Multitangent Functions, Multizeta Values and Holomorphic Dynamics

Renormalization at the confluence of analysis, algebra and geometry

Olivier Bouillot,
Paris XI University.

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Outline

During all the talk, $\text{seq}(E)$ will denote the set of all the sequences (or equivalently words) constructed over E .

This is often written by $E^\star$, but, for technical reasons, we prefer use $\text{seq}(E)$.

Also, we will systematically use the notation $\mathbb{N}_k = \mathbb{N} - [\![0\,;\, k-1]\!]$, where $k \in \mathbb{N}$.

1. Introduction to Multitangent Functions: Definition

Definition:

Let $\mathcal{S}^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ ; \ s_1 \geq 2 \text{ et } s_r \geq 2\}$.
For all sequence $\underline{\mathbf{s}} \in \mathcal{S}^\star$, we consider:

$$\begin{array}{rccl} \mathcal{T}e^{s_1, \cdots, s_r} : & \mathbb{C} - \mathbb{Z} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \displaystyle\sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{1}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}} \ . \end{array}$$

Remarks :

1. Multitangent functions are a generalization of Eisenstein series $(r = 1)$.

2. Multitangent functions appear naturally in problems of holomorphic dynamics.

Property :

1. **Differential property.**

   Let $\underline{\mathbf{s}} = (s_1; \cdots s_r) \in \mathcal{S}^\star$.

   The function $\mathcal{T}e^{\underline{s}}$ is holomorphic on $\mathbb{C} - \mathbb{Z}$; it is a uniformly convergent series on any compact subset of $\mathbb{C} - \mathbb{Z}$ and satisfies:

$$\frac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z} = -\sum_{i=1}^{r} s_i \mathcal{T}e^{s_1, \cdots, s_{i-1}, s_i+1, s_{i+1}, \cdots, s_r} .$$

2. **Parity property.**

$$\forall z \in \mathbb{C} - \mathbb{Z} \ , \ \forall \underline{\mathbf{s}} \in \mathcal{S}^\star \ , \ \mathcal{T}e^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||} \mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z) .$$

3. **Symmetrelity.**

   $\mathcal{T}e^\bullet$ is symm<u>e</u>trel:

$$\forall (\underline{\mathbf{u}} ; \underline{\mathbf{v}}) \in (\mathcal{S}^\star)^2 \ , \ \exists E(\underline{\mathbf{u}} ; \underline{\mathbf{v}}) \subset \mathcal{S}^\star \text{ fini} \ , \ \mathcal{T}e^{\underline{\mathbf{u}}} \mathcal{T}e^{v} = \sum_{\underline{\mathbf{w}} \in E(\underline{\mathbf{u}} ; \underline{\mathbf{v}})} \mathcal{T}e^{\underline{\mathbf{w}}} .$$

1. Introduction to Multitangent Functions: Reduction into Monotangent Functions, First Version

Remark : A monotangent function is a multitangent function with length 1 .

Let: $$\mathcal{MZV} = \text{Vect}_{\mathbb{Q}} \left(\mathcal{Z}e^{\underline{\mathbf{s}}} \right)_{\substack{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \\ s_1 \geq 2}} \; .$$

$$m(\underline{\mathbf{s}}) = \max(s_1 \, ; \cdots ; s_r), \text{ for all } \underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \; .$$

Property: Reduction of Convergent Multitangent Functions into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathcal{S}^\star \;,\; \exists (z_2 \, ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{MZV}^{m(\underline{\mathbf{s}}-1)} \;,\; \mathcal{T}e^{\underline{\mathbf{s}}} = \sum_{\substack{\cancel{k=1} \\ k=2}}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k \; .$$

1. Introduction to Multitangent Functions: Examples of Reduction into Monotangent Functions

Weight 4

$$\mathcal{T}e^{2,2} = 2\zeta(2)\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,3} = -3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,2} = 3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,2} = 0 .$$

Weight 5

Weight 6

$$\mathcal{T}e^{2,4} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} - 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

$$\mathcal{T}e^{3,3} = -\frac{12}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{4,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} + 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} .$$

$$\mathcal{T}e^{2,2,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

$$\mathcal{T}e^{2,1,3} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} + \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{3,1,2} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} - \zeta(3)\mathcal{T}e^{3} .$$

$$\mathcal{T}e^{2,1,1,2} = \frac{4}{5}\zeta(2)^2\mathcal{T}e^{2} .$$

1. Introduction to Multitangent Functions: Projection on Multitangent Function Space

There is a kind of reverse of multitangent reduction:

Conjecture: Projection on Multitangent Function Space

Let: $\mathcal{M}ZV = Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\substack{\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) \\ s_1 \geq 2}}$ and $\mathcal{M}TGF = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}^\star}$.

So, $\mathcal{M}TGF$ is a $\mathcal{M}ZV$-module.

Examples:

$$\mathcal{Z}e^{2}\mathcal{T}e^{2} = \frac{1}{2}\mathcal{T}e^{2,2} \ .$$

$$\mathcal{Z}e^{3,2}\mathcal{T}e^{2} = \frac{1}{4}\mathcal{T}e^{3,2,2} - \frac{1}{4}\mathcal{T}e^{2,2,3} + \frac{7}{120}\mathcal{T}e^{5,2} + \frac{7}{60}\mathcal{T}e^{4,3} - \frac{7}{60}\mathcal{T}e^{3,4} - \frac{7}{120}\mathcal{T}e^{2,5} \ .$$

Let $\mathcal{T} = \left\{ f \in \mathbb{C}\{x\} \; ; \; f(x) = x + \mathcal{O}(x^2) \right\}$.

But: Describe conjugacy class of $\mathcal{T}$.

- At infinity, each tangent to identity difféomorphism is formally conjugated to: $z \longmapsto z + z^{1-p} - \rho z^{1-2p}$, where $p \in \mathbb{N}^*$ and $\rho \in \mathbb{C}$.

  Here, p and ρ are the two formals invariants.

- We are interesseted by the typical conjugacy class : $(p\,;\rho) = (1\,;0)$, i.e.

$$f(z) = z + 1 + \mathcal{O}\left(\frac{1}{z^2}\right) \; .$$

  We also let $I : z \longmapsto z + 1$.

2. A Few Remember about Analytic Invariants: Definition

- $\exists f^* \in z + \mathbb{C}\left[\left[\frac{1}{z}\right]\right]$, $f^* \circ f = l \circ f^*$.

  $\exists {}^*f \in z + \mathbb{C}\left[\left[\frac{1}{z}\right]\right]$, $f \circ {}^*f = {}^*f \circ l$.

- $\pi_f^+ = f_+^* \circ {}^*f_-$ commutes with l and is invariant by conjugation.

Definition:

The analytic invariants of f, denoted by $(A_{2in\pi}^+(f))_{n \in \mathbb{Z}^*}$, are the Fourier's coefficients of $\pi_f^+ - id_{\mathbb{C}}$.

Theorem: (J. Ecalle)

Two tangent to identity diffeomorphims are conjugated if and only if they have the same analytic invariants.

Problematic : Understand how these invariants are constructed.

2. A Few Remember about Analytic Invariants: Universal formula

Each invariant $A^{+}_{2in\pi}(f)$ can be develloped like an entire function of f, i.e. the expansion depends of the Taylor's coefficients of f:

Theorem:

Let f be a convergent tangent to identity diffeomorphism, convergent, expressed at infinity like $f(z) = z + 1 + \sum_{n \geq 3} \frac{a_n}{z^{n-1}}$.

Then,

1. there exists some explicit coefficients $\tau^{\bullet}$, defined on $\mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\})$, valued in the algebra $\mathcal{M}TGF$, such that:

$$\pi_f^{+} = \sum_{\underline{\mathbf{S}} \in \mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\})} \tau^{\underline{\mathbf{S}}} \mathcal{A}_{\underline{\mathbf{S}}} .$$

2. for all $n \in \mathbb{Z}^*$, there exists some explicit coefficients $\widehat{\tau}_n^{\bullet}$, defined on $\mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\})$, valued in the algebra $\mathcal{M}ZV$, such that:

$$\forall n \in \mathbb{Z}^*, \; A_n^{+}(f) = \sum_{\underline{\mathbf{S}} \in \mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\})} \widehat{\tau}_n^{\underline{\mathbf{S}}} \mathcal{A}_{\underline{\mathbf{S}}} .$$

2. A Few Remember about Analytic Invariants: Example of an analytic invariant computation

- We choose a test function with a parameter: $f = l \circ g$, where :

$$l(z) = z + 1 \,.$$

$$g(z) = \left(\exp(\alpha z^{-2}\partial_z)\right) \cdot z = z\,(1 + 3\,\alpha z^{-3})^{1/3} \,.$$

Then: $f(z) = z + 1 + \dfrac{\alpha}{z^2} + \mathcal{O}(z^{-3})$.

- π_f^+ is expressed in term of multitangent functions by:

$$\begin{aligned}
\pi_f^+ = &+\alpha \cdot \mathcal{T}e^{2} \\
&-\alpha^2 \cdot \left(\mathcal{T}e^{3,2} - \mathcal{T}e^{2,3}\right) \\
&+\alpha^3 \cdot \Bigg(\frac{2}{3}\mathcal{T}e^{3,3,2} - \frac{4}{3}\mathcal{T}e^{3,2,3} + \frac{2}{3}\mathcal{T}e^{2,3,3} - \frac{1}{3}\mathcal{T}e^{5,3} - \frac{1}{3}\mathcal{T}e^{3,5} \\
&\qquad -\frac{1}{6}\mathcal{T}e^{6,2} - \frac{1}{6}\mathcal{T}e^{2,6} + \mathcal{T}e^{4,2,2} - 2\mathcal{T}e^{2,4,2} + \mathcal{T}e^{2,2,4} + \mathcal{T}e^{4,4}\Bigg) \\
&+\mathcal{O}(\alpha^4)
\end{aligned}$$

3. Renormalization of Multitangent Functions: the First Property

Property: Multitangent Functions Renormalization.

There exists an extension of multitangent functions to $\text{seq}(\mathbb{N}^*)$ such that:

1. $\mathcal{T}e^{\bullet}$ is always symmetr<u>**e**</u>l.
2. $\forall z \in \mathbb{C} - \mathbb{Z}$, $\mathcal{T}e^1(z) = \dfrac{\pi}{\tan(\pi z)}$.

This extension automatically satisfies: the differential property.
the parity property.

<u>Example:</u>

$$\begin{aligned}
\mathcal{T}e^{2,1}(z) &= \mathcal{H}e_+^{2,1}(z) + \mathcal{H}e_+^{2}(z)\Big(\frac{1}{z} + \underbrace{\mathcal{H}e_-^{1}(z)}_{\text{divergent term}}\Big) + \frac{1}{z^2}\underbrace{\mathcal{H}e_-^{1}(z)}_{\text{divergent term}} + \underbrace{\mathcal{H}e_-^{2,1}(z)}_{\text{divergent term}} \\
&= \mathcal{H}e_+^{2,1}(z) + \mathcal{H}e_+^{2}(z)\Big(\frac{1}{z} + \underbrace{\mathcal{H}e_-^{1}(z)}_{\text{divergent term}}\Big) + \frac{1}{z^2}\underbrace{\mathcal{H}e_-^{1}(z)}_{\text{divergent term}} \\
&\quad + \mathcal{H}e_-^{2}(z)\underbrace{\mathcal{H}e_-^{1}(z)}_{\text{term divergent}} - \underbrace{\mathcal{H}e_-^{1,2}(z)}_{\text{convergent term}} - \mathcal{H}e_-^{3}(z) \ .
\end{aligned}$$

3. Renormalization of Multitangent Functions: Generating Functions $\mathcal{T}ig^{\bullet}$

Theorem: The Generating Functions $\mathcal{T}ig^{\bullet}$.

Let:

$$\begin{cases} Qig^{\emptyset}(z) = 0 \ . \\ Qig^{\binom{\varepsilon_1}{v_1}}(z) = -\mathcal{T}e^{\binom{\varepsilon_1}{1}}(v_1 - z) \ . \\ Qig^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}}(z) = 0 \ , \text{ si } r \geq 2 \ . \end{cases}$$

$$\begin{cases} \delta^{\emptyset} = 0 \ . \\ \delta^{\binom{\varepsilon_1, \cdots, \varepsilon_r}{v_1, \cdots, v_r}} = \begin{cases} \dfrac{(i\pi)^r}{r!} \mathbb{1}_{\{0\}}(\varepsilon_1) \cdots \mathbb{1}_{\{0\}}(\varepsilon_r) & \text{, if } r \text{ is even.} \\ 0 & \text{, if } r \text{ is odd.} \end{cases} \end{cases}$$

Then:

$$\boxed{\mathcal{T}ig^{\bullet}(z) = \delta^{\bullet} + \mathcal{Z}ig^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(z) \times \mathcal{Z}ig_{-}^{\lfloor\bullet} \ .}$$

3. Renormalization of Multitangent Functions: Reduction into Monotangent Functions, second version

Property: Reduction into Monotangent Functions.

$$\forall \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) \ , \ \exists (z_1 ; \cdots ; z_{m(\underline{\mathbf{s}})}) \in \mathcal{MZV}^{m(\underline{\mathbf{s}})} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = \delta^{\underline{\mathbf{s}}} + \sum_{k=1}^{m(\underline{\mathbf{s}})} z_k \mathcal{T}e^k \ ,$$

where $\delta^{\underline{\mathbf{s}}} = \begin{cases} \dfrac{(i\pi)^r}{r!} & \text{, if } \underline{\mathbf{s}} = 1^{[r]} \text{ and if } r \text{ is even.} \\ 0 & \text{, else.} \end{cases}$

Important remark: $z_1 = 0 \iff \underline{\mathbf{s}} \neq 1^{[r]}$ or $\begin{cases} \underline{\mathbf{s}} = 1^{[r]} \ . \\ r \text{ is even} \ . \end{cases}$

3. Renormalization of Multitangent Functions: Examples of Reduction into Monotangent Functions

Weight 2

$$\mathcal{T}e^{1,1} = -3\,\zeta(2)\ .$$

Weight 3

$$\mathcal{T}e^{1,2} = 0\ .$$

$$\mathcal{T}e^{2,1} = 0\ .$$

$$\mathcal{T}e^{1,1,1} = -\zeta(2)\mathcal{T}e^{1}\ .$$

Weight 4

$$\mathcal{T}e^{1,3} = -\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{3,1} = -\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,1,2} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,2,1} = 0\ .$$

$$\mathcal{T}e^{2,1,1} = -\frac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$$

$$\mathcal{T}e^{1,1,1,1} = \frac{3}{2}\zeta(2)^{2}\ .$$

Recall the important remark: $z_1 = 0 \iff \underline{\mathbf{s}} \neq 1^{[r]}$ or $\begin{cases} \underline{\mathbf{s}} = 1^{[r]} \ . \\ r \text{ is even} \ . \end{cases}$

We have:

$$\begin{aligned} \mathcal{T}e^{1,1,2} &= \left(-2\mathcal{Z}e^{2,1} + \mathcal{Z}e^{2}\mathcal{Z}e^{1} - \mathcal{Z}e^{1,2}\right)\mathcal{T}e^{1} + \mathcal{Z}e^{1,1}\mathcal{T}e^{2} \\ &= \left(-2\mathcal{Z}e^{2,1} + \mathcal{Z}e^{2,1} + \mathcal{Z}e^{3}\right)\mathcal{T}e^{1} + \frac{1}{2}\left(\left(\mathcal{Z}e^{1}\right)^{2} - \mathcal{Z}e^{2}\right)\mathcal{T}e^{2} \\ &= \left(\mathcal{Z}e^{3} - \mathcal{Z}e^{2,1}\right)\mathcal{T}e^{1} - \frac{1}{2}\mathcal{Z}e^{2}\mathcal{T}e^{2} \ . \end{aligned}$$

So, by the cancellation of the $\mathcal{T}e^{1}$ term, we obtain:

$$\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3} \ .$$

Conclusion

1. Multitangent functions naturally appears from analysis questions to go immediatly to algebra.

2. Multitangent functions seem to be an interesting functional model for the study of multizetas values.

3. Using divergent multitangent functions gives us more informations than using convergent multitangent functions.