An example of generalization of the Bernoulli numbers and polynomials to any dimension

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Tuesday, 29th March 2016 .

Introduction

Definition:

The numbers $\mathcal{Z}e^{s_1,\cdots,s_r}$ defined by

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots{n_r}^{s_r}} ,$$

where $s_1,\cdots,s_r \in \mathbb{C}$ such that $\Re(s_1+\cdots+s_k) > k$, $k \in [\![1\,;\,r]\!]$, are called multiple zeta values.

Fact: There exists at least three different ways to renormalize multiple zeta values at negative integers.

$$\mathcal{Z}e_{MP}^{0,-2}(0) = \frac{7}{720} \quad , \quad \mathcal{Z}e_{GZ}^{0,-2}(0) = \frac{1}{120} \quad , \quad \mathcal{Z}e_{FKMT}^{0,-2}(0) = \frac{1}{18} .$$

Question: Is there a group acting on the set of all possible multiple zeta values renormalisations?

Main goal: Define multiple Bernoulli numbers in relation with this.

Outline

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Two Equivalent Definitions of Bernoulli Polynomials / Numbers

Bernoulli numbers:	**Bernoulli polynomials:**
By a generating function: $$\frac{t}{e^t - 1} = \sum_{n \geq 0} b_n \frac{t^n}{n!} .$$	By a generating function: $$\frac{te^{xt}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!} .$$
By a recursive formula: $$\begin{cases} b_0 = 1 , \\ \forall n \in \mathbb{N} , \ \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 . \end{cases}$$	By a recursive formula: $$\begin{cases} B_0(x) = 1 , \\ \forall n \in \mathbb{N} , \ B'_{n+1}(x) = (n+1) B_n(x) , \\ \forall n \in \mathbb{N}^* , \ \int_0^1 B_n(x) \, dx = 0 . \end{cases}$$
First examples: $$b_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \cdots$$	First examples: $$\begin{aligned} B_0(x) &= 1 , \\ B_1(x) &= x - \frac{1}{2} , \\ B_2(x) &= x^2 - x + \frac{1}{6} , \\ &\vdots \end{aligned}$$

Elementary properties satisfied by the Bernoulli polynomials and numbers

P1 $b_{2n+1} = 0 \; \textit{if} \; n > 0.$

P2 $B_n(0) = B_n(1) \; \textit{if} \; n > 1.$

P3 $$\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0, \;\; m > 0.$$

P4 $$\begin{cases} B_n'(z) = nB_{n-1}(z) \text{ if } n > 0. \\ B_n(x+y) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} \text{ for all } n. \end{cases}$$

P5 $B_n(x+1) - B_n(x) = nx^{n-1}$, for all n.

P6 $(-1)^n B_n(1-x) = B_n(x)$, for all n.

P7 $$\sum_{k=0}^{N-1} k^n = \frac{B_{n+1}(N) - B_{n+1}(0)}{n+1} .$$

P8 $$\int_a^x B_n(t)\, dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1} .$$

P9 $$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \text{ for all } m > 0 \text{ and } n \geq 0.$$

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Definition:

The Hurwitz Zeta Function is defined, for $\Re e\ s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s,z) = \sum_{n>0} \frac{1}{(n+z)^s} \ .$$

Property:

H1 $$\begin{cases} \dfrac{\partial \zeta}{\partial z}(s,z) = -s\zeta(s+1,z). \\ \zeta(s,x+y) = \displaystyle\sum_{n\geq 0} \binom{-s}{n} \zeta(s+n,x) y^n. \end{cases}$$

H2 $\zeta(s,z-1) - \zeta(s,z) = z^{-s}$.

H3 $\zeta(s,mz) = m^{-s} \sum_{k=0}^{m-1} \zeta\left(s, z + \frac{k}{m}\right)$ if $m \in \mathbb{N}^*$.

Link between the Hurwitz Zeta Function and the Bernoulli polynomials

Property:

$s \longmapsto \zeta(s,z)$ can be analytically extend to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

Remark:

$$\zeta(-n,z) = -\frac{B_{n+1}(z)}{n+1} \text{ for all } n \in \mathbb{N} .$$

$$\zeta(-n,0) = -\frac{b_{n+1}}{n+1} \text{ for all } n \in \mathbb{N} .$$

Related properties:

	Hurwitz zeta function	Bernoulli polynomials
Derivative property	**H1**	**P4**
Difference equation	**H2**	**P5**
Multiplication theorem	**H3**	**P9**

Consequence:

The extension from Bernoulli to multiple Bernoulli polynomials will be done using a generalization of the Hurwitz zeta function: the **Hurwitz multiple zeta functions***.*

Definition of Hurwitz Multiple Zeta Functions

$\mathcal{H}e^{s_1,\cdots,s_r}(z) = \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$, if $z \in \mathbb{C} - \mathbb{N}_{<0}$ and $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, such that $s_1 \geq 2$.

Lemma 1: (B., J. Ecalle, 2012)

For all sequences $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, $s_1 \geq 2$, we have:

$$\mathcal{H}e^{s_1,\cdots,s_r}(z-1) - \mathcal{H}e^{s_1,\cdots,s_r}(z) = \mathcal{H}e^{s_1,\cdots,s_{r-1}}(z)\cdot z^{-s_r} \ .$$

Lemma 2:

The Hurwitz Multiple Zeta Functions multiply by the stuffle product (of $\mathbb{N}^*$).

Reminder: If $(\Omega,+)$ is a semi-group, the stuffle $⊎$ is defined over $\Omega^\star$ by:

$$\begin{cases} \varepsilon ⊎ u = u ⊎ \varepsilon = u \ . \\ ua ⊎ vb = (u ⊎ vb)\, a + (ua ⊎ v)\, b + (u ⊎ v)(a+b) \ . \end{cases}$$

Outline

Reminders on Quasi-symmetric functions

Definition:

Let $\mathrm{x} = \{x_1, x_2, x_3, \cdots\}$ be an infinite commutative alphabet.
A series is said to be quasi-symmetric when its coefficient of $x_1^{s_1} \cdots x_r^{s_r}$ is equals to this of $x_{i_1}^{s_1} \cdots x_{i_r}^{s_r}$ for all $i_1 < \cdots < i_r$.

Example : $M_{2,1}(x_1, x_2, x_3, \ldots) = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_1^2 x_n + \cdots + x_2^2 x_3 + \cdots$
$x_1 x_2^2$ is not in $M_{2,1}$ but in $M_{1,2}$.

Fact 1:

- Quasi-symmetric functions span a vector space: $QSym$.
- A basis of $QSym$ is given by the monomials M_I, for composition $I = (i_1, \cdots, i_r)$:

$$M_{i_1,\cdots,i_r}(X) = \sum_{0<n_1<\cdots<n_r} x_{n_1}^{i_1} \cdots x_{n_r}^{i_r}$$

Fact 2:

- $QSym$ is an algebra whose product is the stuffle product.
- $QSym$ is also a Hopf algebra whose coproduct Δ is given by:

$$\Delta\big(M_{i_1,\cdots,i_r}(\mathrm{x})\big) = \sum_{k=0}^{r} M_{i_1,\cdots,i_k}(\mathrm{x}) \otimes M_{i_{k+1},\cdots,i_r}(\mathrm{x}) \ .$$

Outline

Main Goal

Heuristic:

$$B^{s_1,\cdots,s_r}(z) = \text{Multiple (Divided) Bernoulli Polynomials} = \mathcal{H}e^{-s_1,\cdots,-s_r}(z)\ .$$

$$b^{s_1,\cdots,s_r} = \text{Multiple (Divided) Bernoulli Numbers} = \mathcal{H}e^{-s_1,\cdots,-s_r}(0)\ .$$

We want to define $B^{s_1,\cdots,s_r}(z)$ such that:

- their properties are similar to Hurwitz Multiple Zeta Functions' properties.
- their properties generalize these of Bernoulli polynomials.

Main Goal:

Find some polynomials $B^{s_1,\cdots,s_r}$ such that:

$$\begin{cases} B^n(z) = \dfrac{B_{n+1}(z)}{n+1}\ , \text{ where } n \geq 0\ , \\ B^{n_1,\cdots,n_r}(z+1) - B^{n_1,\cdots,n_r}(z) = B^{n_1,\cdots,n_{r-1}}(z)z^{n_r}\ , \text{ for } \ n_1,\cdots,n_r \geq 0\ , \\ \text{the } B^{n_1,\cdots,n_r} \text{ multiply by the stuffle product.} \end{cases}$$

An algebraic construction

Notation 1:

Let $\mathrm{X} = \{X_1, \cdots, X_n, \cdots\}$ be a (commutative) alphabet of indeterminates.
We denotes:

$$\mathcal{B}^{Y_1,\cdots,Y_r}(z) = \sum_{n_1,\cdots,n_r \geq 0} B^{n_1,\cdots,n_r}(z) \frac{Y_1^{n_1}}{n_1!} \cdots \frac{Y_r^{n_r}}{n_r!} \ ,$$

for all $r \in \mathbb{N}^*$, $Y_1, \cdots, Y_r \in \mathrm{X}$.

Remark: $\mathcal{B}^{Y_1,\cdots,Y_r}(z+1) - \mathcal{B}^{Y_1,\cdots,Y_r}(z) = \mathcal{B}^{Y_1,\cdots,Y_{r-1}}(z) e^{zY_r}$.

Notation 2:

Let $\mathrm{A} = \{a_1, \cdots, a_n, \cdots\}$ be a non-commutative alphabet.
We denotes:

$$\mathfrak{B}(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \mathcal{B}^{X_{k_1},\cdots,X_{k_r}}(z) a_{k_1} \cdots a_{k_r} \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle \ .$$

Remark: $\mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \left(1 + \sum_{k>0} e^{zX_k} a_k\right)$

The abstract construction in the case of quasi-symmetric functions

Let see an analogue of $\mathfrak{B}(z)$ where the multiple Bernoulli polynomials are replaced with the monomial functions $M_I(\mathrm{x})$ of $QSym$:

$$M^{Y_1,\cdots,Y_r}(\mathrm{x}) := \sum_{n_1,\cdots,n_r\geq 0} M_{n_1+1,\cdots,n_r+1}(\mathrm{x}) \frac{Y_1^{n_1}}{n_1!}\cdots\frac{Y_r^{n_r}}{n_r!}\ , \text{ for all } Y_1,\cdots,Y_r\in \mathrm{X}\ .$$

$$\begin{aligned}
\mathfrak{M} &:= 1+\sum_{r>0}\sum_{k_1,\cdots,k_r>0} M^{X_{k_1},\cdots,X_{k_r}}(\mathrm{x})\ a_{k_1}\cdots a_{k_r}\\
&= 1+\sum_{r>0}\sum_{0<p_1<\cdots<p_r}\prod_{i=1}^{r}\left(1+\sum_{k>0}x_n e^{x_n X_k}a_k\right)M^{X_{k_1},\cdots,X_{k_r}}(\mathrm{x})\\
&= \overrightarrow{\prod_{n>0}}\left(1+\sum_{k>0}x_n e^{x_n X_k}a_k\right)\in\mathbb{C}[\![\mathrm{x}]\!][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A}\rangle\!\rangle\ .
\end{aligned}$$

<u>**Computation of the coproduct of $\mathfrak{M}$:**</u> (which does not act on the X's)

$$\Delta M^{Y_1,\cdots,Y_r}(\mathrm{x}) = \sum_{k=0}^{r} M^{Y_1,\ldots,Y_k}(\mathrm{x})\otimes M^{Y_{k+1},\ldots,Y_r}(\mathrm{x})\ .$$

$$\Delta\mathfrak{M} = \mathfrak{M}\otimes\mathfrak{M}\ .$$

Transcription of the multiplication by the stuffle

Property: (J. Y. Thibon, F. Chapoton, J. Ecalle, F. Menous, D. Sauzin, ...)

A family of objects $(B^{n_1,\cdots,n_r})_{n_1,n_2,n_3,\cdots\geq 0}$ multiply by the stuffle product if, and only if, there exists a character χ_z of *QSym* such that

$$\chi_z\big(M_{n_1+1,\cdots,n_r+1}(\mathrm{x})\big) = B^{n_1,\cdots,n_r}(z) \tag{1}$$

Consequences:

1. χ_z can be extended to $QSym[\![\mathrm{X}]\!]$, applying it terms by terms.

$$\chi_z\left(M^{Y_1,\cdots,Y_r}(\mathrm{x})\right) = \mathcal{B}^{Y_1,\cdots,Y_r}(z) \text{ , for all } Y_1,\cdots,Y_r \in \mathrm{X} \text{ .}$$

2. If $B^{n_1,\cdots,n_r}$ multiply the stuffle, $\mathfrak{B} = \chi_z(\mathfrak{M})$ is “group-like” in $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A}\rangle\!\rangle$.

Reformulation of the main goal

Find some polynomials $B^{n_1,\cdots,n_r}$ such that:

$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k}-1} - \dfrac{1}{X_k} \ , \\ \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \ , \text{ where } \mathfrak{E}(z) = 1 + \displaystyle\sum_{k>0} e^{zX_k}\, a_k \ , \\ \mathfrak{B} \text{ is a ``group-like'' element of } \mathbb{C}[z][\![\mathrm{X}]\!]\langle\!\langle \mathrm{A} \rangle\!\rangle \ . \end{cases}$$

Outline

A singular solution

Remainder: $\mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k}\, a_k .$

From a false solution to a singular solution...

$$\mathcal{S}(z) = \overleftarrow{\prod_{n>0}} \mathfrak{E}(z-n) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \frac{e^{z(X_{k_1}+\cdots+X_{k_r})}}{\prod_{i=1}^{r}(e^{X_{k_1}+\cdots+X_{k_i}}-1)} a_{k_1}\cdots a_{k_r} \text{ is a}$$

false solution to system
$$\begin{cases} \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k}-1} - \dfrac{1}{X_k} , \\ \mathfrak{B}(z+1) = \mathfrak{B}(z)\cdot \mathfrak{E}(z) , \\ \mathfrak{B} \text{ is a ``group-like'' element of } \mathbb{C}[z][\![X]\!]\langle\langle A\rangle\rangle . \end{cases}$$

Explanations:

1. $$\begin{aligned} \mathfrak{B}(z) &= \cdots = \mathfrak{B}(z-n)\cdot\mathfrak{E}(z-n)\cdots\mathfrak{E}(z-1) \\ &= \cdots = \left(\lim_{n\longrightarrow+\infty} \mathfrak{B}(z-n)\right)\cdot \overleftarrow{\prod_{n>0}} \mathfrak{E}(z-n) . \end{aligned}$$

2. $\mathcal{S}(z) \in \mathbb{C}[z]((X))\langle\langle A\rangle\rangle$, $\mathcal{S}(z) \notin \mathbb{C}[z][\![X]\!]\langle\langle A\rangle\rangle$.

Heuristic: Find a correction of $\mathcal{S}$, to send it into $\mathbb{C}[z][\![X]\!]\langle\langle A\rangle\rangle$.

Another solution

Fact: If $\Delta(f)(z) = f(z-1) - f(z)$, $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$.

Consequence: There exist a unique family of polynomials such that:

$$\begin{cases} B_0^{n_1,\cdots,n_r}(z+1) - B_0^{n_1,\cdots,n_r}(z) = B_0^{n_1,\cdots,n_{r-1}}(z) z^{n_r} \ . \\ B_0^{n_1,\cdots,n_r}(0) = 0 \ . \end{cases}$$

This produces a series $\mathfrak{B}_0 \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ defined by:

$$\mathfrak{B}_0(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} \mathcal{B}_0^{X_{k_1},\cdots,X_{k_r}}(z)\ a_{k_1}\cdots a_{k_r} \ .$$

Lemma: (B., 2013)

1. The noncommutative series $\mathfrak{B}_0$ is a "group-like" element of $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$.
2. The coefficients of $\mathfrak{B}_0(z)$ satisfy a recurence relation:

$$\begin{cases} \mathcal{B}_0^{Y_1}(z) = \dfrac{e^{zY_1} - 1}{e^{Y_1} - 1} \ , \quad Y_1 \in \mathrm{X} \ . \\ \mathcal{B}_0^{Y_1,\cdots,Y_r}(z) = \dfrac{\mathcal{B}_0^{Y_1+Y_2,Y_3,\cdots,Y_r}(z) - \mathcal{B}_0^{Y_2,Y_3,\cdots,Y_r}(z)}{e^{Y_1} - 1} \ , \quad Y_1,\cdots,Y_r \in \mathrm{X} \ . \end{cases}$$

3. The series $\mathfrak{B}_0$ can be expressed in terms of $\mathcal{S}$: $\mathfrak{B}_0(z) = \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$.

Characterization of the set of solutions

Reminder: A family of multiple Bernoulli polynomials produces a series $\mathfrak{B}$ such that:

$$\begin{cases} \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \text{ , where } \mathfrak{E}(z) = 1 + \sum_{k>0} e^{zX_k} \, a_k \text{ ,} \\ \mathfrak{B} \text{ is a ``group-like'' element of } \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle \text{ ,} \\ \langle \mathfrak{B}(z) | a_k \rangle = \dfrac{e^{zX_k}}{e^{X_k} - 1} - \dfrac{1}{X_k} \text{ .} \end{cases}$$

Proposition: (B. 2013)

Any familly of polynomials which are solution of the previous system comes from a noncommutative series $\mathfrak{B} \in \mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ such that there exists $\mathfrak{b} \in \mathbb{C}[\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$ satisfying:

1. $\langle \mathfrak{b} | A_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0 = \mathfrak{b} \cdot \big(\mathcal{S}(0)\big)^{-1} \cdot \mathcal{S}(z)$.

Theorem: (B., 2013)

The subgroup of "group-like" series of $\mathbb{C}[z][\![\mathrm{X}]\!]\langle\langle \mathrm{A} \rangle\rangle$, with vanishing coefficients in length 1, acts on the set of all possible multiple Bernoulli polynomials, *i.e.* on the set of all possible *algebraic* renormalization.

Outline

New Goal:

From $\mathfrak{B}(z) = \mathfrak{b} \cdot \mathfrak{B}_0$, determine a suitable series $\mathfrak{b}$ such that the reflexion formula

$$(-1)^n B_n(1-z) = B_n(z) \ , n \in \mathbb{N}$$

has a nice generalization.

For a generic series $s \in \mathbb{C}[z][\![\boldsymbol{X}]\!]\langle\!\langle \boldsymbol{A} \rangle\!\rangle$,

$$s(z) = \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_1}, \cdots, X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r} \ ,$$

we consider:

$$\begin{aligned} \overline{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{X_{k_r}, \cdots, X_{k_1}}(z)\, a_{k_1} \cdots a_{k_r} \\ \widetilde{s}(z) &= \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} s^{-X_{k_1}, \cdots, -X_{k_r}}(z)\, a_{k_1} \cdots a_{k_r} \end{aligned}$$

The reflection equation for $\mathfrak{B}_0(z)$

Proposition: (B. 2014)

Let $sg = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} (-1)^r a_{k_1} \cdots a_{k_r} = \left(1 + \sum_{n>0} a_n\right)^{-1}$. Then,

$$\widetilde{\mathcal{S}}(0) = \left(\overline{\mathcal{S}}(0)\right)^{-1} \cdot sg \quad \text{and} \quad \widetilde{\mathcal{S}}(1-z) = \left(\overline{\mathcal{S}}(z)\right)^{-1} .$$

Corollary 1: (B. 2014)

For all $z \in \mathbb{C}$, we have: $sg \cdot \widetilde{\mathfrak{B}}_0(1-z) = \left(\overline{\mathfrak{B}}_0(z)\right)^{-1}$.

Example:

$$\begin{aligned} \mathcal{B}_0^{-X,-Y,-Z}(1-z) &= -\mathcal{B}_0^{X,Y,Z}(z) - \mathcal{B}_0^{X+Y,Z}(z) - \mathcal{B}_0^{X,Y+Z}(z) \\ &\quad -\mathcal{B}_0^{X+Y+Z}(z) + \mathcal{B}_0^{Y,Z}(z) + \mathcal{B}_0^{Y+Z}(z) . \end{aligned}$$

The generalization of the reflection formula

Corollary 2: (B. 2014)

$$\widetilde{\mathfrak{B}}(1-z)\cdot\overline{\mathfrak{B}}(z)=\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}\ . \qquad (2)$$

Remark: $\widetilde{\mathcal{S}}(0)\cdot sg^{-1}\cdot\overline{\mathcal{S}}(0)=1.$

Heuristic:

A reasonable candidate for a multi-Bernoulli polynomial comes from the coefficients of a series $\mathfrak{B}(z)=\mathfrak{b}\cdot\mathfrak{B}_0(z)$ where $\mathfrak{b}$ satisfies:

1. $\langle\mathfrak{b}|a_k\rangle=\dfrac{1}{e^{X_k}-1}-\dfrac{1}{X_k}$
2. $\mathfrak{b}$ is "group-like"
3. $\widetilde{\mathfrak{b}}\cdot sg^{-1}\cdot\overline{\mathfrak{b}}=1$.

Outline

Resolution of an equation

Goal: Characterise the solutions of $\left\{ \begin{array}{l} \widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1 \ . \\ \mathfrak{u} \text{ is “group-like”} \ . \end{array} \right.$

Proposition: (B., 2014)

Let us denote $\sqrt{sg^{-1}} = 1 + \sum_{r>0} \sum_{k_1, \cdots, k_r > 0} \frac{(-1)^r}{2^{2r}} \binom{2r}{r} a_{k_1} \cdots a_{k_r}$ ..

Any “group-like” solution $\mathfrak{u}$ of $\widetilde{\mathfrak{u}} \cdot sg^{-1} \cdot \overline{\mathfrak{u}} = 1$ comes from a “primitive” series $\mathfrak{v}$ satisfying

$$\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0 \ ,$$

and is given by:

$$\mathfrak{u} = exp(\mathfrak{v}) \cdot \sqrt{sg} \ .$$

If moreover $\langle \mathfrak{u} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k}$, then necessarily, we have:

$$\langle \mathfrak{v} | a_k \rangle = \frac{1}{e^{X_k} - 1} - \frac{1}{X_k} + \frac{1}{2} := f(X_k) \ .$$

New goal: Find a nice series $\mathfrak{v}$ satisfying:

1. $\mathfrak{v}$ is "primitive". 2. $\overline{\mathfrak{v}} + \widetilde{\mathfrak{v}} = 0$. 3. $\langle \mathfrak{v} | a_k \rangle = \dfrac{1}{e^{X_k} - 1} - \dfrac{1}{X_k} + \dfrac{1}{2} = f(X_k)$.

Remark: $\langle \mathfrak{v} | a_k \rangle$ is an odd formal series in $X_k \in \mathrm{X}$.

Generalization: $\widetilde{\mathfrak{v}} = -\mathfrak{v}$, so $\overline{\mathfrak{v}} = \mathfrak{v}$.

$\Longrightarrow \langle \mathfrak{v} | a_{k_1} a_{k_2} \rangle = -\frac{1}{2} f(X_{k_1} + X_{k_2})$, but does not determine $\langle \mathfrak{v} | a_{k_1} a_{k_2} a_{k_3} \rangle$.

A restrictive condition:

A natural condition is to have:

there exists $\alpha_r \in \mathbb{C}$ such that $\langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \alpha_r f(X_{k_1} + \cdots + X_{k_r})$.

Now, there is a unique "primitive" series $\mathfrak{v}$ satisfying this condition and the new goal:

$$\langle \mathfrak{v} | a_{k_1} \cdots a_{k_r} \rangle = \frac{(-1)^{r-1}}{r} f(X_{k_1} + \cdots + X_{k_r}) \ .$$

Definition : (B., 2014)

The series $\mathfrak{B}(z)$ and $\mathfrak{b}$ defined by

$$\begin{cases} \mathfrak{B}(z) & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} & = & exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

are noncommutative series of $\mathbb{C}[z][\![X]\!]\langle\langle A \rangle\rangle$ whose coefficients are respectively the exponential generating functions of *multiple Bernoulli polynomials* and *multiple Bernoulli numbers*.

Example:

The exponential generating function of bi-Bernoulli polynomials and numbers are respectively:

$$\begin{aligned} \sum_{n_1,n_2 \geq 0} B^{n_1,n_2}(z) \frac{X^{n_1}}{n_1!} \frac{Y^{n_2}}{n_2!} &= -\frac{1}{2} f(X+Y) + \frac{1}{2} f(X) f(Y) - \frac{1}{2} f(X) + \frac{3}{8} \\ &+ f(X) \, \frac{e^{zY}-1}{e^Y-1} - \frac{1}{2} \, \frac{e^{zY}-1}{e^Y-1} \\ &+ \frac{e^{z(X+Y)}-1}{(e^X-1)(e^{X+Y}-1)} - \frac{e^{zY}-1}{(e^X-1)(e^Y-1)} \, . \end{aligned}$$

Examples of explicit expression for multiple Bernoulli numbers:

Consequently, we obtain explicit expressions like, for $n_1, n_2, n_3 > 0$:

$$b^{n_1,n_2} = \frac{1}{2}\left(\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1} - \frac{b_{n_1+n_2+1}}{n_1+n_2+1}\right).$$

$$\begin{aligned} b^{n_1,n_2,n_3} = &+\frac{1}{6}\frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+1}}{n_2+1}\frac{b_{n_3+1}}{n_3+1} \\ &-\frac{1}{4}\left(\frac{b_{n_1+n_2+1}}{n_1+n_2+1}\frac{b_{n_3+1}}{n_3+1} + \frac{b_{n_1+1}}{n_1+1}\frac{b_{n_2+n_3+1}}{n_2+n_3+1}\right) \\ &+\frac{1}{3}\frac{b_{n_1+n_2+n_3+1}}{n_1+n_2+n_3+1}\ . \end{aligned}$$

Remark: If $n_1 = 0$, $n_2 = 0$ or $n_3 = 0$, the expressions are not so simple...

Table of Multiple Bernoulli Numbers in length 2

$b^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

Outline

Table of Multiple Bernoulli Numbers in length 2

$b^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is “constant”.

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is “constant”.
- “symmetrie” relatively to $p=q$.

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two antidiagonals is "constant".
- "symmetrie" relatively to $p = q$.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$b^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$							
$q=3$							
$q=4$							
$q=5$							
$q=6$							

Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0		0		0		0
$q=3$							
$q=4$	0		0		0		0
$q=5$							
$q=6$	0		0		0		0

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0		0		0
$q=3$		$-\frac{1}{2880}$					
$q=4$	0	$-\frac{1}{504}$	0		0		0
$q=5$		$\frac{1}{6048}$					
$q=6$	0	$\frac{1}{480}$	0		0		0

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$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$		$\frac{1}{480}$		$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$		$-\frac{1}{264}$		$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

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$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$		$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$		$-\frac{1}{264}$		$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$		$-\frac{1}{264}$		$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$		$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

From these previous remarks, let us present an algorithm to compute bi-Bernoulli numbers.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Outline

Properties satisfied by multiple Bernoulli polynomials 1

Proposition: (B., 2013)

The multiple Bernoulli polynomials $B^{n_1,\cdots,n_r}$ multiply the stuffle.

Theorem: (B., 2014)

P'1 All multiple Bernoulli numbers satisfy : $b^{2p_1,\cdots,2p_r} = 0$.

P'2 If $n_r > 0$, $B^{n_1,\cdots,n_r}(0) = B^{n_1,\cdots,n_r}(1)$.

P'5 $B^{n_1,\cdots,n_r}(z+1) - B^{n_1,\cdots,n_r}(z) = B^{n_1,\cdots,n_{r-1}}(z) \cdot z^{n_r}$.

P'6 There exists a reflexion formula for multiple Bernoulli polynoms: $\widetilde{\mathfrak{B}}(1-z) \cdot \overline{\mathfrak{B}}(z) = 1$.

P'7 The truncated multiple sums of powers $S_N^{s_1,\cdots,s_r}$, defined by

$$S_N^{s_1,\cdots,s_r} = \sum_{0 \leq n_r < \cdots < n_1 < N} {n_1}^{s_1} \cdots {n_r}^{s_r}$$

are given by the coefficients of $\mathfrak{B}_0(N)$.

Properties satisfied by multiple Bernoulli polynomials 2

Proposition: (B. 2014)

For all positive integers $n_1, \cdots, n_r$, $b^{n_1, \cdots, n_r} = b^{n_r, \cdots, n_1}$.

Proposition: (B., 2014)

For a series $s(z) \in \mathbb{C}[z][\![\boldsymbol{X}]\!]\langle\langle \boldsymbol{A} \rangle\rangle$, let us define $\Delta(s)(z)$ by:

$$\Delta(s)(z) = \sum_{r \in \mathbb{N}} \sum_{k_1, \cdots, k_r > 0} X_{k_1} \cdots X_{k_r} \langle s(z) | a_{k_1} \cdots a_{k_r} \rangle \; a_{k_1} \cdots a_{k_r} \; .$$

Δ is a derivation, and :

P'4 The derivation of multiple Bernoulli polynomials are given by:

$$\partial_z \mathfrak{B}(z) = \Delta\left(\mathfrak{b} \cdot \mathcal{S}(0)^{-1}\right) \cdot \left(\mathfrak{b} \cdot \mathcal{S}(0)^{-1}\right)^{-1} \cdot \mathfrak{B}(z) + \Delta(\mathfrak{B}(z)) \; .$$

Proposition: (B., 2015)

P'3 The recurrence relation of bi-Bernoulli numbers is (partially) given by:

$$2\left(\sum_{k=0}^{p}\sum_{l=0}^{q}\binom{p}{k}\binom{q}{l}be^{k,l}-be^{p,q}\right)=$$

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{2}-\frac{1}{q+1}\right)$$

$$+\left(\frac{1}{2}-\frac{1}{p+1}\right)be^{q}+be^{p}\left(\frac{1}{2}-\frac{1}{q+1}\right)$$

$$-\left(\frac{1}{2}-\frac{1}{p+1}\right)-\left(\frac{1}{2}-\frac{1}{p+q+1}\right)$$

$$-be^{p}+\frac{3}{4}$$

if $p, q > 0$.

1. We have respectively defined the Multiple (divided) Bernoulli Polynomials and Multiple (divided) Bernoulli Numbers by:

$$\begin{cases} \mathfrak{B}(z) &= exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \cdot (\mathcal{S}(0))^{-1} \cdot \mathcal{S}(z) \\ \mathfrak{b} &= exp(\mathfrak{v}) \cdot \sqrt{\mathcal{S}g} \end{cases}$$

They both multiply the stuffle.

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:
   - the nullity of b_{2n+1} if $n > 0$.
   - the symmetry $B_n(1) = B_n(0)$ if $n > 1$.
   - the difference equation $\Delta(B_n)(x) = nx^{n-1}$.
   - the reflection formula $(-1)^n B_n(1-x) = B_n(x)$.

THANK YOU FOR YOUR ATTENTION !