

Multiple Bernoulli Polynomials and Numbers

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1 Reminders

- Reminders on Bernoulli Polynomials and Numbers
- Reminders on Riemann Zeta Function and Hurwitz Zeta Function

2 The Form of a Multiple Bernoulli Polynomial

3 The General Reflexion Formula of Multiple Bernoulli Polynomial

4 An Example of Multiple Bernoulli Polynomial

Two Equivalent Definitions of Bernoulli Polynomials / Numbers

Bernoulli numbers:

By a generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} b_n \frac{t^n}{n!} .$$

By a recursive formula:

$$\begin{cases} b_0 = 1 , \\ \forall n \in \mathbb{N} , \sum_{k=0}^n \binom{n+1}{k} b_k = 0 . \end{cases}$$

First examples:

$$b_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots$$

Bernoulli polynomials :

By a generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!} .$$

By a recursive formula:

$$\begin{cases} B_0(x) = 1 , \\ \forall n \in \mathbb{N} , B'_{n+1}(x) = (n+1)B_n(x) , \\ \int_0^1 B_n(x) dx = 0 . \end{cases}$$

First examples:

$$\begin{aligned} B_0(x) &= 1 , \\ B_1(x) &= x - \frac{1}{2} , \\ B_2(x) &= x^2 - x + \frac{1}{6} , \\ &\vdots \end{aligned}$$

Some Properties of Bernoulli Polynomials

Property 1: Difference Equation

$\Delta(B_n)(x) = nx^{n-1}$ for all $n \in \mathbb{N}^*$, where $\Delta(f)(z) = f(z+1) - f(z)$.

Property 2: Reflexion Formula

$(-1)^n B_n(1-x) = B_n(x)$ for all $n \in \mathbb{N}$.

Property 3: Expression in the falling factorial basis

$B_{n+1}(x) - b_{n+1} = \sum_{k=0}^n \frac{n+1}{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_{k+1}$ for all $n \in \mathbb{N}$, where:

- $(x)_n = x(x-1) \cdots (x-n+1)$.
- $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denotes the Stirling number of second kind.

On the Riemann Zeta Function and Hurwitz Zeta Function

Definition:

The Riemann Zeta Function and Hurwitz Zeta Function are defined, for $\Re s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s} \quad , \quad \zeta(s, z) = \sum_{n \geq 0} \frac{1}{(n+z)^s} \quad .$$

Property:

$s \mapsto \zeta(s)$ and $s \mapsto \zeta(s, z)$ can be analytically extend to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

Remark:

$$\begin{aligned} \zeta(-n) &= -\frac{b_{n+1}}{n+1} \text{ for all } n \in \mathbb{N} . \\ \zeta(-n, z) &= -\frac{B_{n+1}(z)}{n+1} \text{ for all } n \in \mathbb{N} . \end{aligned}$$

Multiple Zeta Values and Hurwitz Multiple Zeta Functions

Definition of Hurwitz Multiple Zeta Functions

$$\mathcal{H}e^{s_1, \dots, s_r}(z) = \sum_{0 < n_r < \dots < n_1} \frac{1}{(n_1 + z)^{s_1} \dots (n_r + z)^{s_r}}, \text{ if } z \in \mathbb{C} - \mathbb{N}_{\leq 0} \text{ and } (s_1, \dots, s_r) \in (\mathbb{N}^*)^r, \text{ such that } s_1 \geq 2.$$

Lemma:

For all sequences $(s_1, \dots, s_r)$, we have:

$$\Delta_-(\mathcal{H}e^{s_1, \dots, s_r}) = \mathcal{H}e^{s_1, \dots, s_{r-1}} \cdot J^{s_r},$$

where $J^s(z) = z^{-s}$ and $\Delta_-(f)(z) = f(z-1) - f(z)$.

Heuristic:

$$\mathcal{B}e^{s_1, \dots, s_r}(z) = \text{Multiple (Divided) Bernoulli Polynomials} = \mathcal{H}e^{-s_1, \dots, -s_r}(z).$$

$$be^{s_1, \dots, s_r} = \text{Multiple (Divided) Bernoulli Numbers} = \mathcal{H}e^{-s_1, \dots, -s_r}(0).$$

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We want to define $\mathcal{B}e^{s_1, \dots, s_r}(z)$ such that:

- their properties are similar to Hurwitz Multiple Zeta Functions' properties.
- their properties generalize these of Bernoulli polynomials.

Main Goal:

Find some polynomials $\mathcal{B}e^{s_1, \dots, s_r}$ such that:

$$\left\{ \begin{array}{l} \mathcal{B}e^s(z) = \frac{B_{s+1}(z)}{s+1}, \text{ where } s \geq 0 ; \\ \Delta(\mathcal{B}e^{s_1, \dots, s_r})(z) = \mathcal{B}e^{s_1, \dots, s_{r-1}}(z) \cdot z^{s_r}, \text{ where } r \geq 2, \text{ and } s_1, \dots, s_r \geq 0 ; \\ \text{the } \mathcal{B}e^{s_1, \dots, s_r} \text{ are multiplied by the stuffle product.} \end{array} \right.$$

A particular solution of the main goal

Lemma:

The family of difference equations

$$\Delta(\mathcal{B}e^{s_1, \dots, s_r})(z) = \mathcal{B}e^{s_1, \dots, s_{r-1}}(z) \cdot z^{s_r}$$

has a unique family of solutions $\mathcal{B}e_0^{s_1, \dots, s_r} \in z\mathbb{C}[z]$, which are multiplied by the stuffle.

Sketch of proof: $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$.



Examples:

$$\mathcal{B}eeg_0^X(z) = \sum_{p \geq 0} \mathcal{B}e_0^p(z) \frac{X^p}{p!} = \frac{e^{zX} - 1}{e^X - 1}.$$

$$\mathcal{B}eeg_0^{X,Y}(z) = \sum_{p,q \geq 0} \mathcal{B}e_0^{p,q}(z) \frac{X^p}{p!} \frac{Y^q}{q!} = \frac{\mathcal{B}eeg_0^{X+Y}(z) - \mathcal{B}eeg_0^Y(z)}{e^X - 1}.$$

The Form of a Multiple Bernoulli Polynomial

Notations: The previous family $\mathcal{B}e_0^{s_1, \dots, s_r}$ is denoted by $\mathcal{B}e_0^\bullet$.
In general, a family $M^{s_1, \dots, s_r}$ is denoted by $M^\bullet$ and called a **mould**.

Mould product: $M^\bullet = A^\bullet \times B^\bullet \iff M^{s_1, \dots, s_r} = \sum_{k=0}^r A^{s_1, \dots, s_k} B^{s_{k+1}, \dots, s_r}$.

Theorem:

The solutions $\mathcal{B}e^\bullet(z)$, valued in $\mathbb{C}[z]$ and multiplied by the stuffle product, to the difference equation $\Delta(\mathcal{B}e^{s_1, \dots, s_r})(z) = \mathcal{B}e^{s_1, \dots, s_{r-1}}(z) \cdot z^{s_r}$ satisfying $\mathcal{B}e^s(z) = \frac{B_{s+1}(z)}{s+1}$ have an exponential generating function $\mathcal{B}eeg^\bullet(z)$ of the form

$$\mathcal{B}eeg^\bullet(z) = beeg^\bullet \times \mathcal{B}eeg_0^\bullet(z),$$

where:

$$\left\{ \begin{array}{l} beeg^\bullet \text{ is valued in } \mathbb{C} . \\ \text{The } beeg^\bullet\text{'s are multiplied by the stuffle product.} \\ beeg^X = \frac{1}{e^X - 1} - \frac{1}{X} . \end{array} \right.$$

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A New Expression of the Mould $\mathcal{B}eeg_0^\bullet(z)$

New Goal:

From $\mathcal{B}eeg^\bullet(z) = beeg^\bullet \times \mathcal{B}eeg_0^\bullet(z)$, determine a suitable mould $be^\bullet$, or $beeg^\bullet$ such that the reflexion formula is a nice generalization of

$$(-1)^n B_n(1-z) = B_n(z), n \in \mathbb{N}.$$

Definition:

$$Sing^{X_1, \dots, X_r}(z) = \frac{e^{z(X_1 + \dots + X_r)}}{(e^{X_1} - 1)(e^{X_1 + X_2} - 1) \dots (e^{X_1 + \dots + X_r} - 1)}, \text{ for all } z \in \mathbb{C}.$$

Property:

$$\mathcal{B}eeg_0^\bullet(z) = \left(Sing^\bullet(0) \right)^{\times -1} \times Sing^\bullet(z).$$

Key Point of the Reflexion Formula of $Sing^\bullet(z)$

Notations: $M^{-\bullet} : (\omega_1, \dots, \omega_r) \mapsto M^{-\omega_1, \dots, -\omega_r}$.

$$M^{\overleftarrow{\bullet}} : (\omega_1, \dots, \omega_r) \mapsto M^{\omega_r, \dots, \omega_1}.$$

Lemma:

Let $Sg^\bullet$ be the mould defined by: $Sg^{X_1, \dots, X_r} = (-1)^r$.

For all $z \in \mathbb{C}$, we have:

$$Sing^{-\bullet}(0) = \left(Sing^{\overleftarrow{\bullet}}(0) \right)^{\times -1} \times Sg^\bullet, \quad Sing^{-\bullet}(1-z) = \left(Sing^{\overleftarrow{\bullet}}(z) \right)^{\times -1}$$

Sketch of proof: Use $\frac{1}{e^X - 1} + \frac{1}{e^{-X} - 1} + 1 = 0$.

□

Examples:

$$Sing^{-X, -Y}(0) = Sing^{X, Y}(0) + Sing^{X+Y}(0) + Sing^X(0) + 1.$$

$$Sing^{-X, -Y}(1-z) = Sing^{X, Y}(z) + Sing^{X+Y}(z).$$

Corollary:

For all $z \in \mathbb{C}$, we have

$$\mathcal{B}eeg_0^{-\bullet}(1-z) \times \mathcal{B}eeg_0^{\overleftarrow{\bullet}}(z) = 1^{\bullet} + I^{\bullet},$$

$$\text{where } 1^{x_1, \dots, x_r} = \begin{cases} 1 & \text{if } r = 0. \\ 0 & \text{if } r > 0. \end{cases} \quad \text{and} \quad I^{x_1, \dots, x_r} = \begin{cases} 1 & \text{if } r = 1. \\ 0 & \text{if } r \neq 1. \end{cases}$$

Theorem:

For all $z \in \mathbb{C}$, we have:

$$\mathcal{B}eeg^{-\bullet}(1-z) \times \mathcal{B}eeg^{\overleftarrow{\bullet}}(z) = beeg^{-\bullet} \times (1^{\bullet} + I^{\bullet}) \times beeg^{\overleftarrow{\bullet}}.$$

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Goal: Solve the mould equation $\mathcal{X}eeg^\bullet \times (1^\bullet + I^\bullet) \times \mathcal{X}eeg^{\overleftarrow{\bullet}} = 1^\bullet$.

Consequence: Define $\mathcal{B}eeg^\bullet = \mathcal{X}eeg^\bullet \times \mathcal{B}eeg_0^\bullet$, so that:

$$\mathcal{B}eeg^{-\bullet}(1-z) \times \mathcal{B}eeg^{\overleftarrow{\bullet}}(z) = 1^\bullet.$$

Definition:

The mould $\sqrt{\mathcal{S}g^\bullet}$ is defined by:

$$\sqrt{\mathcal{S}g^{x_1, \dots, x_r}} = \frac{(-1)^r}{2^{2r}} \binom{2r}{r}$$

Property:

Any solution $\mathcal{X}eg^\bullet$ of $\mathcal{X}eg^\bullet \times (1^\bullet + I^\bullet) \times \mathcal{X}eg^{\overleftarrow{\bullet}} = 1^\bullet$ comes from a mould $\mathcal{Z}eg^\bullet$ satisfying:

$$\left\{ \begin{array}{l} \mathcal{Z}eg^\bullet + \mathcal{Z}eg^{\overleftarrow{\bullet}} = 0^\bullet, \\ \mathcal{Z}eg^\bullet \text{ is "primitive" ,} \end{array} \right.$$

and is given by: $\mathcal{X}eg^\bullet = \text{Exp}(\mathcal{Z}eg^\bullet) \times \sqrt{Sg^\bullet}$.

Sketch of proof: group-like = $\text{Exp}(\text{primitive})$.



Definition of Multiple Bernoulli Polynomials and Numbers

The mould $\mathcal{Z}eg^\bullet$ defined by

$$\begin{cases} \mathcal{Z}eg^\emptyset := 0, \\ \mathcal{Z}eg^{x_1, \dots, x_r} := \frac{(-1)^{r-1}}{r} \mathcal{Z}eg^{x_1 + \dots + x_r}, \end{cases}$$

where $\mathcal{Z}eg^x = \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}$, satisfies the required conditions:

$$\begin{cases} \mathcal{Z}eg^\bullet + \mathcal{Z}eg^{\overleftarrow{\bullet}} = 0, \\ \mathcal{Z}eg^\bullet \text{ is "primitive" }, \end{cases}$$

Definition:

The moulds $\mathcal{B}eeg^\bullet(z)$ and $\mathcal{b}eeg^\bullet$ are defined by:

$$\mathcal{B}eeg^\bullet(z) = \text{Exp}(\mathcal{Z}eg^\bullet) \times \sqrt{Sg^\bullet} \times (\text{Sing}^\bullet(0))^{\times-1} \times \text{Sing}^\bullet(z).$$

$$\mathcal{b}eeg^\bullet = \text{Exp}(\mathcal{Z}eg^\bullet) \times \sqrt{Sg^\bullet}$$

The coefficients $\mathcal{B}e^\bullet(z)$ and $\mathcal{B}e^\bullet$ of these exponential generating series are called respectively the **Multiple Bernoulli Polynomials and Numbers**.

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$q = 0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q = 1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

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$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.

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$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
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$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.

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$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
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$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
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- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.

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$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
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- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.
- “symmetrie” relatively to $p = q$.

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$q = 0$							
$q = 1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q = 2$							
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$q = 2$	0		0		0		0
$q = 3$							
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$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$		$-\frac{1}{264}$		$\frac{691}{65520}$
$q = 6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$q = 0$							
$q = 1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$		$\frac{691}{65520}$
$q = 6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$q = 0$							
$q = 1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q = 6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$q = 0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q = 1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q = 2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q = 3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q = 4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q = 5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q = 6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Reminders on Stirling Numbers of Second Kind

Definition

The **Stirling number of the second kind**, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, is the number of ways to partition a set of n objects into k non-empty subsets.

Recurrence relation:

$$\left\{ \begin{array}{l} \left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 0 \text{ if } n > 0. \\ \left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\} \end{array} \right.$$

Triangle:

n	0	1	2	3	4	5	6
$k=0$	1						
$k=1$	0	1					
$k=2$	0	1	1				
$k=3$	0	1	3	1			
$k=4$	0	1	7	6	1		
$k=5$	0	1	15	25	10	1	
$k=6$	0	1	31	90	65	15	1

A Recurrence Similar to the Stirling Number of 2nd Kind

Reminder: The falling factorial is defined by: $(z)_n = z(z-1)\cdots(z-n+1)$.

Coefficient in the falling factorial $(z)_n$ of $\mathcal{B}e^{1,1,k}(z) - \mathcal{B}e^{1,1,k}(0)$:

n	1	2	3	4	5	6	7	8	9	10	11
$k=0$	1	0	12	96	36						
$k=1$	0	1	24	300	240	36					
$k=2$	0	1	49	924	1260	420	36				
$k=3$	0	1	99	2821	5964	3360	636	36			
$k=4$	0	1	199	8562	26677	22764	7176	888	36		
$k=5$	0	1	399	25885	115270	140497	65820	13392	1176	36	
$k=6$	0	1	799	78054	486965	817755	535417	159564	22800	1500	36

Recurrence relation: $u_{n+1,k} = ku_{n,k} + u_{n,k-1}$ for all $k > 0$.

1. We have respectively defined the Multiple Bernoulli Polynomials and Multiple Bernoulli Numbers by:

$$\begin{cases} Beeg^{\bullet}(z) &= Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{Sg^{\bullet}} \times (Sing^{\bullet}(0))^{\times-1} \times Sing^{\bullet}(z) . \\ beeg^{\bullet} &= Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{Sg^{\bullet}} \end{cases}$$

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:

- the difference equation $\Delta(B_n)(x) = nx^{n-1}$.
- the reflection formula $(-1)^n B_n(1-x) = B_n(x)$.
- the expression of Bernoulli polynomials in the falling factorial basis.

3. The Multiple Bernoulli Polynomials/Numbers can be multiplied using the stuffle product.

THANK YOU FOR YOUR ATTENTION !