An Embodiment of Quasi-Symmetric Functions: the Hurwitz Multizeta Functions.

Olivier Bouillot,
Marnes la Vallée University, France

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Introduction.

Definition: Hurwitz multizeta functions.

Let $\mathcal{S}_+^\star = \{\underline{\mathbf{s}} = (s_1, \cdots, s_r) \in \mathbb{N}_1^\star \,; r \in \mathbb{N} \text{ and } s_1 \geq 2\}$.

For all $\underline{\mathbf{s}} \in \mathcal{S}_+^\star$:

$$\mathcal{H}e_+^{s_1, \cdots, s_r}(z) = \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ .$$

Aim: Studying the algebra $\mathcal{H}MZF_+ = \text{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\mathcal{S}_+^\star}$.

Theorem:

1. The family $\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\mathcal{S}_+^\star}$ is $\mathbb{C}(z)$-free.
2. $\mathcal{H}MZF_+ \simeq \mathbb{Q}\left\langle \mathcal{L}(y_1; y_2; \cdots) - \{y_1\} \right\rangle$.

Outline

1 A Fundamental Property

2 A Lemma of Rational Fractions and 1-Periodic Functions

3 Proof of the Key Point

1. A Fundamental Property: Definitions and Notations

- **Hurwitz multizeta functions.**

$$\mathcal{H}e_+^{s_1,\cdots,s_r} : z \longmapsto \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \text{ , for all } (s_1,\cdots,s_r) \in \mathcal{S}_+^\star \ .$$

$$\mathcal{H}e_+^{1} : z \longmapsto \sum_{n_1>0} \left(\frac{1}{n_1+z} - \frac{1}{n_1} \right) \ .$$

- **The shift operator.**

$$\tau^{-1}(f)(z) = f(z-1) \ .$$

- **A difference operator.**

$$\Delta_-(f)(z) = f(z-1) - f(z) \ .$$

- **Some auxiliary functions.** For all $(s_1,\cdots,s_r) \in \mathbb{N}_1^\star$:

$$J^{s_1,\cdots,s_r}(z) = \begin{cases} \dfrac{1}{z^{s_1}} & \text{, if } r = 1 \ . \\ 0 & \text{, otherwise} \ . \end{cases}$$

1. A Fundamental Property - Statement and proof

Fundamental Property:

For all sequences $\underline{\mathbf{s}} = (s_1, \cdots, s_r) \in \mathcal{S}_+^\star$, we have:

$$\Delta_-(\mathcal{H}e_+^{s_1,\cdots,s_r}) = \mathcal{H}e_+^{s_1,\cdots,s_{r-1}} \cdot J^{s_r} \ .$$

Sketch of proof:

$$\begin{aligned}
\mathcal{H}e_+^{\underline{\mathbf{s}}}(z-1) &= \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z-1)^{s_1}\cdots(n_r+z-1)^{s_r}} \\
&= \sum_{-1<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \\
&= \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} + \sum_{0=n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \\
&= \mathcal{H}e_+^{\underline{\mathbf{s}}}(z) + \mathcal{H}e_+^{s_1,\cdots,s_{r-1}}(z) \cdot \frac{1}{z^{s_r}} \ .
\end{aligned}$$

□

1. A Fundamental Property - Statements of corollaries

Corollary A:

Each Hurwitz multizeta function is a resurgent function.

Corollary B: the key point.

The family $\left(\mathcal{H}e_{+}^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}}\in\mathcal{S}_{+}^{\star}}$ is $\mathbb{C}(z)$-free.

Both of these are not trivial consequences...

$\Longrightarrow$ For the safety of the audience, we will focus ourself only on the second one.

2. A lemma of rational fractions and 1-periodic functions

Lemma:

Let F be a rational fraction and f a 1-periodic function.

If, for an n-tuple $(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, the equality $F + \sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i = f$ is a valid one, then we necessarily have:

$$\begin{cases} \lambda_1 = \cdots = \lambda_n = 0 \ . \\ F \text{ and } f \text{ are constant functions} \ . \end{cases}$$

Sketch of proof:

- $$\begin{aligned} \sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i \neq 0 \quad & \Longrightarrow \quad \exists N \in \mathbb{N} \ , \ +N \, , \, -N \ \notin \text{poles}(F) \\ & \Longrightarrow \quad -N \in \text{poles}(f) \qquad \Longrightarrow \mathbb{Z} \subset \text{poles}(f) \\ & \Longrightarrow \quad +N \in \text{poles}(f) \qquad \Longrightarrow +N \in \text{poles}(F) \\ & \Longrightarrow \quad \underline{\text{Contradiction}}. \end{aligned}$$
- $$\begin{cases} \sum_{i=1}^{n} \lambda_i \mathcal{H}e_+^i = 0 \Longrightarrow \lambda_1 = \cdots = \lambda_n = 0 \ . \\ F = f = \text{constant functions.} \end{cases}$$

□

3. Proof of the Key Point: Notations

- **Definition of an order in $\mathcal{S}_+^\star$:**

  Definition: $d^\circ(s_1, \cdots, s_r) = s_1 + \cdots + s_r - r$.

  Notation: For all $d \in \mathbb{N}$:

$$\begin{cases} \mathcal{S}_{\leq d}^\star & = & \{\underline{\mathbf{s}} \in \mathcal{S}^\star \; ; \; d^\circ \underline{\mathbf{s}} \leq d\} \\ \mathcal{S}_d^\star & = & \{\underline{\mathbf{s}} \in \mathcal{S}^\star \; ; \; d^\circ \underline{\mathbf{s}} = d\} \end{cases}$$

  Order $<$: degree 1 $<$ degree 2 $<$ degree 3 $< \cdots$.

  For all $d \in \mathbb{N}^*$, inside $\mathcal{S}_d^\star$: length 1 $<$ length 2 $<$ length 3 $< \cdots$.

  Notation: $\mathcal{S}_{d+1}^\star = \{\underline{\mathbf{s}}^n \; ; \; n \in \mathbb{N}^*\}$.

  For all $n \in \mathbb{N}$, $S_n = \{\underline{\mathbf{s}}^i \; ; \; 1 \leq i \leq n\}$.

- **Double induction process:**

$$\mathcal{P}_{d,n} : \text{“the family } \left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}_{\leq d}^\star} \bigcup \left(\mathcal{H}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in S_n} \text{ is } \mathbb{C}(z)\text{-free.”}$$

- Let us suppose that:

$$F_2 \mathcal{H}e_+^2 + F_3 \mathcal{H}e_+^3 + F_{2,1} \mathcal{H}e_+^{2,1} + F_4 \mathcal{H}e_+^4 = G$$
$$\Downarrow$$
$$F_2 = F_3 = F_{2,1} = F_4 = G = 0 \; .$$

- Let us consider the relation:

$$F_2 \mathcal{H}e_+^2 + F_3 \mathcal{H}e_+^3 + F_{2,1} \mathcal{H}e_+^{2,1} + F_4 \mathcal{H}e_+^4 + F_{3,1} \mathcal{H}e_+^{3,1} = G \; ,$$

where F_2 , F_3 , $F_{2,1}$, F_4 , $F_{3,1}$ and G are rational fractions.

Aim: $F_2 = F_3 = F_{2,1} = F_4 = F_{3,1} = G = 0$.

Main idea: Proving by contradiction that $F_{3,1} = 0$.

Remark: If $F_{3,1} \neq 0$, we can suppose that $F_{3,1} = 1$.

Reminder:

$$F_2\mathcal{H}e_+^2 + F_3\mathcal{H}e_+^3 + F_{2,1}\mathcal{H}e_+^{2,1} + F_4\mathcal{H}e_+^4 + \mathcal{H}e_+^{3,1} = G \ ,$$

- Applying Δ_- to this equality:

$$\begin{aligned} &\left(\Delta_-(F_2)\mathcal{H}e_+^2 + \tau^{-1}(F_2)\cdot J^2\right) &&+ \left(\Delta_-(F_3)\mathcal{H}e_+^3 + \tau^{-1}(F_3)\cdot J^3\right) \\ + &\left(\Delta_-(F_{2,1})\mathcal{H}e_+^{2,1} + \tau^{-1}(F_{2,1})\mathcal{H}e_+^2\cdot J^1\right) &&+ \left(\Delta_-(F_4)\mathcal{H}e_+^4 + \tau^{-1}(F_4)\cdot J^4\right) \\ + &\ \mathcal{H}e_+^3\cdot J^1 = \Delta_-(G)\ . \end{aligned}$$

- Using the hypothesis:

$$\begin{cases} \Delta_-(F_2) + \tau^{-1}(F_{2,1})\cdot J^1 = 0\ . \\ \Delta_-(F_3) + J^1 = 0\ . \\ \Delta_-(F_{2,1}) = 0\ . \\ \Delta_-(F_4) = 0\ . \\ \Delta_-(G) = \sum_{k=2}^{4} \tau^{-1}(F_k)\cdot J^k\ . \end{cases}$$

Reminder:

3. Proof of the Key Point: Example to picture the proof 3/3

- Contradiction:

$$\begin{aligned} \Delta_-(F_3) + J^1 = 0 \quad &\Longrightarrow \quad \Delta_-\left(F_3 + \mathcal{H}e_+^1\right) = 0 \\ &\Longrightarrow \quad F_3 + \mathcal{H}e_+^1 = 1 - \text{periodic function} \\ &\Longrightarrow \quad 1 = 0, \textit{ because of the last lemma !!!} \end{aligned}$$

- So, $F_{3,1} = 0$.

  The hypothesis implies now:

$$F_2 = F_3 = F_{2,1} = F_4 = F_{3,1} = G = 0 \; .$$

□

Reminder: $\mathcal{S}^{\star}_{d+1} = \{\underline{\mathbf{s}}^{n} \ ; \ n \in \mathbb{N}^*\}$.

For all $n \in \mathbb{N}$, $S_n = \{\underline{\mathbf{s}}^{i} \ ; \ 1 \leq i \leq n\}$.

- Let us consider the following properties:

$\mathcal{D}_d$: "the family $(\mathcal{H}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{\leq d}}$ is $\mathbb{C}(z)$-free."

$\mathcal{P}_{d,n}$: "the family $(\mathcal{H}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{\leq d}} \bigcup (\mathcal{H}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in S_n}$ is $\mathbb{C}(z)$-free."

- **Aim:** $\forall d \in \mathbb{N}\,,\ \mathcal{P}_{d,n} \Longrightarrow \mathcal{P}_{d,n+1}$,

because:

$$\begin{cases} \mathcal{D}_0 \text{ is true.} \\ \mathcal{D}_{d+1} \text{ is true} \iff \forall n \in \mathbb{N}, \mathcal{P}_{d,n} \text{ is true.} \end{cases}$$

Let us consider:

$$\sum_{\substack{\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{\leq d} \cup S_{n+1} \\ \underline{\mathbf{s}} \neq \emptyset}} F_{\underline{\mathbf{s}}} \cdot \mathcal{H}e^{\underline{\mathbf{s}}} = F \ ,$$

where F and $F_{\underline{\mathbf{s}}}$, $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{\leq d} \cup S_{n+1}$, are rational fractions.

Aim: Proving that $F_{\underline{\mathbf{s}}^{n+1}} = 0$, by contradiction.

Let us denote $\underline{\mathbf{s}}^{n+1} = \underline{\mathbf{u}} \cdot p$, where $\begin{cases} p \geq 1 \ . \\ \underline{\mathbf{u}} \in \mathcal{S}^{\star}_{\leq d+2-p} \ . \end{cases}$

If $F_{\underline{\mathbf{s}}^{n+1}} \neq 0$, we can suppose $F_{\underline{\mathbf{s}}^{n+1}} = 1$.

- **Step 1: writing the system.**

$$\begin{cases} \forall \underline{\mathbf{s}} \in \left(\mathcal{S}^{\star}_{\leq d} \cup S_n\right) - \{\emptyset\} \ , \ \Delta_{-}(F_{\underline{\mathbf{s}}}) + \displaystyle\sum_{\substack{k \in \mathbb{N}^* \\ \underline{\mathbf{s}} \cdot k \in \mathcal{S}^{\star}_{\leq d} \cup S_n}} \tau^{-1}(F_{\underline{\mathbf{s}} \cdot k}) \cdot J^k = 0 \ . \\ \Delta_{-}(F) = \displaystyle\sum_{k=2}^{d+1} \tau^{-1}(F_k) J^k + (1 - \delta_{n,0}) \tau^{-1}(F_{d+2}) J^{d+2} \ . \end{cases}$$

- **Step 2: solving partially the system.**

Lemma:

Let r be a positive integer and $p \geq 2$.
Let us also consider two r-tuples, $(n_1, \cdots, n_r) \in \mathbb{N}^r$ and $(k_1, \cdots, k_r) \in (\mathbb{N}_{\geq 2})^r$ such that $\sum_{i=1}^{r}(k_i - 1) \leq p - 2$.

Then, $F_{\underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdots k_r \cdot 1^{[n_r]}} = \begin{cases} 0 & \text{, if } n_r > 0 \ . \\ \text{cste} & \text{, if } n_r = 0 \ . \end{cases}$

Sketch of proof:

Let $\delta = p - 2 - \sum_{i=1}^{r}(k_i - 1)$; let us suppose here that $\delta = 0$!!!!

Applied to $\underline{\mathbf{v}} \cdot 1^{[n]}$, $n \in \mathbb{N}$, where $\underline{\mathbf{v}} = \underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdots k_r$, the system gives us:

$$\forall n \in \mathbb{N} \ , \ \Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n]}}\right) + \tau^{-1}\left(F_{\underline{\mathbf{v}} \cdot 1^{[n+1]}}\right) J^1 = 0 \ .$$

3. Proof of the Key Point: Proof 4/5

$$\forall n \in \mathbb{N} \ , \ \Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n]}} \right) + \tau^{-1} \left(F_{\underline{\mathbf{v}} \cdot 1^{[n+1]}} \right) J^1 = 0 \ . \qquad (1)$$

$$\begin{aligned}
\exists n_0 \in \mathbb{N} \, , \, F_{\underline{\mathbf{v}} \cdot 1^{[n_0]}} = 0 \quad & \Longrightarrow \quad \Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \right) = 0 \ , \ \text{by } (1) \\
& \Longrightarrow \quad F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \text{ is a 1-periodic function :} \\
& \Longrightarrow \quad F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \text{ is a constant function :} \\
& \qquad\quad F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} = f_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \in \mathbb{C} \\
& \Longrightarrow \quad \Delta_- \left(F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 2]}} + f_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} \mathcal{H}e^1 \right) = 0 \ , \ \text{by } (1) \\
& \Longrightarrow \quad F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 2]}} + f_{\underline{\mathbf{v}} \cdot 1^{[n - 1]}} \mathcal{H}e^1 \text{ is a 1-periodic function} \\
& \Longrightarrow \quad \begin{cases} F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} = f_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 1]}} = 0 \\ F_{\underline{\mathbf{v}} \cdot 1^{[n_0 - 2]}} \text{ is a constant function} \end{cases} \\
& \Longrightarrow \quad \text{and so on.}
\end{aligned}$$

Consequence:

$$F_{\underline{\mathbf{u}} \cdot k_1 \cdot 1^{[n_1]} \cdot \ldots \cdot k_r \cdot 1^{[n_r]}} = \begin{cases} 0 & , \text{ si } n_r > 0 \ . \\ \text{cste} & , \text{ si } n_r = 0 \ . \end{cases}$$

□

- **Step 3:** Highlighting of the contradiction.

We have:

$$\begin{cases} \Delta_-(F_{\underline{\mathbf{u}}}) + \sum_{k=1}^{p-1} \tau^{-1}(F_{\underline{\mathbf{u}}\cdot k}) J^k + J^p = 0 \ . \\ \forall k \in [\![1 \, ; \, p-1]\!] \, , \, F_{\underline{\mathbf{u}}\cdot p-1} = f_{\underline{\mathbf{u}}\cdot p-1} \in \mathbb{C} \ . \end{cases}$$

From

$$\Delta_- \left(F_{\underline{\mathbf{u}}} + \sum_{k=1}^{p-1} f_{\underline{\mathbf{u}}\cdot k} \mathcal{H}e^k + \mathcal{H}e^p \right) = 0 \ ,$$

we deduce:

$$F_{\underline{\mathbf{u}}} + \sum_{k=1}^{p-1} f_{\underline{\mathbf{u}}\cdot k} \mathcal{H}e^k + \mathbf{1} \cdot \mathcal{H}e^p \text{ defines a 1-periodic function.}$$

This contradicts the lemma. Thus, $F_{\underline{\mathbf{s}}^{n+1}} = 0$ and the induction hypothesis gives the conclusion. □

Conclusion

We have proved:

Theorem:

1. The family $\left(\mathcal{H}e_{+}^{\mathbf{s}}\right)_{\underline{\mathbf{s}}\in\mathcal{S}_{+}^{\star}}$ is $\mathbb{C}(z)$-free.
2. $\mathcal{H}MZF_{+} \simeq \mathbb{Q}\left\langle \mathcal{L}(y_1; y_2; \cdots) - \{y_1\} \right\rangle$.

Consequence:

The algebra spanned by the Hurwitz multizeta functions is an embodiment of *QSym*.

THANK YOU FOR YOUR ATTENTION !