Generalization of Bernoulli numbers and polynomials to the multiple case

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Outline

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Definition

Definition:

- *Concrete def.:* A mould is a function with a variable number of variables.
- *Mathematical def. 1:* A mould is a function defined on a monoid.
- *Mathematical def. 2:* A mould is a collection $(f_0, f_1, f_2, \cdots)$ of functions, f_n being a function of n variables.

Typical examples: The addition : $f_n(x_1, \cdots, x_n) = x_1 + \cdots + x_n$.

The Multiple Zeta Values :

$$f_0 : () \longmapsto 1$$

$$f_1 : s_1 \longmapsto \sum_{0<n_1} \frac{1}{n_1^{s_1}}$$

$$f_2 : (s_1, s_2) \longmapsto \sum_{0<n_2<n_1} \frac{1}{n_1^{s_1} n_2^{s_2}}$$

$$\vdots$$

First notations.

Let Ω be an alphabet, and $\Omega^\star$ be the set of all words

- *New notations:*

	Functional notations	Mould notations
Name	f	$M^\bullet$
Evaluation	$f(x)$	$M^{\underline{\omega}}$

- *Why do we need some "new" notations ?*

$\rightsquigarrow$ To mix easily index and exponent in notations.

$\rightsquigarrow$ To understand easily the type of object at first sight.

Mould/comould's contractions.

Moulds might be contracted with dual objects, called **comoulds** (which are also functions with a variable number of variables) :

Definition: Mould - Comould contraction:

The mould-comould contraction of a mould $M^\bullet$ and a comould $B_\bullet$ is:

$$\sum_\bullet M^\bullet B_\bullet := \sum_{\underline{\omega} \in \Omega^\star} M^{\underline{\omega}} B_{\underline{\omega}}$$

(*if the sum is well-defined...*)

For analytical reasons, a mould-comould contraction might be understood to be an **algebra automorphism** or a **derivation**.

Important remark:

All the following mould's definitions come from such an interpretation, in particular for the mould algebra's structure and the mould's symmetries.

Difference between moulds and co-moulds

Moulds:	**Co-moulds:**
$\rightsquigarrow$ needs a commutative $\mathbb{C}$-algebra $\mathbf{C}$, in which moulds take their values	$\rightsquigarrow$ needs a $\mathbf{C}$-algebra $\mathbf{O}$, in which co-moulds take their values
$\rightsquigarrow$ $\mathbf{C}$ = algebra of **coefficients**	$\rightsquigarrow$ $\mathbf{O}$ = algebra of **operators**
$\rightsquigarrow$ mould = any **map** $\Omega^\star \longmapsto \mathbf{C}$	$\rightsquigarrow$ co-mould = any **homomorphism** $\Omega^\star \longmapsto \mathbf{O}$

- To each letter $\omega \in \Omega$, we define a symbol a_ω, which will be, when necessary, specialized to B_ω.

  $\rightsquigarrow$ The symbols a_ω do not commute.

  $\rightsquigarrow$ The symbols a_ω are extended to words:

$$a_{\omega_1 \cdots \omega_r} = a_{\omega_1} \cdots a_{\omega_r} \ .$$

- To each mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$, we define a series $s(M^\bullet) \in \mathbf{C}\langle\langle \mathrm{A} \rangle\rangle$, where $\mathrm{A} = \{a_\omega \ ; \ \omega \in \Omega\}$ by:

$$s(M^\bullet) = \sum_{\underline{\boldsymbol{\omega}} \in \Omega^\star} M^{\underline{\boldsymbol{\omega}}} \, a_{\underline{\boldsymbol{\omega}}} := \sum_\bullet M^\bullet \, a_\bullet \ .$$

  $\rightsquigarrow$ If φ is a specialization morphism defined by $\varphi(a_\omega) = B_\omega$, then:

$$\varphi(s(M^\bullet)) = \sum_\bullet M^\bullet \, B_\bullet \ .$$

Outline

Mould operations.

Let $M^\bullet$ and $N^\bullet$ be two moulds valued in an algebra $\mathbf{C}$, $\lambda \in \mathbf{C}$.

- **Addition:** $\sum_\bullet (M^\bullet + N^\bullet) a_\bullet = \sum_\bullet M^\bullet a_\bullet + \sum_\bullet N^\bullet a_\bullet$

$$S^\bullet = M^\bullet + N^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \,,\; S^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}} + N^{\underline{\boldsymbol{\omega}}} \,.$$

- **Scalar multiplication:** $\sum_\bullet (\lambda M)^\bullet a_\bullet = \lambda \sum_\bullet M^\bullet a_\bullet$

$$(\lambda M)^\bullet = \lambda \cdot M^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \,,\; (\lambda M)^{\underline{\boldsymbol{\omega}}} = \lambda\, M^{\underline{\boldsymbol{\omega}}} \,.$$

- **Mould multiplication:** $\sum_\bullet (M^\bullet \times N^\bullet)\, a_\bullet = \Big(\sum_\bullet M^\bullet a_\bullet\Big)\Big(\sum_\bullet N^\bullet a_\bullet\Big)$

$$P^\bullet = M^\bullet \times N^\bullet \iff \forall \underline{\boldsymbol{\omega}} \in \Omega^\star \,,\; P^{\underline{\boldsymbol{\omega}}} = \sum_{\substack{(\underline{\boldsymbol{\omega}}^1\,;\,\underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \\ \underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}}^1 \cdot \underline{\boldsymbol{\omega}}^2}} M^{\underline{\boldsymbol{\omega}}^1} N^{\underline{\boldsymbol{\omega}}^2} \,.$$

- **Example of mould product computation:** $P^\bullet = M^\bullet \times N^\bullet$

$$\begin{aligned} P^\emptyset &= M^\emptyset N^\emptyset \\ P^{\omega_1} &= M^{\omega_1} N^\emptyset + M^\emptyset N^{\omega_1} \\ P^{\omega_1,\omega_2} &= M^{\omega_1,\omega_2} N^\emptyset + M^{\omega_1} N^{\omega_2} + M^\emptyset N^{\omega_1,\omega_2} \end{aligned}$$

- **Example of operations on moulds:**

$$\begin{aligned} \left(A^\bullet \times B^\bullet\right) \times C^\bullet &= A^\bullet \times \left(B^\bullet \times C^\bullet\right) . \\ A^\bullet \times 1^\bullet &= 1^\bullet \times A^\bullet = A^\bullet . \\ \left(A^\bullet + B^\bullet\right) \times C^\bullet &= A^\bullet \times C^\bullet + B^\bullet \times C^\bullet . \\ A^\bullet \times \left(B^\bullet + C^\bullet\right) &= A^\bullet \times B^\bullet + A^\bullet \times C^\bullet . \\ A^\bullet \times B^\bullet &\neq B^\bullet \times A^\bullet . \end{aligned}$$

Property: Algebraic structure

Let $\mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ be the set of moulds defined over the alphabet Ω and valued in the algebra $\mathbf{C}$.
So, $(\mathcal{M}^\bullet_{\mathbf{C}}(\Omega), +, ., \times)$ is a non commuative, associative, unitary $\mathbf{C}$-algebra.

Outline

Translating the Leibnitz rules

- In practice, the comould satisfies some "Leibnitz rules", which have to be translated in terms of the a_ω:

$$B_\omega(\varphi\psi) = B_\omega(\varphi)\psi + \varphi B_\omega(\psi) \quad \rightsquigarrow \quad \Delta(a_\omega) = a_\omega \otimes 1 + 1 \otimes a_\omega$$

$$B_\omega(\varphi\psi) = \sum_{\substack{\omega_1,\omega_2\in\Omega \\ \omega_1+\omega_2=\omega}} B_{\omega_1}(\varphi)B_{\omega_2}(\psi) \quad \rightsquigarrow \quad \Delta(a_\omega) = \sum_{\substack{\omega_1,\omega_2\in\Omega \\ \omega_1+\omega_2=\omega}} a_{\omega_1} \otimes a_{\omega_2}$$

(where $(\Omega, +)$ is a semi-group)

- Δ is extended to words over A in order to be an algebra homomorphism, and then to $\mathbf{C}\langle\langle \mathrm{A}\rangle\rangle$ by linearity. So,

$(\mathbf{C}\langle\langle \mathrm{A}\rangle\rangle, \cdot, \Delta)$ is a Hopf algebra.

Property:

Let $\varphi : a_\omega \longmapsto B_\omega$ be a specialization morphism.

1. If $\Delta\left(s(M^\bullet)\right) = s(M^\bullet) \otimes s(M^\bullet)$ (*i.e.* is group-like), then $\varphi\left(s(M^\bullet)\right)$ is an automorphism.
2. If $\Delta\left(s(M^\bullet)\right) = s(M^\bullet) \otimes 1 + 1 \otimes s(M^\bullet)$ (*i.e.* is primitive), then $\varphi\left(s(M^\bullet)\right)$ is a derivation.

Definition: shuffle product

For $u, v \in \Omega^\star$ and $a, b \in \Omega$, we define recursively the shuffle product $⧢$ by:

$$\begin{cases} 1 ⧢ u & = & u ⧢ 1 \; = \; u \,. \\ ua ⧢ vb & = & (u ⧢ vb)\, a + (ua ⧢ v)\, b \end{cases}$$

Let $sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2)$ be the set of words which appear in $\underline{\boldsymbol{\omega}}^1 ⧢ \underline{\boldsymbol{\omega}}^2$, counted with their multiplicity.

Lemma:

For any word $\underline{\boldsymbol{\omega}} \in \Omega^\star$, $\Delta(\underline{\boldsymbol{\omega}}) = \sum_{\substack{\underline{\boldsymbol{\omega}}_1, \underline{\boldsymbol{\omega}}_2 \in \Omega^\star \\ \underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}_1, \underline{\boldsymbol{\omega}}_2)}} \underline{\boldsymbol{\omega}}_1 \otimes \underline{\boldsymbol{\omega}}_2$.

Property:

A mould $M^\bullet \in \mathcal{M}_{\mathbb{C}}^\bullet(\Omega)$ satisfies:

1. $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes s(M^\bullet)$ if, and only if, the mould $M^\bullet$ satisfies:

$$\forall(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \quad \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}^1} M^{\underline{\boldsymbol{\omega}}^2} \ . \qquad (1)$$

2. $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes 1 + 1 \otimes s(M^\bullet)$ if, and only if, the mould $M^\bullet$ satisfies:

$$\begin{cases} M^{\emptyset} = 0 \ . \\ \forall(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star - \{\emptyset\})^2 \ , \quad \displaystyle\sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}^1 \,;\, \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = 0 \ . \end{cases} \qquad (2)$$

<u>**Sketch of proof of (1):**</u> $\Delta(s(M^\bullet)) = \displaystyle\sum_{\omega \in \Omega^\star} M^\omega \Delta(\omega) = \sum_{\substack{(\omega, \omega_1, \omega_2) \in (\Omega^\star)^3 \\ \omega \in sh\underline{\mathbf{A}}(\omega_1, \omega_2)}} M^\omega \omega_1 \otimes \omega_2$

$$\Downarrow$$

$$\Delta(s(M^\bullet)) - s(M^\bullet) \otimes s(M^\bullet) = \sum_{(\omega_1, \omega_2) \in (\Omega^\star)^2} \left(\sum_{\substack{\omega \in \Omega \\ \omega \in sh\underline{\mathbf{A}}(\omega_1, \omega_2)}} M^\omega - M^{\omega_1} M^{\omega_2} \right) \omega_1 \otimes \omega_2 \ .$$

Definition

A mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ is called:

1. symmetr**A**l when:

$$\forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \quad \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}^1} M^{\underline{\boldsymbol{\omega}}^2} \ . \tag{3}$$

2. altern**A**l when:

$$\begin{cases} M^{\emptyset} = 0 \ . \\ \forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star - \{\emptyset\})^2 \ , \quad \displaystyle\sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{A}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = 0 \ . \end{cases} \tag{4}$$

Reformulation of the previous property:

A mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ is:

1. symmetr**A**l if, and only if, $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes s(M^\bullet)$.
2. altern**A**l if, and only if, $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes 1 + 1 \otimes s(M^\bullet)$.

Example 1:

Let Ω be a set, and for all $\omega \in \Omega$, let $f_\Omega : [0;1] \longmapsto \mathbb{R}$ be a continuous function.

Let $\mathcal{I}^\bullet$ be the mould defined by:

$$\mathcal{I}^{\omega_1,\cdots,\omega_r} = \int_{0<t_r<\cdots<t_1<1} f_{\omega_1}(t_1)\cdots f_{\omega_r}(t_r)\ dt_1\cdots dt_r .$$

Then, $\mathcal{I}^\bullet$ is symmetr**A**l .

Example 2:

Any mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\Omega)$ such that:

$$l(\omega) \neq 1 \Longrightarrow M^\omega = 0 \ ,$$

is altern**A**l .

The symmetr**E**l case: $\Delta(a_\omega) = \sum_{\substack{\omega_1,\omega_2\in\Omega \\ \omega_1\perp\omega_2=\omega}} a_{\omega_1}\otimes a_{\omega_2}$ 1/3

Definition: stuffle product

Suppose that $(\Omega,+)$ is an additive semi-group. For $u,v\in\Omega^\star$ and $a,b\in\Omega$, we define recursively the stuffle product $⊎$ by:

$$\begin{cases} \varepsilon ⊎ u = u ⊎ \varepsilon = u \,. \\ ua ⊎ vb = (u ⊎ vb)\,a + (ua ⊎ v)\,b + (u ⊎ v)(a+b)\,. \end{cases} \tag{5}$$

Let $sh\mathbf{\underline{E}}(\underline{\boldsymbol{\omega}}^1\,;\,\underline{\boldsymbol{\omega}}^2)$ be the set of words which appear in $\underline{\boldsymbol{\omega}}^1 ⊎ \underline{\boldsymbol{\omega}}^2$, counted with their multiplicity.

Lemma:

For any word $\underline{\boldsymbol{\omega}}\in\Omega^\star$, $\Delta(\underline{\boldsymbol{\omega}}) = \sum_{\substack{\underline{\boldsymbol{\omega}}_1,\underline{\boldsymbol{\omega}}_2\in\Omega^\star \\ \underline{\boldsymbol{\omega}}\in\mathrm{sh}\mathbf{\underline{E}}(\underline{\boldsymbol{\omega}}_1,\underline{\boldsymbol{\omega}}_2)}} \underline{\boldsymbol{\omega}}_1\otimes\underline{\boldsymbol{\omega}}_2\,.$

Property:

A mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ satisfies:

1. $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes s(M^\bullet)$ if, and only if, the mould $M^\bullet$ satisfies:

$$\forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \quad \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}^1} M^{\underline{\boldsymbol{\omega}}^2} \ . \tag{6}$$

2. $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes 1 + 1 \otimes s(M^\bullet)$ if, and only if, the mould $M^\bullet$ satisfies:

$$\begin{cases} M^\emptyset = 0 \ . \\ \forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star - \{\emptyset\})^2 \ , \quad \displaystyle\sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = 0 \ . \end{cases} \tag{7}$$

Definition

A mould $M^\bullet \in \mathcal{M}_{\mathbb{C}}^\bullet(\Omega)$ is called:

1. symmetr$\underline{\mathbf{E}}$l when:

$$\forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star)^2 \ , \quad \sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = M^{\underline{\boldsymbol{\omega}}^1} M^{\underline{\boldsymbol{\omega}}^2} \ . \tag{8}$$

2. altern$\underline{\mathbf{E}}$l when:

$$\begin{cases} M^\emptyset = 0 \ . \\ \forall(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2) \in (\Omega^\star - \{\emptyset\})^2 \ , \quad \displaystyle\sum_{\underline{\boldsymbol{\omega}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{\omega}}^1 ; \underline{\boldsymbol{\omega}}^2)} M^{\underline{\boldsymbol{\omega}}} = 0 \ . \end{cases} \tag{9}$$

Reformulation of the previous property:

A mould $M^\bullet \in \mathcal{M}_{\mathbb{C}}^\bullet(\Omega)$ is:

1. symmetr$\underline{\mathbf{E}}$l if, and only if, $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes s(M^\bullet)$.
2. altern$\underline{\mathbf{E}}$l if, and only if, $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes 1 + 1 \otimes s(M^\bullet)$.

An example of symmetr**E**l and altern**E**l moulds

Example 1:

Let $(u_n)_{n\in\mathbb{N}_{\geq 1}}$ be a sequence of complex numbers such that:

$$\exists \alpha \geq 2,\ u_n \underset{n\longrightarrow+\infty}{=} \mathcal{O}(n^{-\alpha})\ .$$

Let $\mathcal{S}^\bullet \in \mathcal{M}^\bullet_{\mathbb{R}}(\mathbb{N}_{\geq 1})$ be the mould defined by:

$$\mathcal{S}^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1<+\infty} u_{n_1}^{s_1}\cdots u_{n_r}^{s_2}.$$

Then, $\mathcal{S}^\bullet$ is symmetr**E**l .

Example 2:

The mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{R}}(\Omega)$ defined by:

$$M^\omega = \frac{(-1)^{l(\omega)}}{l(\omega)}\ ,$$

is altern**E**l .

Outline

Formal moulds

Other point of view on moulds: A mould is a collection of functions $(f_0, f_1, f_2, \cdots)$, where $f_i : \Omega^i \longmapsto \mathbf{C}$.

Definition:

A formal mould is a collection of formal series $(S_0, S_1, S_2, \cdots)$, where S_n is a formal power series in n indeterminates (and consequently, S_0 is constant)

Notation: $\mathcal{FM}^\bullet_{\mathbf{C}} = \{\text{formal mould with values in the algebra } \mathbf{C}\}$.

What is the difference between a mould and a formal mould?

Mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbf{C}}(\Omega)$ where $\Omega = (X_1, X_2, \cdots)$.	Formal mould $M^\bullet \in \mathcal{FM}^\bullet_{\mathbf{C}}$
No link between M^{X_1,X_2} and M^{X_2,X_1} !!!	M^{X_1,X_2} and M^{X_2,X_1} are related by the **substitution** of the indeterminates.

Generics example of formal moulds

Definition:

With a mould $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$, we associate two formal moulds $Mog^\bullet$ and $Meg^\bullet$ defined by:

$$\left\{\begin{array}{lcl} Mog^{X_1,\cdots,X_r} & = & \displaystyle\sum_{s_1,\cdots,s_r\in\mathbb{N}} M^{s_1,\cdots,s_r} X_1^{s_1}\cdots X_r^{s_r} \ . \\ Meg^{X_1,\cdots,X_r} & = & \displaystyle\sum_{s_1,\cdots,s_r\in\mathbb{N}} M^{s_1,\cdots,s_r} \frac{X_1^{s_1}}{s_1!}\cdots\frac{X_r^{s_r}}{s_r!} \ . \end{array}\right.$$

Definition:

The secondary symmetries of a mould are the symmetries of its corresponding formal moulds.

Theorem: (Ecalle, $\sim 80's$)

Let $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$ be a mould.

1. $M^\bullet$ is symmetr**A**l if, and only if, $Mog^\bullet$ is symmetr**A**l .
2. $M^\bullet$ is altern**A**l if, and only if, $Mog^\bullet$ is altern**A**l .

Theorem: (B., 2015)

Let $M^\bullet \in \mathcal{M}^\bullet_{\mathbb{C}}(\mathbb{N})$ be a mould.

1. $M^\bullet$ is symmetr**A**l if, and only if, $Meg^\bullet$ is symmetr**A**l .
2. $M^\bullet$ is altern**A**l if, and only if, $Meg^\bullet$ is altern**A**l .
3. $M^\bullet$ is symmetr**E**l if, and only if, $Meg^\bullet$ is symmetr**E**l .
4. $M^\bullet$ is altern**E**l if, and only if, $Meg^\bullet$ is altern**E**l .

Algebraic construction to prove our result on secondary symmetries

1. Let us consider:

   $\mathbf{X} = \{X_1, X_2, \cdots\}$ be an infinite set of indeterminates.

   $$\mathbf{Y} = \mathbb{N}\mathbf{X} = \left\{ \sum_{x \in \mathbf{X}} \lambda_x x \ ; \ (\lambda_X)_{X \in \mathbf{X}} \in \mathbb{N}^{\mathbf{X}} \text{ has finitely nonzero terms} \right\}.$$

   $\mathrm{A} = \{A_y \ ; \ y \in \mathbf{Y}\}$.

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   $\mathrm{A} = \{A_y \ ; \ y \in \mathbf{Y}\}.$
2. We extend the definition of $Mog^{\bullet}$ and $Meg^{\bullet}$ to words constructed over $\mathrm{A}^{\star}$:
   $$Mog^{A_{y_1}, \cdots, A_{y_r}} := Mog^{y_1, \cdots, y_r} \qquad , \qquad Meg^{A_{y_1}, \cdots, A_{y_r}} := Meg^{y_1, \cdots, y_r}$$

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3. We define a secondary formal mould/comould contraction, *i.e.* to a formal mould $FM^\bullet \in \mathcal{FM}^\bullet_{\mathbf{C}}$, we associate a series $S(FM^\bullet) \in \mathbf{C}[\![\mathbf{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle$ by:
   $$S(FM^\bullet) = \sum_{\underline{\boldsymbol{\omega}} \in \mathrm{A}^\star} FM^{\underline{\boldsymbol{\omega}}} A_{\underline{\boldsymbol{\omega}}} := \sum_{\bullet} FM^\bullet A_\bullet \ .$$

Algebraic construction to prove our result on secondary symmetries

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4. We define a map Δ as before, *i.e.* a coproduct, by:

   <u>*Case AI:*</u> $\quad \Delta(A_y) = A_y \otimes 1 + 1 \otimes A_y \qquad \Delta(\underline{\boldsymbol{A}}) = \displaystyle\sum_{\substack{\underline{\boldsymbol{B}},\underline{\boldsymbol{C}} \in \mathrm{A}^\star \\ \underline{\boldsymbol{A}} \in sh\mathbf{A}(\underline{\boldsymbol{B}},\underline{\boldsymbol{C}})}} \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{C}}$

   <u>*Case EI:*</u> $\quad \Delta(A_y) = \displaystyle\sum_{\substack{u,v \in \mathbf{Y} \\ u+v=y}} A_u \otimes A_v \qquad \Delta(\underline{\boldsymbol{A}}) = \displaystyle\sum_{\substack{\underline{\boldsymbol{B}},\underline{\boldsymbol{C}} \in \mathrm{A}^\star \\ \underline{\boldsymbol{A}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{B}},\underline{\boldsymbol{C}})}} \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{C}}$

Algebraic construction to prove our result on secondary symmetries

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4. We define a map Δ as before, *i.e.* a coproduct, by:

   <u>*Case AI:*</u> $\quad \Delta(A_y) = A_y \otimes 1 + 1 \otimes A_y \qquad \Delta(\underline{\boldsymbol{A}}) = \displaystyle\sum_{\substack{\underline{\boldsymbol{B}}, \underline{\boldsymbol{C}} \in \mathrm{A}^\star \\ \underline{\boldsymbol{A}} \in sh\mathbf{A}(\underline{\boldsymbol{B}}, \underline{\boldsymbol{C}})}} \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{C}}$

   <u>*Case EI:*</u> $\quad \Delta(A_y) = \displaystyle\sum_{\substack{u, v \in \mathbf{Y} \\ u+v=y}} A_u \otimes A_v \qquad \Delta(\underline{\boldsymbol{A}}) = \displaystyle\sum_{\substack{\underline{\boldsymbol{B}}, \underline{\boldsymbol{C}} \in \mathrm{A}^\star \\ \underline{\boldsymbol{A}} \in sh\underline{\mathbf{E}}(\underline{\boldsymbol{B}}, \underline{\boldsymbol{C}})}} \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{C}}$

5. Now, we just have to adapt the proof of

   $\Delta(s(M^\bullet)) = s(M^\bullet) \otimes s(M^\bullet) \iff M^\bullet$ is symmetr<u>**A**</u>l /symmetr<u>**E**</u>l .

Exercices

1. Adapt the previous construction to have a better understanding of the symmetr**I**lity . Then prove that:
   a. $M^\bullet$ is symmetr**E**l $\iff$ $Mog^\bullet$ is symmetr**I**l .
   b. $M^\bullet$ is altern**E**l $\iff$ $Mog^\bullet$ is altern**I**l .

2. Let $\delta_{a,b}$ be the Kronecker symbol.

   Let $M^\bullet$ be a mould such that:

$$\begin{cases} M^a M^b &= M^{a,b} + M^{b,a} + \delta_{a,b} M^b \\ M^a M^{b,c} &= M^{a,b,c} + M^{b,a,c} + M^{b,c,a} + \delta_{a,b} M^{a,c} + \delta_{a,c} M^{b,a} \\ M^{a,b} M^{c,d} &= M^{a,b,c,d} + M^{a,c,b,d} + M^{a,c,d,b} + M^{c,d,a,b} + M^{c,a,d,b} + M^{c,a,b,d} \\ & \quad + \delta_{a,c}(M^{c,b,d} + M^{c,d,b}) + \delta_{a,c}\delta_{b,d} M^{c,d} \\ & \vdots \end{cases}$$

   a. Define a (primary) mould symmetry satisfied by $M^\bullet$.
   b. What are its corresponding secondary symmetries ?

Outline

Why multiple Bernoulli numbers and polynomials?

Definition:

The numbers $\mathcal{Z}e^{s_1,\cdots,s_r}$ defined by

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots {n_r}^{s_r}} ,$$

where $s_1, \cdots, s_r \in \mathbb{C}$ such that $\Re(s_1 + \cdots + s_k) > k$, $k \in [\![1; r]\!]$, are called multiple zeta values.

Fact: There exist at least three different ways to renormalise multiple zeta values at negative integers.

$$\mathcal{Z}e^{0,-2}_{MP}(0) = \frac{7}{720} \quad , \quad \mathcal{Z}e^{0,-2}_{GZ}(0) = \frac{1}{120} \quad , \quad \mathcal{Z}e^{0,-2}_{FKMT}(0) = \frac{1}{18} .$$

Question: Is there a group acting on the set of all possible multiple zeta values renormalisations?

Main goal: Define multiple Bernoulli numbers in relation with this.

Definition:

The Riemann and Hurwitz zeta functions are defined, for $\Re e\, s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s} \quad , \quad \zeta(s,z) = \sum_{n \geq 0} \frac{1}{(n+z)^s} \ .$$

Property:

$s \longmapsto \zeta(s)$ and $s \longmapsto \zeta(s,z)$ can be analytically extend to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

Remark:

$$\zeta(-n) = -\frac{b_{n+1}}{n+1} \text{ for all } n \in \mathbb{N} \ .$$

$$\zeta(-n,z) = -\frac{B_{n+1}(z)}{n+1} \text{ for all } n \in \mathbb{N} \ .$$

Hurwitz Multiple Zeta Functions

Definition: Hurwitz multiple zeta functions

Let $z \in \mathbb{C} - \mathbb{N}_{<0}$ and $(s_1, \cdots, s_r) \in (\mathbb{N}_{\geq 1})^r$, such that $s_1 \geq 2$.
The Hurwitz multiple zeta functions are defined by:

$$\mathcal{H}e^{s_1, \cdots, s_r}(z) = \sum_{0 < n_r < \cdots < n_1} \frac{1}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \ .$$

Lemma: (B., Ecalle, 2013)

Let Δ_- be the difference operator: $\Delta_-(f)(z) = f(z-1) - f(z)$
Then: $\Delta_-(\mathcal{H}e^{\bullet}) = \mathcal{H}e^{\bullet} \times J^{\bullet}$, where $J^{s_1, \cdots s_r}(z) = \begin{cases} z^{-s_1} & \text{if } r = 1. \\ 0 & \text{otherwise.} \end{cases}$

Heuristic:

$\mathcal{B}e^{s_1, \cdots, s_r}(z) =$ Multiple (divided) Bernoulli polynomials $= \mathcal{H}e^{-s_1, \cdots, -s_r}(z)$.

$be^{s_1, \cdots, s_r} =$ Multiple (divided) Bernoulli numbers $= \mathcal{H}e^{-s_1, \cdots, -s_r}(0)$.

Some Properties of Bernoulli Polynomials

Property 1: Difference Equation

$$\Delta\left(\frac{B_n}{n}\right)(x) = x^{n-1} \text{ for all } n \in \mathbb{N}^*, \text{ where } \Delta(f)(z) = f(z+1) - f(z)\ .$$

Property 2: Reflexion Formula

$$(-1)^n B_n(1-x) = B_n(x) \text{ for all } n \in \mathbb{N}.$$

Main Goal

We want to define a mould $\mathcal{B}e^{\bullet}(z) \in \mathcal{M}^{\bullet}_{\mathbb{Q}[z]}(\mathbb{N})$ such that:

- its values have properties similar to Hurwitz Multiple Zeta Functions' properties.
- its values have properties generalizing these of Bernoulli polynomials.

Main Goal:

Find $\mathcal{B}e^{\bullet} \in \mathcal{M}^{\bullet}_{\mathbb{C}[z]}(\mathbb{N})$ such that:

$$\begin{cases} \mathcal{B}e^{s}(z) = \dfrac{B_{s+1}(z)}{s+1} \text{ , where } s \geq 0 \text{ ;} \\ \\ \Delta\left(\mathcal{B}e^{\bullet}\right)(z) = \mathcal{B}e^{\bullet} \times J^{\bullet}(z) \\ \\ \mathcal{B}e^{\bullet} \text{ is symmetr}\underline{\mathbf{E}}\text{l .} \end{cases}$$

A singular solution

$$\left\{\begin{array}{l} \mathcal{B}e^{s}(z) = \dfrac{B_{s+1}(z)}{s+1}\ \text{, where } s \geq 0 \\[2ex] \Delta\left(\mathcal{B}e^{\bullet}\right)(z) = \mathcal{B}e^{\bullet} \times J^{\bullet}(z) \\[2ex] \mathcal{B}e^{\bullet} \text{ is symmetr}\underline{\mathbf{E}}\text{l .} \end{array}\right. \iff \left\{\begin{array}{l} \mathcal{B}eeg^{X}(z) = \dfrac{e^{zX}}{e^{X}-1} - \dfrac{1}{X} \\[2ex] \Delta\left(\mathcal{B}eeg^{\bullet}\right)(z) = \mathcal{B}eeg^{\bullet} \times Jeg^{\bullet}(z) \\[2ex] Beeg^{\bullet} \text{ is symmetr}\underline{\mathbf{E}}\text{l .} \end{array}\right.$$

Reminders:

1. $Jeg^{X}(z) = e^{zX}$ and $Jeg^{X_1,\cdots,X_r} = 0$ if $r \neq 1$.
2. $\Delta\left(\dfrac{e^{zX}}{e^{X}-1}\right) = e^{zX}$.

From a false solution to a singular solution...

$\mathcal{S}ing^{X_1,\cdots,X_r}(z) = \dfrac{e^{z(X_{k_1}+\cdots+X_{k_r})}}{\prod_{i=1}^{r}(e^{X_{k_1}+\cdots+X_{k_i}}-1)}$ is a false solution to the system we are solving...:

$$\mathcal{S}ing^{\bullet}(z) \notin \mathcal{FM}^{\bullet}_{\mathbb{C}[X]} \text{ but } \mathcal{S}ing^{\bullet}(z) \in \mathcal{FM}^{\bullet}_{\mathbb{C}[\![X]\!]} \ .$$

Another solution

Fact: If $\Delta(f)(z) = f(z-1) - f(z)$, $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$.

Consequence: There exists a unique family of polynomials $(Be^{s_1,\cdots,s_r})$ such that:

$$\begin{cases} Be_0^{n_1,\cdots,n_r}(z+1) - Be_0^{n_1,\cdots,n_r}(z) = Be_0^{n_1,\cdots,n_{r-1}}(z)z^{n_r} \ . \\ Be_0^{n_1,\cdots,n_r}(0) = 0 \ . \end{cases}$$

This produces $\mathcal{B}e_0^{\bullet} \in \mathcal{M}_{\mathbb{C}[z]}^{\bullet}(\mathbb{N})$ and $\mathcal{B}eeg_0^{\bullet} \in \mathcal{FM}_{\mathbb{C}[z]}^{\bullet}$.

Lemma: (B., 2013)

1. The moulds $\mathcal{B}e_0^{\bullet}(z)$ and $\mathcal{B}eeg_0^{\bullet}(z)$ are symmetr**E**l .
2. The mould $\mathcal{B}eeg_0^{\bullet}(z)$ satisfies a recurence relation:

$$\begin{cases} \mathcal{B}e_0^{Y_1}(z) = \dfrac{e^{zY_1} - 1}{e^{Y_1} - 1} \\ \mathcal{B}e_0^{Y_1,\cdots,Y_r}(z) = \dfrac{\mathcal{B}e_0^{Y_1+Y_2,Y_3,\cdots,Y_r}(z) - \mathcal{B}e_0^{Y_2,Y_3,\cdots,Y_r}(z)}{e^{Y_1} - 1} \end{cases}$$

3. $\mathcal{B}eeg_0^{\bullet}(z) = \left(\mathcal{S}ing^{\bullet}(0)\right)^{\times -1} \times \mathcal{S}ing^{\bullet}(z)$.

Characterization of the set of solutions

Proposition: (B. 2013)

Any familly of polynomials which are solution of the previous system comes from a formal mould $\mathcal{B}eeg^{\bullet}$ valued in $\mathbb{C}[z]$ such that there exists a formal mould $beeg^{\bullet}$ valued in $\mathbb{C}$ satisfying:

1. $beeg^{X} = \dfrac{1}{e^X - 1} - \dfrac{1}{X}$
2. $beeg^{\bullet}$ is symmetr**E**l
3. $\mathcal{B}eeg^{\bullet}(z) = beeg^{\bullet} \times \mathcal{B}eeg_0^{\bullet}(z) = beeg^{\bullet} \times \big(\mathcal{S}ing^{\bullet}(0)\big)^{\times -1} \times \mathcal{S}ing^{\bullet}(z)$.

Theorem: (B., 2013)

The subgroup of symmetr**E**l formal moulds with vanishing coefficients in length 1, acts on the set of all possible multiple Bernoulli polynomials, *i.e.* on the set of all possible *algebraic* renormalization of multiple zeta values at negative integers.

Using the Reflexion Formula of Bernoulli polynomials

Notations: $M^{-\bullet} : (\omega_1, \cdots, \omega_r) \longmapsto M^{-\omega_1, \cdots, -\omega_r}$.

$$M^{\overleftarrow{\bullet}} : (\omega_1, \cdots, \omega_r) \longmapsto M^{\omega_r, \cdots, \omega_1} .$$

Lemma: (B., 2013)

Let $\mathcal{S}g^\bullet$ be the mould defined by: $\mathcal{S}g^{X_1, \cdots, X_r} = (-1)^r$.
For all $z \in \mathbb{C}$, we have:

$$\mathcal{S}ing^{-\bullet}(0) = \left(\mathcal{S}ing^{\overleftarrow{\bullet}}(0)\right)^{\times -1} \times \mathcal{S}g^\bullet \quad , \quad \mathcal{S}ing^{-\bullet}(1-z) = \left(\mathcal{S}ing^{\overleftarrow{\bullet}}(z)\right)^{\times -1} .$$

Sketch of proof: Use recursively $\dfrac{1}{e^X - 1} + \dfrac{1}{e^{-X} - 1} + 1 = 0$. □

Examples:

$$\mathcal{S}ing^{-X,-Y}(0) = \mathcal{S}ing^{X,Y}(0) + \mathcal{S}ing^{X+Y}(0) + \mathcal{S}ing^{X}(0) + 1 .$$

$$\mathcal{S}ing^{-X,-Y}(1-z) = \mathcal{S}ing^{X,Y}(z) + \mathcal{S}ing^{X+Y}(z) .$$

Reflexion Formula for Multiple Bernoulli Polynomials

Theorem: (B., 2013)

For all $z \in \mathbb{C}$, we have:

$$\mathcal{B}eeg^{-\bullet}(1-z) \times \mathcal{B}eeg^{\overleftarrow{\bullet}}(z) = beeg^{-\bullet} \times (1^{\bullet} + l^{\bullet}) \times beeg^{\overleftarrow{\bullet}} \ ,$$

where $1^{x_1,\cdots,x_r} = \begin{cases} 1 & \text{if } r = 0 \ . \\ 0 & \text{if } r > 0 \ . \end{cases}$ and $l^{x_1,\cdots,x_r} = \begin{cases} 1 & \text{if } r = 1 \ . \\ 0 & \text{if } r \neq 1 \ . \end{cases}$

Heuristic:

$$\mathcal{S}ing^{-\bullet}(1-z) \times \mathcal{S}ing^{\overleftarrow{\bullet}}(z) = 1^{\bullet}$$

Consequently, we will define $beeg^{\bullet}$ such that:

$$beeg^{-\bullet} \times (1^{\bullet} + l^{\bullet}) \times beeg^{\overleftarrow{\bullet}} = 1^{\bullet}$$

Consequently, we will define $beeg^{\bullet}$ such that:

$$beeg^{-\bullet} \times (1^{\bullet} + I^{\bullet}) \times beeg^{\overleftarrow{\bullet}} = 1^{\bullet}$$

$$\iff \quad beeg^{-\bullet} \times (\mathcal{S}g^{\bullet})^{\times -1} \times beeg^{\overleftarrow{\bullet}} = 1^{\bullet}$$

$$\iff \quad ceeg^{-\bullet} \times ceeg^{\overleftarrow{\bullet}} = 1^{\bullet} \text{ , where } ceeg^{\bullet} = beeg^{\bullet} \times (\mathcal{S}g^{\bullet})^{\times -\frac{1}{2}}$$

$$\iff \quad Exp(deg^{-\bullet}) \times Exp(deg^{\overleftarrow{\bullet}}) = 1^{\bullet} \text{ , where } ceeg^{\bullet} = Exp(deg^{\bullet})$$

$$\iff \quad deg^{-\bullet} + deg^{\overleftarrow{\bullet}} = O^{\bullet} \ .$$

Definition of Multiple Bernoulli Polynomials and Numbers

The mould $deg^{\bullet}$ defined by

$$\begin{cases} deg^{\emptyset} := 0 \ , \\ deg^{x_1, \cdots, x_r} := \dfrac{(-1)^{r-1}}{r} deg^{x_1 + \cdots + x_r} \ , \end{cases}$$

where $deg^{X} = \dfrac{1}{e^{X}-1} - \dfrac{1}{X} + \dfrac{1}{2}$, satifies the required conditions:

$$\begin{cases} deg^{\bullet} + deg^{\overleftarrow{-\bullet}} = 0 \ , \\ deg^{\bullet} \text{ is altern}\underline{\mathbf{E}}\text{l} \ . \end{cases}$$

Definition: (B., 2014)

The moulds $\mathcal{B}eeg^{\bullet}(z)$ and $beeg^{\bullet}$ are defined by:

$$\mathcal{B}eeg^{\bullet}(z) = Exp(deg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \times (\mathcal{S}ing^{\bullet}(0))^{\times -1} \times \mathcal{S}ing^{\bullet}(z) \ .$$

$$beeg^{\bullet} = Exp(deg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}}$$

The coefficients $\mathcal{B}e^{\bullet}(z)$ and $\mathcal{B}e^{\bullet}$ of these exponential generating series are called respectively the **Multiple Bernoulli Polynomials and Numbers**.

Examples of explicit expression for multiple Bernoulli numbers

Consequently, we obtain explicit expressions like, for $n_1, n_2, n_3 > 0$:

$$be^{n_1,n_2} = \frac{1}{2}\left(\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1} - \frac{b_{n_1+n_2+1}}{n_1+n_2+1}\right).$$

$$\begin{aligned} be^{n_1,n_2,n_3} &= +\frac{1}{6}\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1}\frac{b_{n_3+1}}{n_3+1} \\ &\quad -\frac{1}{4}\left(\frac{b_{n_1+n_2+1}}{n_1+n_2+1}\frac{b_{n_3+1}}{n_3+1} + \frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+n_3+1}}{n_2+n_3+1}\right) \\ &\quad +\frac{1}{3}\frac{b_{n_1+n_2+n_3+1}}{n_1+n_2+n_3+1}. \end{aligned}$$

Remark: If $n_1 = 0$, $n_2 = 0$ or $n_3 = 0$, the expressions are not so simple...

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is “constant”.
- “symmetry” relatively to $p = q$.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.

Conclusion

1. We have respectively defined the Multiple Bernoulli Polynomials and Multiple Bernoulli Numbers by:

$$\begin{cases} \mathcal{B}eeg^{\bullet}(z) & = & Exp(\lceil eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \times \left(\mathcal{S}ing^{\bullet}(0)\right)^{\times -1} \times \mathcal{S}ing^{\bullet}(z) \ . \\ beeg^{\bullet} & = & Exp(\lceil eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \end{cases}$$

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:
   - the nullity of b_{2n+1} if $n > 0$:

$$b^{2p_1, \cdots, 2p_r} = 0 \ .$$

   - the symmetry $B_n(1) = B_n(0)$ if $n > 1$:

$$\text{if } n_r > 0, \ \ B^{n_1, \cdots, n_r}(0) = B^{n_1, \cdots, n_r}(1) \ .$$

   - the difference equation $\Delta(B_n)(x) = nx^{n-1}$:

$$B^{n_1, \cdots, n_r}(z+1) - B^{n_1, \cdots, n_r}(z) = B^{n_1, \cdots, n_{r-1}}(z) \cdot z^{n_r} \ .$$

   - the reflection formula $(-1)^n B_n(1-x) = B_n(x)$:

$$\mathcal{B}eeg^{-\bullet}(1-z) \cdot \mathcal{B}eeg^{\overleftarrow{\bullet}}(z) = 1^{\bullet} \ .$$

THANK YOU FOR YOUR ATTENTION !