathcal{S}^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ ; \ s_1 \geq 2 \text{ et } s_r \geq 2\}$.
For all sequence $\underline{\mathbf{s}} \in \mathcal{S}^\star$, we consider:

$$\begin{array}{rccl} \mathcal{T}e^{s_1,\cdots,s_r} : & \mathbb{C} - \mathbb{Z} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \displaystyle\sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{1}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} \ . \end{array}$$

Remarks :

1. Multitangent functions are a generalization of Eisenstein series $(r = 1)$.

2. Multitangent functions appear naturally in problems of holomorphic dynamics.

Introduction to Multitangent Functions: First Properties.

Property :

1. **Differential property.**

   Let $\underline{\mathbf{s}} = (s_1; \cdots s_r) \in \mathcal{S}^\star$.

   The function $\mathcal{T}e^{\underline{s}}$ is holomorphic on $\mathbb{C} - \mathbb{Z}$; it is a uniformly convergent series on any compact subset of $\mathbb{C} - \mathbb{Z}$ and satisfies:

$$\frac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z} = -\sum_{i=1}^{r} s_i \mathcal{T}e^{s_1, \cdots, s_{i-1}, s_i+1, s_{i+1}, \cdots, s_r} .$$

2. **Parity property.**

$$\forall z \in \mathbb{C} - \mathbb{Z} \ , \ \forall \underline{\mathbf{s}} \in \mathcal{S}^\star \ , \ \mathcal{T}e^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||} \mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z) .$$

3. **Symmetrelity.**

   $\mathcal{T}e^{\bullet}$ is symm$\underline{\text{e}}$trel.

Introduction to Multitangent Functions: Reduction into Monotangent Functions, First Version.

Remark : A monotangent function is a multitangent function with length 1 .

Let: $$\mathcal{MZV} = \mathrm{Vect}_{\mathbb{Q}} \left(\mathcal{Z}e^{\underline{\mathbf{s}}} \right)_{\substack{\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) \\ s_1 \geq 2}} \ .$$

$$m(\underline{\mathbf{s}}) = \max(s_1 \, ; \cdots ; s_r), \text{ for all } \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) \ .$$

Property: Reduction of Convergent Multitangent Functions into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathcal{S}^\star \ , \ \exists (z_2 \, ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{MZV}^{m(\underline{\mathbf{s}})-1} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = \sum_{\substack{\cancel{k=1} \\ k=2}}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k \ .$$

Sketch of proof:

1. Partial fraction expansion of $\dfrac{1}{(n_1 + X)^{s_1} \cdots (n_r + X)^{s_r}}$.

2. Using an analytic argument:
$$\forall z \in \mathbb{C} - \mathbb{R} \ , \ |\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq 4r \left(\frac{2}{|\Im m \ z|} \right)^{s_1 + \cdots + s_r - r - 1} \frac{e^{-\pi |\Im m \ z|}}{1 - e^{-\pi |\Im m \ z|}} \ .$$

Introduction to Multitangent Functions: Examples of Reduction into Monotangent Functions.

Weight 4

$$\mathcal{T}e^{2,2} = 2\zeta(2)\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,3} = -3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,2} = 3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,2} = 0 .$$

Weight 5

Weight 6

$$\mathcal{T}e^{2,4} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} - 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

$$\mathcal{T}e^{3,3} = -\frac{12}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{4,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} + 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

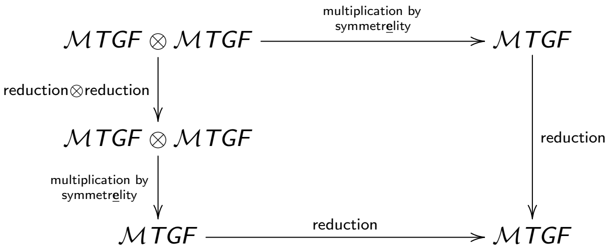
$$\mathcal{T}e^{2,2,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,1,3} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} + \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,1,2} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} - \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,1,2} = \frac{4}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: a Commutative Diagram.



Property:

The symmetr<u>e</u>lity of $\mathcal{T}e^{\bullet}$ and the precedent commutative diagram show the symmetr<u>e</u>lity of $\mathcal{Z}e^{\bullet}$.

Remark: We obtain other relations between multizeta values, for example some of regularization.

Relations Between Multizeta Values Obtained via the Study of Multitangent Functions: For weight 4.

Multitangent functions give us the following relations between multizetas values:

- By the commutative diagram:

$$6\mathcal{Z}e^{2,2} + 8\mathcal{Z}e^{4} = 5\left(\mathcal{Z}e^{2}\right)^2 .$$

$$2\mathcal{Z}e^{2,2} + \mathcal{Z}e^{4} = \left(\mathcal{Z}e^{2}\right)^2 .$$

- By the absence of composant $\mathcal{T}e^{1}$:

$$2\mathcal{Z}e^{2,2} + 4\mathcal{Z}e^{3,1} = \left(\mathcal{Z}e^{2}\right)^2 .$$

Consequences: : For the weight 4, we find all the symmetr**e**lity and symmetr**a**lity relations. We find sufficiently many relations to deduce those of regularization.

Problem : We are not able to find Euler relation: $\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3}$ $\cdots$

Multitangent Functions Renormalization: the property.

Theorem: Multitangent Functions Renormalization.

There exists an extension of multitangent functions to $\mathrm{seq}(\mathbb{N}^*)$ such that:

1. $\mathcal{T}e^\bullet$ is always symmetr<u>**e**</u>l.
2. $\forall z \in \mathbb{C} - \mathbb{Z}$, $\mathcal{T}e^1(z) = \dfrac{\pi}{\tan(\pi z)}$.

This extension automatically satisfies: the differential property.
the parity property.

It also satisfy the reduction property.

The removal to the right algorithm does not apply:

$$\mathcal{T}e^{1,2}(z) = \underbrace{\mathcal{T}e^{1}(z)}_{\substack{\text{known by}\\\text{hypothesis}}} \times \underbrace{\mathcal{T}e^{2}(z)}_{\substack{\text{convergent}\\\text{multitangent}\\\text{function}}} - \underbrace{\mathcal{T}e^{2,1}(z)}_{\substack{\text{problematics}\\=\text{unknown}}} - \underbrace{\mathcal{T}e^{3}(z)}_{\substack{\text{convergent}\\\text{multitangent}\\\text{function}}} \quad .$$

<u>**Sketch of proof:**</u>

1. $\mathcal{T}e^\bullet = \mathcal{H}e^\bullet_+ \times \mathcal{C}e^\bullet \times \mathcal{H}e^\bullet_-$.
2. Renormalize $\mathcal{H}e^\bullet_\pm$ by the removal to the right/left algorithm.
3. $\cdots$

Multitangent Functions Renormalization: problem.

Problem: Divergent multitangent functions are not written in an internal way...

We would like to express divergent multitangent functions with convergent multitangent functions.

Let us consider colored multizeta values and colored multitangent functions:

For $\binom{\underline{\varepsilon}}{\underline{s}} \in \text{seq}\left(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*\right)$ and $e_k = e^{-2i\pi\varepsilon_k}$, for $k \in [\![1 \, ; \, n]\!]$, we denote:

$$\mathcal{Z}e^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r}} = \sum_{1 \leq n_r < \cdots < n_1} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{{n_1}^{s_1} \cdots {n_r}^{s_r}} .$$

$$\mathcal{T}e^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{s_1, \cdots, s_r}}(z) = \sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} .$$

Theorem: The Generating Functions $\mathcal{T}ig^\bullet$.

Let:

$$\begin{cases} Qig^{\emptyset}(z) = 0 \ . \\ Qig^{\binom{\varepsilon_1}{v_1}}(z) = -Te^{\binom{\varepsilon_1}{1}}(v_1 - z) \ . \\ Qig^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}}(z) = 0 \ , \text{si } r \geq 2 \ . \end{cases}$$

$$\begin{cases} \delta^{\emptyset} = 0 \ . \\ \delta^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}} = \begin{cases} \dfrac{(i\pi)^r}{r!} \mathbb{1}_{\{0\}}(\varepsilon_1) \cdots \mathbb{1}_{\{0\}}(\varepsilon_r) & \text{, if } r \text{ is even.} \\ 0 & \text{, if } r \text{ is odd.} \end{cases} \end{cases}$$

Then:

$$\mathcal{T}ig^\bullet(z) = \delta^\bullet + \mathcal{Z}ig_+^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(z) \times \mathcal{Z}ig_-^{\lfloor\bullet} \ .$$

Multitangent Functions Renormalization: Reduction into Monotangent Functions, second version.

Property: Reduction into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)\ ,\ \exists (z_1 ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{MZV}^{m(\underline{\mathbf{s}})}\ ,\ \mathcal{T}e^{\underline{\mathbf{s}}} = \delta^{\underline{\mathbf{s}}} + \sum_{k=1}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k\ ,$$

where $\delta^{\underline{\mathbf{s}}} = \begin{cases} \dfrac{(i\pi)^r}{r!} & \text{, if } \underline{\mathbf{s}} = 1^{[r]} \text{ and if } r \text{ is even.} \\ 0 & \text{, else.} \end{cases}$

Important remark: $z_1 = 0 \iff \underline{\mathbf{s}} \neq 1^{[r]}$ or $\begin{cases} \underline{\mathbf{s}} = 1^{[r]}\ . \\ r \text{ is even}\ . \end{cases}$

Multitangent Functions Renormalization: Examples of Reduction into Monotangent Functions.

Weight 2

$$\mathcal{T}e^{1,1} = -3\,\zeta(2)\ .$$

Weight 3

$$\mathcal{T}e^{1,2} = 0\ .$$

$$\mathcal{T}e^{2,1} = 0\ .$$

$$\mathcal{T}e^{1,1,1} = -\zeta(2)\mathcal{T}e^{1}\ .$$

Weight 4

$$\mathcal{T}e^{1,3} = -\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{3,1} = -\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,1,2} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,2,1} = 0\ .$$

$$\mathcal{T}e^{2,1,1} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,1,1,1} = \frac{3}{2}\zeta(2)^{2}\ .$$

We have:

$$\begin{aligned}\mathcal{T}e^{1,1,2} &= \left(-2\mathcal{Z}e^{2,1} + \mathcal{Z}e^{2}\mathcal{Z}e^{1} - \mathcal{Z}e^{1,2}\right)\mathcal{T}e^{1} + \mathcal{Z}e^{1,1}\mathcal{T}e^{2} \\ &= \left(-2\mathcal{Z}e^{2,1} + \mathcal{Z}e^{2,1} + \mathcal{Z}e^{3}\right)\mathcal{T}e^{1} + \frac{1}{2}\left(\left(\mathcal{Z}e^{1}\right)^{2} - \mathcal{Z}e^{2}\right)\mathcal{T}e^{2} \\ &= \left(\mathcal{Z}e^{3} - \mathcal{Z}e^{2,1}\right)\mathcal{T}e^{1} - \frac{1}{2}\mathcal{Z}e^{2}\mathcal{T}e^{2}\ .\end{aligned}$$

So, by the cancellation of the $\mathcal{T}e^{1}$ term, we obtain:

$$\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3}\ .$$

Conclusion:

1. Multitangent functions seem to be an interesting functional model for the study of multizetas values.
2. There is a deep link between multizeta values and multitangent functions, **the reduction into monotangent functions** ; another important link, but a conjectural one, is **projection on multitangent function space**.
3. Convergent multitangent functions don't allow us to find the full dimorphy of multizeta values: we find only $1 + \frac{1}{2}$ symmetries.
4. The missing relations seem to be retrievable, but in a more complicated way...