The Algebra of Multitangent Functions

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Abstract

Multizeta values are numbers appearing in many different contexts. Unfortunately, there arithmetics remains mostly out of reach.

In this article, we define a functional analogue of the algebra of multizetas values, namely the algebra of multitangent functions, which are 1-periodic functions defined by a process formally similar to multizeta values.

We introduce here the fundamental notions of reduction into monotangent functions, projection onto multitangent functions and that of trifactorisation, giving a way of writing a multitangent function in terms on Hurwitz multizeta functions. This explains why the multitangent algebra is a functional analogue of the algebra of multizeta values. We then discuss the most important algebraic and analytic properties of these functions and their consequences on multizeta values, as well as their regularization in the divergent case.

Each property of multitangent has a pendant on the side of multizeta values. This allowed us to propose new conjectures, which have been checked up to the weight 18.

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1. Introduction

1.1. The Riemann zeta function at positive integers

An interesting problem, but still unsolved and probably out of reach today, is to determine the polynomial relations over $\mathbb{Q}$ between the numbers

$\zeta(2)$, $\zeta(3)$, $\zeta(4)$, $\cdots$, where the Riemann zeta function ζ can be defined by the convergent series

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

in the domain $\Re e\, s > 1$.

Thanks to Euler, we know the classical formula for all even integers s:

$$\zeta(s) = -\frac{(2\pi)^s}{2} \frac{|B_s|}{s!} \ .$$

From this, one can see that $\mathbb{Q}[\zeta(2), \zeta(4), \zeta(6), \cdots] = \mathbb{Q}[\pi^2]$. Now, Lindemann's theorem on the transcendence of π concludes the discussion for s even, as the last ring is of transcendence degree 1 .

Euler failed to give such a formula for $\zeta(3)$. Actually, the situation is quite more complicated concerning the values of the Riemann zeta function at odd integers. Essentially, nothing is known about their arithmetics. One had to wait the end of the twentieth century to see the first results:

1. In 1979, Roger Apéry proved that $\zeta(3)$ is an irrational number (see [1]) ;
2. In 2000, Tanguy Rivoal proved there are infinitely many numbers in the list $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\cdots$ which are irrational numbers (see [29]) ;
3. in 2004, Wadim Zudilin showed that there is at least one number in the list $\zeta(3)$, $\zeta(5)$, $\cdots$, $\zeta(11)$ which is irrational (see [40]) .

One conjectures that each number $\zeta(s)$, $s \geq 2$, is a transcendental number. To be more precise, the following conjecture is expected:

Conjecture 1. *The numbers π , $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\cdots$ are algebraically independent over $\mathbb{Q}$.*

1.2. The multizeta values

The notion of multizeta value has been introduced in order to study questions related to this conjecture. Multizeta values are a multidimensional generalization of the values of the Riemann zeta function ζ at positive integers, defined by:

$$\mathcal{Z}e^{s_1, \cdots, s_r} = \sum_{0 < n_r < \cdots < n_1} \frac{1}{{n_1}^{s_1} \cdots {n_r}^{s_r}} \ ,$$

for all sequences of $\mathcal{S}_b^\star = \{(s_1; \cdots; s_r) \in \mathrm{seq}(\mathbb{N}^*) \, ; s_1 \geq 2\}$.

Their first introduction dates back to the year 1775 when Euler studied in his famous article [21] the case of length 2. In this work, he proved numerous remarkable relations between these numbers, like $\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3}$ or more generally:

$$\forall p \in \mathbb{N}^* \ , \ \sum_{k=1}^{p-1} \mathcal{Z}e^{p+1-k,k} = \mathcal{Z}e^{p+1} \ .$$

Although they sporadically appeared in the mathematics as well as in the physics literature, we can say that they were forgotten during the XIX$^{\text{th}}$ century and during most of the XX$^{\text{th}}$ century. In the last 70's, these numbers have been reintroduced by Jean Ecalle in holomorphic dynamics. He used them as auxiliary coefficients in order to construct some geometrical and analytical objects, such as solutions of differential equations with specific dynamical properties. During the late 80's, multizeta values appeared in many different contexts. They have been the object of an enormous renewed interest, which has then been massive and decisive. Finally, these numbers began to be studied for themselves.

Today, multizeta values arise in many different areas like in:

1. Number theory (search for relations between multizeta values, in order to study the hypothetical algebraic independence of values of Riemann's zeta function ; arithmetical dimorphy) : see [17], [36], [39] for example.
2. Quantum groups, knot theory or mathematical physics (with the Drinfeld associator which has multizeta values as coefficients): see [5], [6], [24] ou [26].
3. Resurgence theory and analytical invariants (in many cases, these invariants are expressed in term of series of multizetas values) : see [3] and [4]
4. the study of Feynman diagrams : see [5], [6] or [26].
5. the study of $\mathbb{P}^1 - \{0; 1; \infty\}$ (through the Grothendieck-Ihara program): see [23], [25], [27] for example.
6. the study of the "absolute Galois group": see [24] for example.

In regard of Conjecture 1, one of the important questions is the understanding of the relations between multizeta numbers. There are numerous relations between these numbers, coming in particular from their representation as iterated series or as iterated itegral. Let us remind what is the seond representation.

It is now a well-known fact that multizeta values has a representation as an iterated integral. This can be seen in the following way. If we consider the 1-differential forms

$$\omega_0 = \frac{dt}{t} \text{ and } \omega_1 = \frac{dt}{1-t} \ ,$$

the iterated integral

$$\mathcal{W}a^{\alpha_1,\cdots,\alpha_r} = \int_{0<t_1<\cdots<t_r<1} \omega_{\alpha_1}\cdots\omega_{\alpha_r}$$

is well defined when $(\alpha_1;\cdots;\alpha_r) \in \{0;1\}^r$ satisfied $\alpha_1 = 1$ and $\alpha_r = 0$. This allows us to defined a symmetr$\underline{\boldsymbol{a}}$l mould, denoted by $\mathcal{W}a^\bullet$.

It is easy to see that the moulds $\mathcal{Z}e^\bullet$ and $\mathcal{W}a^\bullet$ are related each others by :

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \mathcal{W}a^{1,0^{[s_r-1]},\cdots,1,0^{[s_1-1]}} \ ,$$

for all sequences $(s_1;\cdots;s_r) \in \text{seq}(\mathbb{N}^*)$ stisfying $s_1 \geq 2$.

Among others, the relation coming from the symmetr$\underline{\boldsymbol{a}}$lity and symmetr$\underline{\boldsymbol{e}}$lity relations are particularly important. These two types of relations allow us to express a product of two multizeta values as a $\mathbb{Q}$-linear combination of multizeta values in two different ways. One conjectures that these two families (up to a regularization process) spans all the other relations between these numbers (see [36] or [39]). This conjecture, out of reach today, would in particular show the absence of relations between multizeta values of different weights, and so the transcendence of the numbers $\zeta(s)$, $s \geq 2$.

1.3. On multitangent functions

In this article, we will present an algebra of functions, the algebra of multitangent functions, which is in a certain sense a good analogue of the algebra of the multizeta values. Before we give the definition, let us mention two ideas which underlie the definition of multitangent functions.

First, we know that one of the essential ideas of the explicit calculation of $\zeta(2n)$, where $n \in \mathbb{N}$, is a symmetrization of the set of summation, that is to say a transformation which allows us to transform a sum over $\mathbb{N}$ into a sum over $\mathbb{Z}$. Here, the transformation comes from the expansion of the cotangent function. By the same idea, we are able to compute numerous sums of the form $\displaystyle\sum_{m\in\mathbb{N}^*} \frac{\omega^{mr}}{m^r}$, where ω is a root of unity.

Consequently, it is a natural idea to try to symmetrize the summation simplex of multizeta values.

Next, some well-known ideas are interesting to stress out. One knows that working with numbers imposes a certain rigidity, while working with functions, which will be evaluated after to a particular point, gives more flexibility. One also knows that working with periodic functions gives us access to a whole panel of methods.

The simplest suggestion of a functional model of multizeta values is to consider the Hurwitz multizeta functions:

$$z \longmapsto \mathcal{H}e_+^{s_1,\cdots,s_r}(z) = \sum_{0<n_1<\cdots<n_r} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ ,$$

for all sequences of $(s_1,\cdots,s_r) \in \mathcal{S}_b^\star$. The advantage of these functions is to have a very simple link with the multizetas values:

$$\mathcal{H}e_+^{s_1,\cdots,s_r}(0) = \mathcal{Z}e^{s_1,\cdots,s_r} \ ,$$

where $(s_1,\cdots,s_r) \in \mathcal{S}_b^\star$.

Unfortunately, this choice seems not to be the best one, according to the previous remarks: these functions are not periodic and the set of summation is not symmetric... So, we are led to modify the model by considering the functions:

$$z \longmapsto \mathcal{T}e^{s_1,\cdots,s_r}(z) = \sum_{-\infty<n_1<\cdots<n_r<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ ,$$

for all sequences of $\mathcal{S}_{b,e}^\star = \{(s_1;\cdots;s_r) \in \text{seq}(\mathbb{N}^*)\,;\, s_1 \geq 2 \text{ and } s_r \geq 2\}$.

Obviously, these are 1-periodic functions and the set of summation is a symmetric set. Nevertheless, what is gained on one side is obviously lost on the other one: in spite of similar expressions, the link with multizeta values is not so clear. Indeed, this link does exist and is actually stronger than the one with Hurwitz multizeta functions (see §3 and §7.5.3) .

We are going to refer to these functions as "*multitangent functions*". The prefix "multi" characterizes the summation set in more than one variable; the suffix "tangent" comes from the link between Einsenstein series and the cotangent function. A more representative name would have been "multiple cotangent functions" or "multicotangent functions", but we preferred to simplify it by forgetting the syllable "co", which doesn't alter its quintessence.

To the best of our knowledge, this family of functions had never been studied from the point of view of special functions, even if it is an interesting and completely natural mathematical object. There are, actually, three good reasons to study such a family of functions, in an algebraic as well as in an analytical way:

1. The multitangent functions seem to have appeared for the first time in resurgence theory and holomorphic dynamics, in a book of Jean Ecalle (see [14], vol. 2 as well as [3] or the survey [4]). Consequently, these functions have some direct applications.
2. The multitangent functions are deeply linked to multizeta values, at least because of an evidence formal similarity. In a naive approach, we can raise the same questions as for multizeta values, but this time for multitangent functions.
3. The multitangent functions are a multidimensional generalization of the Eisenstein series, which have been used by Eisenstein to develop his theory of trigonometric functions in his famous article of 1847 (see [20] or [37] for a modern approach). So, interesting facts may emerge from this generalization.

1.4. Eisenstein series

The series considered by Eisenstein are defined for all $z \in \mathbb{C} - \mathbb{Z}$ by:

$$\varepsilon_k(z) = \sum_{m \in \mathbb{Z}} \frac{1}{(z+m)^k} \ ,$$

where $k \in \mathbb{N}^*$.

As Eisenstein himself said, "the fundamental properties of these simply-periodic functions reveal themselves through consideration of a single identity" (see [20]):

$$\frac{1}{p^2 q^2} = \frac{1}{(p+q)^2} \left(\frac{1}{p^2} + \frac{1}{q^2} \right) + \frac{2}{(p+q)^3} \left(\frac{1}{p} + \frac{1}{q} \right) .$$

From this, he would obtain some identities, which are non trivial at a first sight, between these series. About the ingenuity and the virtuosity of

Eisenstein, André Weil compared his work with one of the most difficult works, even today, of the last period of creation of Beethoven: the Diabelli variations. It is a work of art based from the most harmless theme which may be and which, during the variations following one another, will generate a prodigious and extremely rich musical universe which is full of delicacy, but also at the same time full of pianos and compositional virtuosity. The parallel to show the beauty of the results obtained by Eisenstein is crystal clear.

In his variations, Eisenstein obtained, in particular, the following relations:

$$\begin{aligned} {\varepsilon_2}^2(z) &= \varepsilon_4(z) + 4\zeta(2)\varepsilon_2(z) \ . && (1) \\ \varepsilon_3(z) &= \varepsilon_1(z)\varepsilon_2(z) \ . && (2) \\ 3\varepsilon_4(z) &= \varepsilon_2(z)^2 + 2\varepsilon_1(z)\varepsilon_3(z) \ . && (3) \end{aligned}$$

Eisenstein also proved that each of his series is in fact a polynomial with real coefficients in ε_1 . In our study of the algebraic relations between multitangent functions, we will find another proof of the relations (1) , (2) and (3) . These are particular cases of more general relations: the relation (1) is a mix of the relations of symmetr$\underline{\boldsymbol{e}}$lity and of the reduction of multitangent function into monotangent functions, while relations (2) and (3) are the archetype of relations of symmetr$\underline{\boldsymbol{e}}$lity for divergent multitangent functions.

Let us mention that although Weil preferred in [37] the notation ε_k in honour of Eisenstein, from now on, we will systematically use the notation $\mathcal{T}e^s$ coming from multitangent functions. Also, in connection with the name "multitangent functions", we shall name them "monotangent functions" in order to mean the sequence is of length one.

1.5. Results proved in this article

Because of the three fundamental reasons evoked before, we have initiated a complete study of multitangent functions. The first important properties (see §2 and 5) are:

Property 1. 1. *The mould*[1] $\mathcal{T}e^\bullet$ *of multitangent function is a symmetr$\underline{e}$l mould, that is, for all sequences $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ in $\mathcal{S}^\star_{b,e}$, we have*

$$\mathcal{T}e^{\underline{\boldsymbol{\alpha}}}(z)\mathcal{T}e^{\underline{\boldsymbol{\beta}}}(z) = \sum_{\gamma \in sh\underline{e}(\underline{\boldsymbol{\alpha}},\underline{\boldsymbol{\beta}})} \mathcal{T}e^{\underline{\boldsymbol{\gamma}}}(z) \ , \text{ where } z \in \mathbb{C} - \mathbb{Z} \ .$$

2. *There are many $\mathbb{Q}$-linear relations between of multitangent functions.*

In one word, the first point allows us to find more than one half of all the known algebraic relations between multizeta values (the relation of symmetr$\underline{\boldsymbol{e}}$lity and a few of double-shuffle relations), while the second point allows us to find exactly the others algebraic relations between multizeta values (the relation of symmetr$\underline{\boldsymbol{a}}$lity and the other double-shuffle relations).

We will also see that each multitangent function has a simple expression in term of multizeta values and monotangent functions. We will also determine that a sort of converse is true: the algebra of multitangent functions is a module over the algebra of multizeta values. The first property is called the "*reduction into monotangent functions*" (see §3), while the second property is called "*projection onto multitangent functions*" (see §4).

Theorem 1. *(Reduction into monotangent functions)*
For all sequences $\underline{\mathbf{s}} = (s_1; \cdots; s_r) \in seq(\mathbb{N}^*)$, there exists an explicit family $(z_k^{\underline{\mathbf{s}}})_{k \in [\![0 ; M]\!]} \in \left(Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\sigma}})_{\underline{\sigma} \in \mathcal{S}_b^\star}\right)^{M+1}$, with $M = \max\limits_{i \in [\![1 ; r]\!]} s_i$, such that:

$$\mathcal{T}e^{\underline{\mathbf{s}}}(z) = z_0^{\underline{\mathbf{s}}} + \sum_{k=1}^{max(s_1;\cdots;s_r)} z_k^{\underline{\mathbf{s}}} \mathcal{T}e^k(z) \text{ , where } z \in \mathbb{C} - \mathbb{Z} \text{ .}$$

Moreover, if $\underline{\mathbf{s}} \in \mathcal{S}_{b,e}^\star$, then $z_0^{\underline{\mathbf{s}}} = z_1^{\underline{\mathbf{s}}} = 0$.

From an algebraic point of view, let us define some algebras related to the first point of the property 1:

$$\mathcal{M}ZV_{CV} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{Z}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}_b^\star} \quad \text{and} \quad \mathcal{M}ZV_{CV,p} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{Z}e^{\underline{\mathbf{s}}}\right)_{\substack{\underline{\mathbf{s}} \in \mathcal{S}_b^\star \\ ||\underline{\mathbf{s}}||=p}} ,$$

$$\mathcal{H}MZF_{CV,+} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}_b^\star} \quad \text{and} \quad \mathcal{H}MZF_{CV,+,p} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_+^{\underline{\mathbf{s}}}\right)_{\substack{\underline{\mathbf{s}} \in \mathcal{S}_b^\star \\ ||\underline{\mathbf{s}}||=p}} ,$$

$$\mathcal{H}MZF_{CV,-} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_-^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}_e^\star} \quad \text{and} \quad \mathcal{H}MZF_{CV,-,p} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_-^{\underline{\mathbf{s}}}\right)_{\substack{\underline{\mathbf{s}} \in \mathcal{S}_e^\star \\ ||\underline{\mathbf{s}}||=p}} ,$$

$$\mathcal{M}TGF_{CV} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{T}e^{\underline{\mathbf{s}}}\right)_{\underline{\mathbf{s}} \in \mathcal{S}_{b,e}^\star} \quad \text{and} \quad \mathcal{M}TGF_{CV,p} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{T}e^{\underline{\mathbf{s}}}\right)_{\substack{\underline{\mathbf{s}} \in \mathcal{S}_b^\star \\ ||\underline{\mathbf{s}}||=p}} ,$$

$$\mathcal{H}MZV_{CV,\pm} = \mathrm{Vect}_{\mathbb{Q}}\left(\mathcal{H}e_+^{\underline{\mathbf{s}}^1}\mathcal{H}e_-^{\underline{\mathbf{s}}^2}\right)_{\substack{\underline{\mathbf{s}}^1 \in \mathcal{S}_b^\star \\ \underline{\mathbf{s}}^2 \in \mathcal{S}_e^\star}} ,$$

[1]See the appendix for a brief introduction to mould notations and calculus.

where $p \in \mathbb{N}$, $\mathcal{S}_e^\star = \{(s_1; \cdots ; s_r) \in \text{seq}(\mathbb{N}^*) \, ; s_r \geq 2\}$ and the weight of a sequence $\underline{\mathbf{s}} = (s_1, \cdots, s_r) \in \mathbb{N}^\star$ is defined by:

$$||\underline{\mathbf{s}}|| = s_1 + \cdots + s_r \ .$$

Using this notation, we can state the following:

Theorem 2. *(Projection onto multitangent functions)*
The following assertions are equivalent:

1. *For all non negative integer* p , $\mathcal{M}TGF_{CV,p} = \bigoplus_{k=0}^{p-2} \mathcal{M}ZV_{CV,p-k} \cdot \mathcal{T}e^k$.
2. $\mathcal{M}TGF_{CV}$ *is a* $\mathcal{M}ZV_{CV}$*-module.*
3. *For all sequence* $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_e^\star$, $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2 \in \mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$.

We will see that the duality reduction/projection is a very important process (see §5). In one sentence, we can sum up all the study by saying:

"the algebra of multitangent functions is a functional analogue of the algebra of multizeta values: each result on multizeta values has a translation in the algebra of multitangent functions, and conversely."

We can also sum up this study by the following diagram:

$$\begin{tikzcd}
\mathcal{M}ZV_{CV} \arrow[d, shift right, dotted, "\text{projection}"'] & \mathcal{H}MZF_{+,CV} \arrow[l, "\text{evaluation at 0}"'] \arrow[d, hook] \\
\mathcal{M}TGF_{CV} \arrow[u, shift right, "reduction"'] \arrow[r, hook, "\text{trifactorization}"] & \mathcal{H}MZF_{\pm,CV}
\end{tikzcd}$$

In this diagram, which will be constructed throughout the article as an evolutive one, the trifactorization is an explicit expression of each multitangent function in term of Hurwitz multizeta functions. Using it, we will be able to regularize divergent multitangent functions (see §7), that is to say multitangent functions depending of a sequence $\underline{\mathbf{s}} \in \mathbb{N}_1^\star - \mathcal{S}_{b,e}^\star$. This explains that we allow such sequences in the Theorem 7 .

We will also see some analytical properties of the multitangent functions (see 6), such as their Fourier expansion or their upper bound on the half-plane, which would be useful for direct applications. Finally, we will perform some explicit calculation (see section 6) to obtain:

Property 2. *Let $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$.*
Let us also set E the floor function and define for $(k;n) \in \mathbb{N} \times \mathbb{N}^*$ the functions $t_{k,n}$ by:

$$\forall x \in \mathbb{R} \ , \ t_{k,n}(x) = \begin{cases} \cos^{(n-1)}(x) & , \text{ if } k \text{ is odd.} \\ \sin^{(n-1)}(x) & , \text{ if } k \text{ is even.} \end{cases}$$

Then, we consider the moulds $sg^\bullet$, $e^\bullet$ and $s^\bullet$, which are $\mathbb{C}$-valued and defined over the alphabet $\Omega = \{1; -1\}$:

$$sg^{\underline{\varepsilon}} = \prod_{k=1}^{n} \varepsilon_k, \qquad s^{\underline{\varepsilon}} = \sum_{k=1}^{n} \varepsilon_k, \qquad e^{\underline{\varepsilon}} = \sum_{k=1}^{n} \varepsilon_k e^{(2k-1)\frac{i\pi}{n}} .$$

Then, for all $z \in \mathbb{C} - \mathbb{Z}$, we have:

$$\mathcal{T}e^{n^{[k]}}(z) = \frac{(-1)^{n-1+E(\frac{kn+1}{2})}\pi^{kn}}{(kn)!(2\sin(\pi z))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}}(e^{\underline{\varepsilon}})^{kn} t_{kn,n}(s^{\underline{\varepsilon}}\pi z) \ .$$

2. Definition of the multitangent functions and its first properties

Let us begin with a general lemma which immediately shows, if a certain condition holds, that a mould defined as an iterated sum of holomorphic functions is a symmetr$\underline{e}$l mould valued in the algebra of holomorphic functions. This will give us the analytical definition of multitangent functions, but this will also be useful for dealing with the Hurwitz multizeta functions in the sequel. In the case of multizeta values, it gives the well-known convergence criterion.

As a consequence of this lemma, we will obtain four elementary, but fundamental, properties of multitangent functions.

2.1. A lemma on symmetr$\underline{e}$l moulds

This is a first version of this lemma, for classical sums, that is to say when the summation index varies from N to $+\infty$, when $N \in \mathbb{N}$:

Lemma 1. *(Definition of symmetrel moulds, version* 1.*)*

Let $\mathcal{U}$ be an open set of $\mathbb{C}$, $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions on $\mathcal{U}$ and $N \in \mathbb{N}$.
We assume that for all compact subsets K of $\mathcal{U}$,

$$||f_n||_{\infty,K} \underset{n\longrightarrow+\infty}{=} \mathcal{O}\left(\frac{1}{n}\right) .$$

Then, for all sequences $\underline{\mathbf{s}} \in seq(\mathbb{C}) - \{\emptyset\}$, of length r, satisfying

$$\begin{cases} \Re e\,(s_1) > 1, \\ \quad\vdots \\ \Re e\,(s_1 + \cdots + s_r) > r, \end{cases} \tag{4}$$

we have:

1. *The function* $Fe_N^{\mathbf{s}} : \begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \displaystyle\sum_{N<n_r<\cdots<n_1<+\infty} (f_{n_1}(z))^{s_1}\cdots(f_{n_r}(z))^{s_r} \end{array}$
   is well defined on $\mathcal{U}$.
2. *$Fe_N^{\mathbf{s}}$ is holomorphic on $\mathcal{U}$ and for all $z \in \mathcal{U}$:*

$$(Fe_N^{\mathbf{s}})'(z) = \sum_{N<n_r<\cdots<n_1<+\infty} \left(\prod_{i=1}^{r} (f_{n_i})^{s_i}\right)' .$$

Moreover, $Fe_N^{\bullet}$ a symmetrel mould defined on the set of sequences $\underline{\mathbf{s}} \in seq(\mathbb{C})$ satisfying (4), valued in $\mathcal{H}(\mathcal{U})$, if we set $Fe_N^{\emptyset} = 1$.

All the interest of this lemma is to give in one result an absolute convergence criterion for iterated sum as well as to give the symmetr$\underline{\boldsymbol{e}}$l character. So, from now on, each time we will consider a mould which satisfies the hypothesis of this lemma and its second version, we will just say it will be a symmetr$\underline{\boldsymbol{e}}$l mould without further explanation.

In the following proof, we will just indicate the reason of the conditions imposed to obtain absolute convergence of the series and the holomorphy of $Fe_N^{\mathbf{s}}$. Nevertheless, we will prove in detail the symmetr$\underline{\boldsymbol{e}}$lity of $Fe_N^{\mathbf{s}}$ even if it is also elementary and a direct consequence of a calculation made by Michael Hoffman (see [22], page 485) .

Proof. Points 1 and 2 can be proved simultaneously because the series which defines $Fe_N^{\underline{\mathbf{s}}}$ is normally convergent on every compact subset of $\mathcal{U}$. Thus, the classical theorem of Weierstrass for limit of sequences of holomorphic functions concludes the proposition. Actually, if K is a compact subset of $\mathcal{U}$, there exists $M_K > 0$ such that for all $n \in \mathbb{N}$:

$$||f_n||_{\infty,K} \leq \frac{M_K}{n+1} .$$

Besides, for $z \in K$, we can write $f_n(z) = r_n(z)e^{i\theta_n(z)}$ with $r_n(z) \geq 0$ and $\theta_n(z) \in]-\pi;\pi]$. Thus: $|f_n(z)^i| = e^{-\theta_n(z)} \in [0;e^{\pi}]$.

In particular, we obtain: $|f_n(z)^s| \leq \dfrac{{M_K}^{\Re e\, s} e^{\pi \Im m\, s}}{(n+1)^{\Re e\, s}}$. Therefore, there exists a constant $C > 0$ satisfying:

$$\begin{aligned} \sum_{N<n_r<\cdots<n_1} ||f_{n_1}^{s_1}\cdots f_{n_r}^{s_r}||_{\infty,K} &\leq \sum_{N<n_r<\cdots<n_1} \frac{C}{(n_1+1)^{\Re e\, s_1}\cdots(n_r+1)^{\Re e\, s_r}} \\ &\leq C\mathcal{Z}e^{\Re e\, s_1,\cdots,\Re e\, s_r} < +\infty . \end{aligned}$$

Let $N \in \mathbb{N}$. We will show the symmetr$\underline{\boldsymbol{e}}$lity of $Fe_N^{\bullet}(z)$ by an induction process. To be precise, we will show the equality $Fe_N^{\underline{\mathbf{s}}^1}(z)Fe_N^{\underline{\mathbf{s}}^2}(z) = \displaystyle\sum_{\gamma \in sh\underline{\boldsymbol{e}}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} Fe_N^{\gamma}(z)$, with sequences $\underline{\mathbf{s}}^1$ and $\underline{\mathbf{s}}^2$ of $\text{seq}(\mathbb{C})$ satisfying (4) . The induction is over the integer $l(\underline{\mathbf{s}}^1) + l(\underline{\mathbf{s}}^2)$.

Before starting[2], let us remind that, if $\underline{\mathbf{s}} \in \text{seq}(\mathbb{C})$ satisfy (4), then, by definition of $Fe_N^{\bullet}$, we have:

$$Fe_N^{\underline{\mathbf{s}}} = \sum_{p>N} (f_p)^{s_k} Fe_p^{\underline{\mathbf{s}}^{<r}} .$$

Anchor step: Let $(\underline{\mathbf{u}};\underline{\mathbf{v}}) \in (\text{seq}(\mathbb{C}))^2$ satisfying (4) and $l(\underline{\mathbf{u}}) = l(\underline{\mathbf{v}}) = 1$.

Writing $\underline{\mathbf{u}} = (u)$ and $\underline{\mathbf{v}} = (v)$, we successively have, for $N \in \mathbb{N}$:

$$\begin{aligned}
Fe_N^{\underline{\mathbf{u}}} Fe_N^{\underline{\mathbf{v}}} &= \left(\sum_{p>N} (f_p)^u\right)\left(\sum_{q>N} (f_q)^v\right) \\
&= \sum_{p>q>N} (f_p)^u (f_q)^v + \sum_{p=q>N} (f_p)^u (f_q)^v + \sum_{q>p>N} (f_p)^u (f_q)^v \\
&= \sum_{q>N} (f_q)^v Fe_q^u(z) + Fe_N^{u+v}(z) + \sum_{p>N} (f_p)^u Fe_p^v(z) \\
&= Fe_N^{u,v} + Fe_N^{u+v} + Fe_N^{v,u} = \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}};\underline{\mathbf{v}})} Fe_N^{\underline{\mathbf{w}}} \ .
\end{aligned}$$

Thus, the property is initialised.

<u>Induction step:</u> Let us suppose that the result is proved for all sequences $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ of seq($\mathbb{C}$) satisfying (4) and such that $l(\underline{\mathbf{u}}) + l(\underline{\mathbf{v}}) \geq 2$.

In the same way as for length 1 and by the use of the induction hypothesis, if $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ are of length k and l respectively, we successively have:

[2] Let us remind that if $\underline{\mathbf{s}} = (s_1, \cdots, s_r)$, the notation $\underline{\mathbf{s}}^{\leq k}$ refers to the sequence $(s_1, \cdots, s_k)$ of the first k terms of $\underline{\mathbf{s}}$, while $\underline{\mathbf{s}}^{<k}$ refers to the empty sequence when $k = 1$ or the sequence of the first $(k-1)$ terms of $\underline{\mathbf{s}}$ if $k \geq 2$.
For this notation, see the annex on mould calculus.

$$
\begin{aligned}
Fe_N^{\underline{\mathbf{u}}} Fe_N^{\underline{\mathbf{v}}} &= \sum_{p>q>N} (f_p)^{u_k} (f_q)^{v_l} Fe_p^{\underline{\mathbf{u}}^{\le k-1}} Fe_q^{\underline{\mathbf{v}}^{\le l-1}} + \sum_{n=p=q>N} (f_n)^{u_k+v_l} Fe_n^{\underline{\mathbf{u}}^{\le k-1}} Fe_n^{\underline{\mathbf{v}}^{\le l-1}} \\
&\quad + \sum_{q>p>N} (f_p)^{u_k} (f_q)^{v_l} Fe_p^{\underline{\mathbf{u}}^{\le k-1}} Fe_q^{\underline{\mathbf{v}}^{\le l-1}} \\
&= \sum_{q>N} (f_q)^{v_l} Fe_q^{\underline{\mathbf{v}}^{\le l-1}} Fe_q^{\underline{\mathbf{u}}} + \sum_{n>N} (f_n)^{u_k+v_l} Fe_n^{\underline{\mathbf{u}}^{\le k-1}} Fe_n^{\underline{\mathbf{v}}^{\le l-1}} \\
&\quad + \sum_{p>N} (f_p)^{u_k} Fe_p^{\underline{\mathbf{u}}^{\le k-1}} Fe_p^{\underline{\mathbf{v}}} \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}};\underline{\mathbf{v}}^{\le l-1})} \left(\sum_{q>N} (f_q)^{v_l} Fe_q^{\underline{\mathbf{w}}} \right) + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}^{\le k-1};\underline{\mathbf{v}}^{\le l-1})} \left(\sum_{n>N} (f_n)^{u_k+v_l} Fe_n^{\underline{\mathbf{w}}} \right) \\
&\quad + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}^{\le k-1};\underline{\mathbf{s}})} \left(\sum_{p>N} (f_p)^{u_k} Fe_p^{\underline{\mathbf{w}}} \right) \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}};\underline{\mathbf{v}}^{\le l-1})\cdot v_l} Fe_N^{\underline{\mathbf{w}}} + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}^{\le k-1};\underline{\mathbf{v}}^{\le l-1})\cdot (u_k+v_l)} Fe_N^{\underline{\mathbf{w}}} + \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}}^{\le k-1};\underline{\mathbf{v}})\cdot u_k} Fe_N^{\underline{\mathbf{w}}} \\
&= \sum_{\underline{\mathbf{w}} \in sh\underline{e}(\underline{\mathbf{u}};\underline{\mathbf{v}})} Fe_N^{\underline{\mathbf{w}}} \ .
\end{aligned}
$$

Thus, by induction, for all sequences $\underline{\mathbf{s}}^1$ and $\underline{\mathbf{s}}^2$ of $\mathrm{seq}(\mathbb{C})$ satisfying (4), we have:

$$
Fe_N^{\underline{\mathbf{s}}^1} Fe_N^{\underline{\mathbf{s}}^2} = \sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} Fe_N^{\underline{\boldsymbol{\gamma}}} \text{ where } N \in \mathbb{N} \ .
$$

In other words, for all $z \in \mathcal{U}$ and $N \in \mathbb{N}$, the mould $Fe_N^{\bullet}(z)$ is a symmetr$\underline{\boldsymbol{e}}$l one.

We obtain, as a corollary, the second version of this lemma, but for sums over all integers:

Lemma 2. *(Definition of symmetrel moulds, version* 2.*)*

Let $\mathcal{U}$ *be an open set of* $\mathbb{C}$, $(f_n)_{n\in\mathbb{Z}}$ *a sequence of holomorphic functions*

on $\mathcal{U}$.
We assume that for all compact subsets K *of* $\mathcal{U}$*,*

$$||f_n||_{\infty,K} \underset{n\longrightarrow\pm\infty}{=} \mathcal{O}\left(\frac{1}{|n|}\right) .$$

1. *Then, for all sequences* $\underline{\mathbf{s}} \in seq(\mathbb{C}) - \{\emptyset\}$*, of length* r*, satisfying*

$$\forall k \in [\![\, 1\, ;\, r\,]\!], \begin{cases} \Re e\,(s_1 + \cdots + s_k) > k, \\ \Re e\,(s_r + \cdots + s_{r-k+1}) > k, \end{cases} \tag{5}$$

the function $\begin{array}{rccl} Fe^{\underline{\mathbf{s}}}: & \mathcal{U} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \displaystyle\sum_{-\infty<n_r<\cdots<n_1<+\infty} \big(f_{n_1}(z)\big)^{s_1} \cdots \big(f_{n_r}(z)\big)^{s_r} \end{array}$
is well defined on $\mathcal{U}$*, holomorphic on* $\mathcal{U}$ *and satisfy:*

$$\forall z \in \mathcal{U}, (Fe^{\underline{\mathbf{s}}})'(z) = \sum_{-\infty<n_r<\cdots<n_1<+\infty} \left(\prod_{i=1}^{r} (f_{n_i}(z))^{s_i}\right)' .$$

2. *Moreover,* $Fe^{\bullet}_N$ *a symmetrel mould defined on the set of sequences* $\underline{\mathbf{s}} \in seq(\mathbb{C})$ *satisfying (12), valued in* $\mathcal{H}(\mathcal{U})$ *, if we set* $Fe^{\emptyset}_N = 1$ *.*

Proof. The lemma for definition of symmetrel moulds, version 1, has several consequences:
• The mould $Fe^{\bullet}$ can be factorised:

$$Fe^{\bullet}(z) = Fe^{\bullet}_{+}(z) \times Ce^{\bullet}(z) \times Fe^{\bullet}_{-}(z) , \tag{6}$$

where, for all $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{C})$ satisfying (5), the functions $Fe^{\underline{\mathbf{s}}}_{+}$, $Ce^{\underline{\mathbf{s}}}$ and $Fe^{\underline{\mathbf{s}}}_{-}$ are defined on $\mathcal{U}$ by:

$$\begin{array}{rcl} Fe^{\underline{\mathbf{s}}}_{+}(z) & = & \displaystyle\sum_{0<n_k<\cdots<n_1<+\infty} \prod_{i=1}^{l(\underline{\mathbf{s}})} (f_{n_i}(z))^{s_i} . \\ Ce^{\underline{\mathbf{s}}}(z) & = & \begin{cases} 1 & \text{, if } l(\underline{\mathbf{s}}) = 0 . \\ (f_0(z))^{s_1} & \text{, if } l(\underline{\mathbf{s}}) = 1 . \\ 0 & \text{, otherwise.} \end{cases} \\ Fe^{\underline{\mathbf{s}}}_{-}(z) & = & \displaystyle\sum_{-\infty<n_r<\cdots<n_k<0} \prod_{i=k}^{l(\underline{\mathbf{s}})} (f_{n_i}(z))^{s_i} . \end{array}$$

Actually, let us set $Fe^{\underline{\mathbf{s}}}_{+0}(z) = \sum_{0\leq n_r<\cdots<n_1<+\infty} \prod_{i=k}^{l(\underline{\mathbf{s}})} (f_{n_i}(z))^{s_i}$ where $z \in \mathcal{U}$ and $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{C})$ satisfying (5) . In the definition of $Fe^{\bullet}_{+0}(z)$, we obtain by isolating the summation index n_r when it is equal to 0:

$$\begin{aligned} Fe^{\underline{\mathbf{s}}}_{+0}(z) &= \sum_{0=n_r<n_{r-1}<\cdots<n_1<+\infty} \prod_{i=k}^{l(\underline{\mathbf{s}})} (f_{n_i}(z))^{s_i} + \sum_{0<n_r<n_{r-1}<\cdots<n_1<+\infty} \prod_{i=k}^{l(\underline{\mathbf{s}})} (f_{n_i}(z))^{s_i} \\ &= (f_0(z))^{s_r}\, Fe^{\underline{\mathbf{s}}^{\leq r-1}}_{+}(z) + Fe^{\underline{\mathbf{s}}}_{+}(z) = \left(Fe^{\bullet}_{+}(z) \times Ce^{\bullet}(z)\right)^{\underline{\mathbf{s}}}. \end{aligned}$$

In the same way, we show that $Fe^{\bullet}(z) = Fe^{\bullet}_{+0}(z) \times Fe^{\bullet}_{-}(z)$, which implies the trifactorisation (6) .

• Since $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{C})$ satisfies (5), $\underline{\mathbf{s}}$ and $\overleftarrow{\underline{\mathbf{s}}}$ satisfy (4) . The lemma for definition of symmetrel moulds, version 1, shows us that the functions $Fe^{\underline{\mathbf{s}}^{\leq k}}_{+}$ and $Fe^{\underline{\mathbf{s}}^{\geq k}}_{-}$ are well defined and holomorphic on $\mathcal{U}$ and that their derivatives can be calculated by a term by term process.

Thus, $Fe^{\underline{\mathbf{s}}}$ is well defined and holomorphic on $\mathcal{U}$, with a derivative which is the summation of the summand derivatives.

• Moreover, according to the first version of this lemma, $Fe^{\bullet}_{+}$ and $Fe^{\bullet}_{-}$ are symmetr$\underline{e}$l moulds, as well as $Ce^{\bullet}$. Since the mould product of symmetr$\underline{e}$l moulds defines a symmetr$\underline{e}$l mould, we deduce that $Fe^{\bullet}$ is a symmetr$\underline{e}$l mould for all $z \in \mathcal{U}$.

2.2. Application: definition of multitangent functions

Let us consider $\mathcal{U} = \mathbb{C} - \mathbb{Z}$ and for $n \in \mathbb{Z}$, the functions

$$\begin{array}{rccc} f_n : & \mathcal{U} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \dfrac{1}{n+z} \end{array} .$$

It is clear that, for all compact subsets K of $\mathbb{C} - \mathbb{Z}$,

$$||f_n||_{\infty,K} \underset{n\longrightarrow\pm\infty}{=} \mathcal{O}\left(\frac{1}{|n|}\right) .$$

The lemma for definition of symmetrel moulds, version 2, allows us to define a symmetr$\underline{e}$l mould, denoted $\mathcal{T}e^{\bullet}$, defined by:

$$\begin{array}{rcll} \mathcal{T}e^{\underline{\mathbf{s}}}: & \mathbb{C}-\mathbb{Z} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \displaystyle\sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} . \end{array}$$

This mould, which will be called *the mould of multitangent functions*, is defined, a priori, for all sequences

$$\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e} = \left\{\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) ; s_1 \geq 2 \text{ and } s_{l(\underline{\mathbf{s}})} \geq 2\right\}$$

and is valued in the algebra of holomorphic functions defined on $\mathbb{C}-\mathbb{Z}$.

2.3. First properties of multitangent functions

Here are the first elementary properties satisfied by the multitangent functions. These are consequences of Lemma 2 or of a simple change of variables in the summations (third point):

Property 3.
1. *The function $\mathcal{T}e^{\underline{\mathbf{s}}}$ is well-defined for sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$.*
2. *The function $\mathcal{T}e^{\underline{\mathbf{s}}}$ is holomorphic on $\mathbb{C}-\mathbb{Z}$ for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$, it is a uniformly convergent series on every compact subset of $\mathbb{C}-\mathbb{Z}$ and satisfies, for all $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ and all $z \in \mathbb{C}-\mathbb{Z}$:*

$$\frac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z}(z) = -\sum_{i=1}^{l(\underline{\mathbf{s}})} s_i \mathcal{T}e^{s_1,\cdots,s_{i-1},s_i+1,s_{i+1},\cdots,s_{l(\underline{\mathbf{s}})}}(z) .$$

3. *For all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ and all $z \in \mathbb{C}-\mathbb{Z}$ we have:*

$$\mathcal{T}e^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||} \mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z) .$$

4. *For all $z \in \mathbb{C}-\mathbb{Z}$, $\mathcal{T}e^{\bullet}(z)$ is symmetr$\underline{e}$l , that is, for all sequences $(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}) \in (\mathcal{S}^{\star}_{b,e})^2$:*

$$\mathcal{T}e^{\underline{\boldsymbol{\alpha}}}(z)\mathcal{T}e^{\underline{\boldsymbol{\beta}}}(z) = \sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{e}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})} \mathcal{T}e^{\underline{\boldsymbol{\gamma}}}(z) .$$

We will speak respectively of the *differentiation property* and the *parity property* to refer to the formula of the second point and that of the third point.

3. Reduction into monotangent functions

The aim of this section is to show a non trivial link between multitangent functions and multizeta values. More precisely, we will show that all (convergent) multitangent functions can be expressed in terms of multizeta values and monotangent functions[3]. In order to do this, we will proceed classically, that is, we will perform a partial fraction expansion (in the variable z) and then sum then after a reorganisation of the terms.

Let us remark that this idea had already been mentioned by Jean Ecalle (cf [14], p. 429) .

3.1. A partial fraction expansion

Let us fix a positive integer r, a family of positive integers $\underline{\mathbf{s}} = (s_i)_{1 \leq i \leq r}$ and finaly a family of complex numbers $\underline{\mathbf{a}} = (a_i)_{1 \leq i \leq r}$, where the a_i are pairwise distinct. Let us also consider the rational fraction defined by

$$F_{\underline{\mathbf{a}},\underline{\mathbf{s}}}(X) = \frac{1}{(X+a_1)^{s_1} \cdots (X+a_r)^{s_r}} \ .$$

We know that the partial fraction expansion of $F_{\underline{\mathbf{a}},\underline{\mathbf{s}}}(X)$ can be written in the following way:

$$F_{\underline{\mathbf{a}},\underline{\mathbf{s}}}(X) = \sum_{i=1}^{r} \sum_{j=0}^{s_i - 1} \frac{1}{j!} \frac{\left(F_{\underline{\mathbf{a}}^{\leq i-1} \cdot \underline{\mathbf{a}}^{\geq i+1}, \underline{\mathbf{s}}^{\leq i-1} \cdot \underline{\mathbf{s}}^{\geq i+1}}\right)^{(j)}(-a_i)}{(X+a_i)^{s_i - j}} \ .$$

With the previous notations, an easy computation shows that, for all $k \in \mathbb{N}$, we have:

$$\frac{(-1)^k}{k!} F^{(k)}_{\underline{\mathbf{a}},\underline{\mathbf{s}}}(X) = \sum_{\substack{n_1, \cdots, n_r \geq 0 \\ n_1 + \cdots + n_r = k}} \frac{\binom{s_1+n_1-1}{s_1-1} \cdots \binom{s_r+n_r-1}{s_r-1}}{(X+a_1)^{s_1+n_1} \cdots (X+a_r)^{s_r+n_r}} \ .$$

To conclude this subsection, let us introduce three notations:

[3]Let us recall that a monotangent function is a multitangent function of length 1 .

$$\varepsilon_i^{\underline{\mathbf{s}},\underline{\mathbf{k}}} = (-1)^{s_1+\cdots+s_{i-1}+s_{i+1}+\cdots+s_r+k_1+\cdots+k_{i-1}+k_{i+1}+\cdots+k_r} ,$$

$${}^iD_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}(\underline{\mathbf{a}}) = \left(\prod_{l=1}^{i-1}(a_i - a_l)^{s_l+k_l}\right)\left(\prod_{l=i+1}^{r}(a_l - a_i)^{s_l+k_l}\right) ,$$

$${}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}} = \left(\prod_{l=1}^{i-1}(-1)^{k_l}\right)\left(\prod_{l=i+1}^{r}(-1)^{s_l}\right)\left(\prod_{\substack{l=1\\ l\neq i}}^{r}\binom{s_l+k_l-1}{s_l-1}\right) .$$

Of course, in the previous notations ${}^iD_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}(a)$ and ${}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}$, the sequence $\underline{\mathbf{k}}$ has the same length than $\underline{\mathbf{a}}$ and an i^{th} index which does not intervene.

So, we finally have the following partial fraction expansion:

$$F_{\underline{\mathbf{a}},\underline{\mathbf{s}}}(X) = \sum_{i=1}^{r}\sum_{k=0}^{s_i-1}\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r\geq 0}} \frac{\varepsilon_i^{\underline{\mathbf{s}},\underline{\mathbf{k}}}}{(X+a_i)^{s_i-k}}\frac{{}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}}{{}^iD_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}(\underline{\mathbf{a}})} , \tag{7}$$

3.2. Expression of a multitangent function in terms of multizeta values and monotangent functions

Plugging (7) in the definition of a multitangent function, we can exchange the multiple summation (from the definition of a multitangent) with the finite summation (from the partial fraction expansion), because of the absolute convergence, and then sum by decomposing the multiple summation into three terms. Then, the following are successively equal to $\mathcal{T}e^{\underline{\mathbf{s}}}(z)$, if $\underline{\mathbf{s}} \in \mathcal{S}_{b,e}^{\star}$:

$$\left(\sum_{i=1}^{r}\sum_{k=0}^{s_i-1}\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r\geq 0}}\sum_{-\infty<n_r<\cdots<n_1<+\infty}\right)\left(\frac{\varepsilon_i^{\underline{\mathbf{s}},\underline{\mathbf{k}}}}{(z+n_i)^{s_i-k}}\frac{{}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}}{{}^iD_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}(\underline{\mathbf{n}})}\right)$$

$$=\left(\sum_{i=1}^{r}\sum_{k=0}^{s_i-1}\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r\geq 0}}\sum_{n_i\in\mathbb{Z}}\sum_{\substack{(n_1;\cdots;n_{i-1})\in\mathbb{Z}^{i-1}\\ n_i<n_{i-1}<\cdots<n_1}}\sum_{\substack{(n_{i+1};\cdots;n_r)\in\mathbb{N}^{r-i}\\ n_r<\cdots<n_{i+1}<n_i}}\right)\left(\frac{\varepsilon_i^{\underline{\mathbf{s}},\underline{\mathbf{k}}}}{(z+n_i)^{s_i-k}}\frac{{}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}}{{}^iD_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}(\underline{\mathbf{n}})}\right)$$

$$=\left(\sum_{i=1}^{r}\sum_{k=0}^{s_i-1}\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r\geq 0}}\right)\left({}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}\sum_{n_i\in\mathbb{Z}}\left(\frac{\varepsilon_i^{\underline{\mathbf{s}},\underline{\mathbf{k}}}}{(z+n_i)^{s_i-k}}\left[\sum_{\substack{(n_{i+1};\cdots;n_r)\in\mathbb{Z}^{r-i}\\ -\infty\leq n_r<\cdots<n_{i+1}<n_i}}\frac{1}{{}^iD_{\underline{\mathbf{k}}^{\geq i}}^{\underline{\mathbf{s}}^{\geq i}}(\underline{\mathbf{n}})}\right]\left[\sum_{\substack{(n_1;\cdots;n_{i-1})\in\mathbb{Z}^{i-1}\\ n_i<n_{i-1}<\cdots<n_1\leq+\infty}}\frac{1}{{}^iD_{\underline{\mathbf{k}}^{\leq i}}^{\underline{\mathbf{s}}^{\leq i}}(\underline{\mathbf{n}})}\right]\right)\right).$$

So, we have the following relation:

$$\mathcal{T}e^{\underline{\mathbf{s}}}(z)=\sum_{i=1}^{r}\sum_{k=0}^{s_i-1}\mathcal{Z}_{i,k}^{\underline{\mathbf{s}}}\mathcal{T}e^{s_i-k}(z)\ ,$$

where

$$\mathcal{Z}_{i,k}^{\underline{\mathbf{s}}}=\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r\geq 0}}{}^iB_{\underline{\mathbf{k}}}^{\underline{\mathbf{s}}}\mathcal{Z}e^{s_r+k_r,\cdots,s_{i+1}+k_{i+1}}\mathcal{Z}e^{s_1+k_1,\cdots,s_{i-1}+k_{i-1}}\ .$$

The divergent monotangent $\mathcal{T}e^1 : z\longmapsto \dfrac{\pi}{\tan(\pi z)}$ seems to appear in this relation. Nevertheless, the $\mathcal{T}e^1$ coefficient is necessarily null. Indeed, it is not difficult to see that all (convergent) multitangent function decrease exponentially to 0when $z\longrightarrow +\infty$ with $\Re e\, z\neq 0$, for example . So, we obtain:

Theorem 3. *(Reduction into monotangent functions, version 1)*
For all sequence $\underline{\mathbf{s}}\in\mathcal{S}_{b,e}^{\star}$, we have:

$$\mathcal{T}e^{\underline{\mathbf{s}}}(z)=\sum_{i=1}^{r}\sum_{k=2}^{s_i}\mathcal{Z}_{i,s_i-k}^{\underline{\mathbf{s}}}\mathcal{T}e^{k}(z)\ \textit{where}\ z\in\mathbb{C}-\mathbb{Z}\ .$$

3.3. Tables of convergent multitangent functions

With a suitable computer algebra software, we can easily generate a table of multitangent functions up to a fixed weight. Different tables can be computed:

1. those given by the previous theorem ;
2. those obtained from the first ones, as soon as we have downloaded a table of exact values of the multizeta values (see [30] for this purpose) ;
3. those obtained from the first ones, as soon as we have downloaded a table of numerical values of the multizeta values (see [2] or [11] for this purpose) ;
4. those obtained from the first ones, after a linearization of products of multizeta values (the choice of linearization by symmetr$\underline{\boldsymbol{e}}$lity is more natural in this context than using the symmetr$\underline{\boldsymbol{a}}$lity) .

Table 1 contains some examples of such tables. Some boxes in it are empty, which means the expression is the same than in the previous column. Let us immediately remark that there are a lot of $\mathbb{Q}$-linear relations between multitangents and all of them all absolutely non trivial. Here are two of them which are easy to state, but the second one is still quite mysterious:

$$\mathcal{T}e^{2,1,2} = 0 \ . \tag{8}$$

$$3\mathcal{T}e^{2,2,2} + 2\mathcal{T}e^{3,3} = 0 \ . \tag{9}$$

We will study this in detail in Section 5.3.

3.4. Linear independence of monotangent functions

At this stage, let us authorize a little incursion in the world of the arithmetic of multitangent functions, a quite obscur world. We have the following lemma. Although it is a simple one, which admits many different proofs, it will be a fundamental lemma which will be used here and there repeatedly in this article.

Lemma 3. *The monotangent functions are $\mathbb{C}$-linearly independant.*

We give a proof based on the differentiation property of multitangent functions. It is possible to prove this using the Fourier coefficients of monotangent functions or by looking at the poles of monotangent functions, etc.

PROOF. Let us suppose the familly $(\mathcal{T}e^n)_{n\in\mathbb{N}^*}$ is not $\mathbb{C}$-free.

So, we would have acces to an integer $r \geq 2$, a r-tuple of integers $(n_1; \cdots; n_r)$ satisfying $0 < n_1 < \cdots < n_r$ and a r-tuple of non all zeros complex numbers $(\lambda_{n_1}; \cdots; \lambda_{n_r})$ such that:

$$\sum_{k=1}^{r} \lambda_{n_k} \mathcal{T}e^{n_k} = 0 \ .$$

Using the differentiation property of multitangent functions, we would obtain:

$$\sum_{k=1}^{r} \frac{(-1)^{n_k-1}\lambda_{n_k}}{(n_k-2)!} \frac{\partial^{n_k-1}\mathcal{T}e^1}{\partial z^{n_k-1}} = 0 \ .$$

So, $\mathcal{T}e^1$ would satisfy a linear differential equation with constant coefficients, and therefore could be written as a $\mathbb{C}$-linear combination of exponential polynomials. This would allow us to obtain an analytic continuation over all $\mathbb{C}$ of the cotangent function.

Because such an analytic continuation is impossible, we have proved that monotangent functions are $\mathbb{C}$-free. □

Since we have just seen that there exists a lot of linear relations between multitangent functions, we know that this lemma can not be extended to multitangent functions. In fact, since we know the Eisenstein relation $\mathcal{T}e^2(z)\mathcal{T}e^1(z) = \mathcal{T}e^3(z)$, we can affirm that monotangent functions are not algebraically independent, even if we restrict to convergent monotangent functions:

$$2\Big(\mathcal{T}e^3\Big)^2 = 3\mathcal{T}e^2\mathcal{T}e^4 - \Big(\mathcal{T}e^2\Big)^3 \ .$$

3.5. A first approach to algebraic structure of $\mathcal{MTGF}$

Recall that we have denoted by $\mathcal{MTGF}_{CV}$ the algebra, over the field of rational numbers, spanned by all the functions $\mathcal{T}e^{\underline{\mathbf{s}}}$, $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$. Now, we will be more precise: for $p \in \mathbb{N}$, we will denote by $\mathcal{MTGF}_{CV,p}$ the $\mathbb{Q}$-algebra spanned by all the functions $\mathcal{T}e^{\underline{\mathbf{s}}}$, with sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ of weight p . In the same way, we will denote by $\mathcal{MZV}_{CV,p}$ the $\mathbb{Q}$-algebra spanned by all the numbers $\mathcal{Z}e^{\underline{\mathbf{s}}}$, with sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b}$ of weight p . Then, $\mathcal{MZV}_{CV}$ will be the $\mathbb{Q}$-algebra spanned by all the numbers $\mathcal{Z}e^{\underline{\mathbf{s}}}$, with sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b}$.

So, the reduction into monotangent functions, together with the previous lemma, yields the following corollary[4]:

[4]The notation $E \cdot \alpha$ denotes the set $\{e \cdot \alpha; e \in E\}$.

Corollary 1. *For all* $p \in \mathbb{N}$, $\mathcal{MTGF}_{CV,p} \subseteq \displaystyle\bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k$.

Proof. Because of the symmetr$\underline{e}$lity of $\mathcal{Z}e^{\bullet}$, we know that each vector space $\mathcal{MVZ}_p$ is a $\mathbb{Q}$-algebra. So, the reduction process gives the inclusion $\mathcal{MTGF}_{CV,p} \subseteq \displaystyle\sum_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k$. But, from the previous lemma, it is quite clear that $\displaystyle\sum_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k = \bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k$. $\square$

3.6. Consequences

An easy consequence of this section is that each property on multitangent functions that will be proven will have implications on multizeta values. The process will often be like this: we express the fact we are studying in terms of multitangent functions, then we reduce all the multitangents into monotangent functions and finally use the $\mathbb{C}$-linear independence of the monotangent functions to conclude something on multizeta values.

The following diagram will evolve throughout the article to explain how the multitangent functions are linked to the multizeta values.

$$\begin{tikzcd} \mathcal{MZV}_{CV} \\ \mathcal{MTGF}_{CV} \arrow[u, "reduction"] \end{tikzcd}$$

Figure 1: Links between multizeta values and multitangent functions

To illustrate this idea, let us show how a calculation on multitangent functions, which will be done in section 8, gives us a calculation of multizeta values. For this purpose, let us consider the following formal power series:

$$\mathcal{Z}_2 \;=\; \sum_{p\geq 0} \mathcal{Z}e^{2^{[p]}} X^p \quad , \quad \mathcal{T}_2(z) \;=\; \sum_{p \geq 0} \mathcal{T}e^{2^{[p]}}(z) X^p \ .$$

The Property 14 will show that, in $\mathbb{C}[\![\sqrt{X}]\!]$, we have:

$$\mathcal{T}_2(z) = 1 + \sum_{k \geq 1} \frac{2^{2k-1}\pi^{2k-2}}{(2k)!} X^k \mathcal{T}e^2(z) = 1 + \frac{\mathrm{ch}(2\pi\sqrt{X}) - 1}{2\pi^2} \mathcal{T}e^2(z) \ . \quad (10)$$

On the other hand, the reduction into monotangent functions implies

$$\mathcal{T}_2(z) = 1 + X{\mathcal{Z}_2}^2 \mathcal{T}e^2(z) \ , \quad (11)$$

because we have, for all $p \in \mathbb{N}$: $\mathcal{T}e^{2^{[p]}} = \sum_{i=1}^{p} \mathcal{Z}e^{2^{[p-i]}} \mathcal{Z}e^{2^{[i-1]}} \mathcal{T}e^2(z)$.

From the last two equations, we therefore obtain: $\mathcal{Z}_2 = \dfrac{\mathrm{sh}^2(\pi\sqrt{X})}{\pi\sqrt{X}}$, that is:

$$\forall n \in \mathbb{N} \ , \ \mathcal{Z}e^{2^{[n]}} = \frac{\pi^{2n}}{(2n+1)!} \ .$$

4. Projection onto multitangent functions

4.1. A second approach to the algebraic structure of $\mathcal{MTGF}$

We have just seen that for all $p \in \mathbb{N}$:

$$\mathcal{MTGF}_{CV,p} \subseteq \bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k \ .$$

The table of convergent multitangents that we have established up to weight 18 shows that the inclusion is in fact an equality. This is why we conjecture that the equality holds for all $p \in \mathbb{N}$.

Conjecture 2. *For all $p \in \mathbb{N}$, $\mathcal{MTGF}_{CV,p} = \bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k$.*

4.2. A structure of $\mathcal{MZV}$-module

Recall the notation $\mathcal{S}_b^\star = \{\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*); \underline{\mathbf{s}} = \emptyset \text{ or } s_1 \geq 2\}$.

If Conjecture 2 is true, then $\mathcal{MZV}_p \cdot \mathcal{T}e^q$ can be seen as a subset of $\mathcal{MTGF}_{CV,p+q}$ for all integers $(p;q) \in \mathbb{N}^2$. This lead us to the following conjecture:

Conjecture 3. 1. *For all sequences $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_e^\star$ and $\underline{\mathbf{s}} \in \mathcal{S}_{b,e}^\star$, we have:*

$$\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^{\underline{\mathbf{s}}} \in \mathcal{MTGF}_{CV,||\underline{\boldsymbol{\sigma}}||+||\underline{\mathbf{s}}||} \ .$$

2. *$\mathcal{MTGF}_{CV}$ is a $\mathcal{MZV}_{CV}$-module.*

Of course, the second point is a direct consequence of the first one, but it is the one which interests us in theory ; in practice, we will favour the first one because of the weight homogeneity. What is important is the equivalence between the conjectures 2 and 3:

Property 4. *Conjecture 2 is equivalent to Conjecture 3.*

Proof. 1. Let us suppose that Conjecture 2 holds.

Thus, according to the reduction into monotangent functions and the symmetr$\underline{e}$lity of the mould $\mathcal{Z}e^\bullet$, it follows, for $(\underline{\boldsymbol{\sigma}};\underline{\mathbf{s}}) \in \mathcal{S}_e^\star \times \mathcal{S}_{b,e}^\star$, that

$$\begin{aligned}\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^{\underline{\mathbf{s}}} &= \sum_{k=2}^{\max(s_1;\cdots;s_r)} \left(\sum_i c_i \mathcal{Z}e^{\underline{\mathbf{s}}_{i,k}}\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\right)\mathcal{T}e^k, \text{where } \underline{\mathbf{s}}_{i,k} \in \mathcal{S}_b^\star \ , \\ &= \sum_{k=2}^{\max(s_1;\cdots;s_r)} \left(\sum_i c_i \mathcal{Z}e^{\underline{\boldsymbol{\sigma}}_{i,k}}\right)\mathcal{T}e^k, \text{where } \underline{\boldsymbol{\sigma}}_{i,k} \in \mathcal{S}_b^\star \ .\end{aligned}$$

According to Conjecture 3, we are now able to write each term of the form $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}_{i,k}}\mathcal{T}e^k$ in $\mathcal{MTGF}_{CV,||\underline{\boldsymbol{\sigma}}||+||\underline{\mathbf{s}}||}$ that is, $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^{\underline{\mathbf{s}}} \in \mathcal{MTGF}_{CV,||\underline{\boldsymbol{\sigma}}||+||\underline{\mathbf{s}}||}$. This concludes that $\mathcal{MTGF}_{CV}$ is a $\mathcal{MZV}_{CV}$-module.

2. Let us suppose that Conjecture 3 holds.

According to Conjecture 3, for all sequences $\underline{\mathbf{s}} \in \mathcal{S}_b^\star$ and all integer $k \geq 2$, we are able to express $\mathcal{Z}e^{\underline{\mathbf{s}}}\mathcal{T}e^k$ in $\mathcal{MTGF}_{CV,||\underline{\mathbf{s}}||+k}$. In other words, for all $p \in \mathbb{N}$:

$$\bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k \subseteq \mathcal{MTGF}_{CV,p} \ .$$

We now conclude to the equality

$$\mathcal{M}TGF_{CV,p} = \bigoplus_{k=0}^{p-2} \mathcal{M}ZV_{CV,p-k} \cdot \mathcal{T}e^k$$

for all $p \in \mathbb{N}$ from Corollary 1 $\square$

The first point of Conjecture 3 can be a bit reduced. We will restrict to the following statement:

Conjecture 4. *For all sequences* $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_b^\star$, $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2 \in \mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$.

From the differentiation property, it is easy to see that we have not lost information.

Property 5. *Conjecture 4 is equivalent to Conjectures 2 and 3.*

Proof. It is sufficient to show that Conjecture 4 implies the first point of Conjecture 3.
So, assume that Conjecture 4 holds ; that is to say that we are able to express each expression of the type $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2$ in $\mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$. Using the differentiation property, we find explicitly an expression of $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^k$ in $\mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+k}$ for all $k \geq 2$. In other words, we have proved the first point of Conjecture 3, which is the desired conclusion. $\square$

4.3. About the projection of a multizeta value onto multitangent functions

For $k \in \mathbb{N}$, $k \geq 2$, an explicit expression of $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^k \in \mathcal{M}TGF_{2,||\underline{\boldsymbol{\sigma}}||+k}$ will be called a projection of the multizeta $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}$ onto the space of multitangent functions, or a projection onto multitangents to shorten the terminology.

4.3.1. A converse of the reduction into monotangent functions

The relations of projections onto multitangent functions can be considered as a converse of the reduction into monotangent functions in the following sense: according to Conjecture 3, each property that will be proven on multizeta values will have implications on multitangent functions.

The process will often be like this: we express the fact we are studying in terms of multizeta functions, then we multiply the different relations by a monotangent function and finally project all of these onto multitangent functions to conclude something on multitangent functions.

The diagram on the figure 2 completes the figure 1. An arrow indicates a link between two algebras while an arrow in dotted lines indicates a hypothetical link.

$$\begin{tikzcd} \mathcal{M}ZV_{CV} \arrow[d, shift left, "reduction"] \\ \mathcal{M}TGF_{CV} \arrow[u, dotted, shift left, "projection"] \end{tikzcd}$$

Figure 2: Links between multizeta values and multitangent functions

4.3.2. How to find a projection onto multitangents in practice ?

The idea is to proceed by induction on the weight of $\underline{\boldsymbol{\sigma}}$. Let us suppose that we know how to express $\mathcal{Z}e^{\underline{\mathbf{s}}}\mathcal{T}e^2$ in $\mathcal{M}TGF_{CV,p+2}$ for all sequences $\underline{\mathbf{s}}$ of weight $p < ||\underline{\boldsymbol{\sigma}}||$. According to the differentiation property, we know how to express $\mathcal{Z}e^{\underline{\mathbf{s}}}\mathcal{T}e^{||\underline{\boldsymbol{\sigma}}||-||\underline{\mathbf{s}}||+2}$ in $\mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$.

We write the reduction into monotangent functions of $\mathcal{T}e^{\underline{\mathbf{s}}}$, for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star_{b,e}$ of weight $||\underline{\boldsymbol{\sigma}}|| + 2$, as:

$$\left(\sum_{i \in E(\underline{\mathbf{s}})} \mathcal{Z}e^{\underline{\mathbf{s}}^i} \right) \mathcal{T}e^2 + \cdots \ .$$

Here, the set $E(\underline{\mathbf{s}})$ is finite and has only sequences of $\mathcal{S}^\star_b$ of weight $||\underline{\boldsymbol{\sigma}}||$; the dotted stand for some elements of $\mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$. In order to express $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2$ in $\mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$, the idea is to find out a linear relation with rational coefficients between $\displaystyle\sum_{i \in E(\underline{\mathbf{s}})} \mathcal{Z}e^{\underline{\mathbf{s}}^i}$ which is equal to $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}$.

4.3.3. Some examples

As an example, let us express $\mathcal{Z}e^{\underline{\mathbf{s}}}\mathcal{T}e^2$ in $\mathcal{M}TGF_{2,||\underline{\mathbf{s}}||+2}$, for all sequences satisfying $||\underline{\mathbf{s}}|| \leq 5$:

The table 1 gives us $\mathcal{Z}e^2\mathcal{T}e^2 = \dfrac{1}{2}\mathcal{T}e^{2,2}$.

Moreover, we know that all multizeta values of weight 3 and 4 can be respectively expressed in terms of $\mathcal{Z}e^3$ and $(\mathcal{Z}e^2)^2$. Hence, we only have to consider the quantities $\mathcal{Z}e^3\mathcal{T}e^2$ and $(\mathcal{Z}e^2)^2\mathcal{T}e^2$.

The table 1 gives us the reduction into monotangent functions of weight 5 and 6. We can, for example, choose the expressions

$$\begin{array}{rcl}\mathcal{Z}e^3\mathcal{T}e^2 & = & \frac{1}{6}\mathcal{T}e^{3,2} - \frac{1}{6}\mathcal{T}e^{2,3}. \\ (\mathcal{Z}e^2)^2\mathcal{T}e^2 & = & -\frac{5}{12}\mathcal{T}e^{3,3}. \end{array}$$

The case $||\underline{\mathbf{s}}|| = 5$ is a bit more complicated. Indeed, a classical conjecture is that the $\mathbb{Q}$-vector space spanned by multizeta values of weight 5 is a 2 dimensional vector space and one of its base is $(\mathcal{Z}e^5; \mathcal{Z}e^2\mathcal{Z}e^3)$. What is certain, is that its dimension is bounded by 2 . Therefore, it is sufficient to express $\mathcal{Z}e^5\mathcal{T}e^2$ and $\mathcal{Z}e^2\mathcal{Z}e^3\mathcal{T}e^2$ in $\mathcal{MTGF}_{2,7}$. An easy calculation based on reduction into monotangent functions gives us the following projections:

$$\begin{array}{rcl}\mathcal{Z}e^5\mathcal{T}e^2 & = & \frac{1}{30}\left(\mathcal{T}e^{2,5} + 2\mathcal{T}e^{3,4} - 2\mathcal{T}e^{4,3} - \mathcal{T}e^{5,2}\right) . \\ \mathcal{Z}e^2\mathcal{Z}e^3\mathcal{T}e^2 & = & \frac{1}{12}\mathcal{T}e^{3,2,2} - \frac{1}{12}\mathcal{T}e^{2,2,3} \\ & & +\frac{1}{24}\mathcal{T}e^{2,5} + \frac{1}{12}\mathcal{T}e^{3,4} - \frac{1}{12}\mathcal{T}e^{4,3} - \frac{1}{24}\mathcal{T}e^{5,2} . \end{array}$$

To complete our calculation, we only have to use the exact expressions of multizeta values in terms of those just considered (see for instance [30]) . The table 2 gives us the complete table of projection onto multitangents of all the multizeta values of weight at most 5 .

4.3.4. An abstract formalization of the method

Recall that all multizeta values of weight n can be expressed as a $\mathbb{Q}$-linear combination of a finite number of them, which are called the irreducible multizeta values. Let us denote by c_n the number of irreducibles of weight n.

A theorem proved by Goncharov and Terasoma shows that if $(d_n)_{n\in\mathbb{N}}$ is the sequence defined by

$$\begin{cases} d_1 = 0 \ , \\ d_2 = d_3 = 1 \ , \\ \forall n \in \mathbb{N}, d_{n+3} = d_{n+1} + d_n \ , \end{cases}$$

then we have: $c_n \leq d_n$ for all $n \in \mathbb{N}$. A conjecture due to Zagier[5] asserts that $(c_n)_{n\in\mathbb{N}}$ satisfies the same recurrence relation as $(d_n)_{n\in\mathbb{N}}$, that is, $c_n = d_n$ for all $n \in \mathbb{N}$.

Let $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_b^\star$ be of weight p. Our objective is to express $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2$ in $\mathcal{M}TGF_{CV,p+2}$.

For this, first begin to write all the reductions into multitangents of weight $p+2$ and of valuation at least 2. Then, isolate from the same side of the sign $=$ the components $\mathcal{T}e^2$. Now, in all these expressions, we can express each multizeta value in terms of the c_p corresponding irreducibles. Since each term $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^k$, $k \geq 3$, which appear in a reduction into monotangents can be expressed by induction in $\mathcal{M}TGF_{CV,p+2}$, we will have written some $\mathbb{Q}$-linear equations with a left-hand side composed of terms of the form $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2$ and a right-hand side expressed in $\mathcal{M}TGF_{CV,p+2}$.

As an example, let us do it for weight 4. Each multizeta value of weight 4 can be written in term of $\mathcal{Z}e^4$, so we obtain the system:

$$\left\{\begin{array}{rcl}
4\mathcal{Z}e^4\mathcal{T}e^2 & = & \mathcal{T}e^{2,4} + 2\mathcal{Z}e^3\mathcal{T}e^3 - \mathcal{Z}e^2\mathcal{T}e^4. \\
-6\mathcal{Z}e^4\mathcal{T}e^2 & = & \mathcal{T}e^{3,3}. \\
4\mathcal{Z}e^4\mathcal{T}e^2 & = & \mathcal{T}e^{4,2} - 2\mathcal{Z}e^3\mathcal{T}e^3 - \mathcal{Z}e^2\mathcal{T}e^4. \\
4\mathcal{Z}e^4\mathcal{T}e^2 & = & \mathcal{T}e^{2,4} + 2\mathcal{Z}e^3\mathcal{T}e^3 - \mathcal{Z}e^2\mathcal{T}e^4. \\
-\mathcal{Z}e^4\mathcal{T}e^2 & = & \mathcal{T}e^{2,1,3} - \mathcal{Z}e^3\mathcal{T}e^3. \\
4\mathcal{Z}e^4\mathcal{T}e^2. & = & \mathcal{T}e^{2,2,2} \\
-\mathcal{Z}e^4\mathcal{T}e^2. & = & \mathcal{T}e^{3,1,2} + \mathcal{Z}e^3\mathcal{T}e^3 \\
2\mathcal{Z}e^4\mathcal{T}e^2. & = & \mathcal{T}e^{2,1,1,2}
\end{array}\right.$$

If we see the quantity $\mathcal{Z}e^4\mathcal{T}e^2$ as a formal variable, this system is con-

[5]Concerning this well-known conjecture, we refer the reader to the works of P. Deligne and A. B. Goncharov ([13]), of T. Terasoma ([35]) as well as the recent works of F. Brown ([7] and [8]) .

nected to the column matrix ${}^t(4;-6;4;4;-1;4;-1;2)$, which has rank 1 . Of course, the second equation can be chosen to produce the simplest relations when we will have to derive it.

The first difficulty is to find out the dimension of the matrix we have to deal with. This is exactly the question of the number of irreducible multizeta values of weight n, which is hypothetically solved by the conjecture of Zagier. Of course, we can bypass this difficulty by treating all the possible values of c_n, i.e. $1,\cdots,d_n$, but this is not really satisfying. Nevertheless, we know that, if $A \in \mathcal{M}_{p,q}(\mathbb{Q})$ has rank q and if its column are denoted by $a_1,\cdots,a_q$, then the matrix with columns $a_1-\alpha a_2, a_3,\cdots,a_r$, $\alpha \in \mathbb{Q}$, has rank $q-1$. Applying this principle to our matrix (eventually more once), it is sufficient to consider the case where $c_n = d_n$, i.e. to suppose that Zagier's conjecture holds.

If we suppose that Zagier's Conjecture holds, the second difficulty is now to evaluate the rank of the matrix...

The table 3 shows us a submatrix of those obtained, using the values of multizeta values given by [30], for the weight $1 = c_4$, $2 = c_5$, $2 = c_6$, $3 = c_7$. Our table of multitangents up to weight 18 shows easily that the equality $c_n = d_n$ is valid up to $n = 16$. Consequently, Conjecture 4, and then Conjectures 2 and 3 , hold up to weight 18 .

4.4. About unit cleansing of multitangent functions

Let us call valuation of a sequence of positive integers, the smallest integers composing this sequence. Following [19], we know that every multizeta value, even if it is a divergent one, can be expressed as a $\mathbb{Q}$-linear combination of multizeta values with a valuation at least 2 , the same weight and a length which might be lesser. Moreover, this expression is unique up to the relations of symmetr$\underline{\boldsymbol{e}}$lity . For a proof of this fact, see for instance the recent article [19]. The most famous example of such a relation is due to Euler: $\mathcal{Z}e^{2,1} = \mathcal{Z}e^{3}$.

Such an expression can be called a “unit cleansing of multizeta values”.

4.4.1. A conjecture about cleansing of multitangent functions

We conjecture a similar result relatively to multitangent functions. Denoting $\mathcal{MTGF}_2 = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}}\in \mathrm{seq}(\mathbb{N}_2)}$, the vector space spanned by multitangents with valuation at least 2, and $\mathcal{MTGF}_{2,p} = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\substack{\underline{\mathbf{s}}\in \mathrm{seq}(\mathbb{N}_2)\\ ||\underline{\mathbf{s}}||=p}}$, the subspace of convergent multitangent functions with valuation at least 2 and weight p, this can be written:

Conjecture 5. *For all sequences* $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$*,* $\mathcal{T}e^{\underline{\mathbf{s}}} \in \mathcal{MTGF}_{2,||\underline{\mathbf{s}}||}$ *.*

For example, the simplest convergent multitangent, which is not cleaned and not the null function, is $\mathcal{T}e^{2,1,3}$. Its unit cleansing is given by the following relations:

$$\begin{aligned}
\mathcal{T}e^{2,1,3} &= \frac{1}{4}\mathcal{T}e^{4,2} - \frac{1}{4}\mathcal{T}e^{2,4} + \frac{1}{6}\mathcal{T}e^{3,3} \\
&= \mathcal{T}e^{4,2} - \frac{1}{4}\mathcal{T}e^{2,4} - \frac{1}{4}\mathcal{T}e^{2,2,2} \\
&= \mathcal{T}e^{4,2} - \frac{1}{4}\mathcal{T}e^{2,4} + \frac{1}{15}\mathcal{T}e^{3,3} - \frac{3}{20}\mathcal{T}e^{2,2,2} \ .
\end{aligned}$$

As this example shows us, there is no uniqueness of such a cleansing. This is, of course, due to the many relations between multitangent functions. Here, the responsible relation is $3\mathcal{T}e^{2,2,2} + 2\mathcal{T}e^{3,3} = 0$, which is the prototype of a more general relation between multitangent functions: $\forall k \in \mathbb{N}^*$, $3\mathcal{T}e^{2^{[3k]}} + (-1)^k 2\mathcal{T}e^{3^{[2k]}} = 0$. This relation is immediately obtained from the reduction into monotangent functions, or by the evaluation of $\mathcal{T}e^{n^{[p]}}$ that will be given in section 8

The table 4 gives us more examples of unit cleansing for multitangent functions.

4.4.2. On projection onto unit-free multitangent functions

By analogy with Conjecture 4, it is quite natural to consider the following conjecture:

Conjecture 6. *For all sequences* $\underline{\boldsymbol{\sigma}} \in \mathcal{S}^{\star}_{b}$, $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^{2} \in \mathcal{MTGF}_{2,||\underline{\boldsymbol{\sigma}}||+2}$ *.*

Conjectures 4 and 6 are probably equivalent but it is sufficient for us to know that Conjecture 6 implies Conjecture 4.

Of course, what we have said on the abstract formalization of the method is also valid in this case. The only modification is to consider multitangent functions of $\mathcal{MTGF}_{2,||\underline{\boldsymbol{\sigma}}||+2}$ instead of $\mathcal{MTGF}_{CV,||\underline{\boldsymbol{\sigma}}||+2}$. So, we will obtain a linear system with f_{n+2} equations[6]and c_n unknown ; the matrix we will obtain has size $f_{n+2} \times c_n$.

The table 3 shows us the obtained matrix[7], using the values of multizeta values given by [30], for the weight $1 = c_4$, $2 = c_5$, $2 = c_6$, $3 = c_7$. Again, the table of multitangents up to weight 18 shows easily that the equality $c_n = d_n$ is valid up to $n = 16$. Consequently, Conjecture 6 holds up to the weight 18 .

It leads to think that:

Conjecture 7. *Let $(c_n)_{n\in\mathbb{N}}$ be the sequence defined by:*

$$\begin{cases} c_1 = 0, c_2 = c_3 = 1. \\ \forall n \in \mathbb{N}, c_{n+3} = c_{n+1} + c_n. \end{cases}$$

If $p \geq 2$, the $f_{p+2} \times c_p$ matrix obtained by the previous process from all the sequences of weight p has rank c_p .

This new conjecture is equivalent to Conjecture 6 and hence implies Conjecture 5 as well as Conjectures 2 and 3 .

Here is a quantitative argument to support this last conjecture, and consequently all of the other conjectures of this section:

The first matrices we have obtained contain lots of 0's. This allows us to say they have "highly" rank c_n ; we are hinting that if this matrix have rank c_n, this is not by chance. It results from the large number of equations and from the small number of unknowns (in comparison to the other one) , but also from the repartition of the large number of zeros which forced the column vectors to be linearly independent.

Moreover, we know:

$$f_{p+2} \approx 0,281,62^{p+2} \qquad , \qquad c_p \approx 0,411,32^p \ .$$

Thus, the more p will be tall, the more there will be "chances" to find some linearly independent rows. Therefore, the conjecture will be probably more true.

[6]Here, $(f_n)_{n\in\mathbb{N}}$ denote the classical Fibonacci series.

[7]That's why, in section 4.3.4, we refer oursef to the same table as submatrix of these we must have. It was the submatrix obtained by considering only multitangent functions of $\mathcal{MTGF}_{2,||\boldsymbol{\sigma}||+2}$.

4.4.3. Unit cleansing of divergent multitangent functions

Let us finish this section by a little anticipation on a later section. We will see in Section 7 that there exists a regularization process allowing us to define multitangent functions for sequences $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)$ which begin or finish by a 1 . These functions will be expressed by the reduction into monotangent functions, with a small non-zero correction which will be a power of π in a few cases. So, according to Conjecture 4, each divergent multitangent function can be expressed as a $\mathbb{Q}$-linear combination of unit-free convergent multitangent functions.

Therefore, Conjecture 5 can be generalised to:

Conjecture 8. *For all sequences* $\underline{\mathbf{s}} \in seq(\mathbb{N}^*)$, $\mathcal{T}e^{\underline{\mathbf{s}}} \in \mathcal{MTGF}_{2,||\underline{\mathbf{s}}||}$.

5. Algebraic properties

5.1. Is $\mathcal{MTGF}_{CV}$ a graded algebra ?

Many conjectures have been stated about multizeta values. These are deep ones, but seem to be completely out of reach nowadays. We will see a first application of the dual process of reduction and projection. Thanks to it, we will state a new conjecture, which is related to a hypothetical structure of graded algebra. Then, we will see two simple examples where it is impossible to have non-trivial $\mathbb{Q}$-linear relations between multitangent functions of different weights.

5.1.1. Hypothetical absence of $\mathbb{Q}$-linear relations between different weight

Let us remind the following well-known conjecture on multizeta values:

Conjecture 9. *There is no null $\mathbb{Q}$-linear relation between multizeta values of different weights:*

$$\mathcal{MZV}_{CV} = \sum_{p \in \mathbb{N}} \mathcal{MZV}_{CV,p} = \bigoplus_{p \in \mathbb{N}} \mathcal{MZV}_{CV,p} \ .$$

In other words, $\mathcal{MZV}_{CV}$ is a graded $\mathbb{Q}$-algebra.

Let us remark that this conjecture implies in particular the transcendence of all the numbers $\mathcal{Z}e^{s}$, where $s \geq 2$. Because of the reduction/projection process, we can state the analogue conjecture for the multitangent functions:

Conjecture 10. *There is no null $\mathbb{Q}$-linear relation between multitangent functions of different weights:*

$$\mathcal{M}TGF_{CV} = \sum_{p\in\mathbb{N}} \mathcal{M}TGF_{CV,p+2} = \bigoplus_{p\in\mathbb{N}} \mathcal{M}TGF_{CV,p+2} \ .$$

In other words, $\mathcal{M}TGF_{CV}$ is a graded $\mathbb{Q}$-algebra.

In Section 4, we have conjectured that for all sequences $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_b^\star$, we have $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2 \in \mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$. The following property explains how these two conjectures are related.

Property 6. 1. *Conjecture 9 implies Conjecture 10.*
2. *Conjectures 3 and 10 imply Conjecture 9.*

Proof. 1. Suppose that there exists a $\mathbb{Q}$-linear relation between multitangent functions of different weights. So, there exists a a family of non zero $\mathbb{Q}$-linear combination of multitangent functions $(t_i)_{i\in I} \in \prod_{i\in I} \mathcal{M}TGF_{CV,i}$ such that

$$\sum_{i\in I} t_i = 0 \ ,$$

where I is a finite set.

Each t_i is a $\mathbb{Q}$-linear combination of convergent multitangents of weight i that we can suppose to be nonzero. By reduction into monotangent functions, for all terms t_i, there exist a familly $(z_{i,j})_{j\in[\![1\,;\,i]\!]}$ of multizeta values, $z_{i,j}$ being of weight j, such that:

$$t_i = \sum_{k=2}^{i} z_{i,i-k}\mathcal{T}e^k.$$

Let us remark that, for a fixed i in I, there exists $z_{i,j} \neq 0$: indeed, if the contrary holds, we would have $t_i = 0$. Thus, denoting by M the greatest element of I, we can write:

$$0 = \sum_{i\in I} t_i = \sum_{i\in I}\sum_{k=2}^{i} z_{i,i-k}\mathcal{T}e^k = \sum_{k=2}^{M}\left(\sum_{\substack{i\in I\\ i\geq k}} z_{i,i-k}\right)\mathcal{T}e^k.$$

Consequently, the linear independence of the monotangent functions implies that for all $k \in [\![2\,;\,M]\!]$:

$$\sum_{\substack{i \in I \\ i \geq k}} z_{i,i-k} = 0 \ .$$

We have obtained a non trivial $\mathbb{Q}$-linear relation between multizeta values of different weight. Thus: $\displaystyle\sum_{k \in \mathbb{N}^*} \mathcal{Z}_k \neq \bigoplus_{k \in \mathbb{N}^*} \mathcal{Z}_k$.
We have therefore shown that:

$$\sum_{k \in \mathbb{N}^*} \mathcal{T}_k \neq \bigoplus_{k \in \mathbb{N}^*} \mathcal{T}_k \Longrightarrow \sum_{k \in \mathbb{N}^*} \mathcal{Z}_k \neq \bigoplus_{k \in \mathbb{N}^*} \mathcal{Z}_k \ .$$

2. Suppose now that there exists some non trivial $\mathbb{Q}$-linear relation between of multizeta values of different weight. So, there exist two famillies, one of sequences $\underline{\mathbf{s}}^1, \cdots, \underline{\mathbf{s}}^n$ in $\mathcal{S}_b^\star$ and the second of non-zero rational numbers $c_1, \cdots, c_n$ such that:

$$\sum_{i=1}^{n} c_i \mathcal{Z}e^{\underline{\mathbf{s}}^i} = 0 \ .$$

The map $i \longmapsto ||\underline{\mathbf{s}}^i||$ is supposed non constant.
Thus:

$$\sum_{i=1}^{n} c_i \mathcal{Z}e^{\underline{\mathbf{s}}^i} \mathcal{T}e^2(z) = 0 \ , \text{ where } z \in \mathbb{C} - \mathbb{Z} \ .$$

Conjecture 3 gives us, for all $i \in [\![1\,;\,n]\!]$, an expression of $c_i \mathcal{Z}e^{\underline{\mathbf{s}}^i} \mathcal{T}e^2$ as a $\mathbb{Q}$-linear combination of multitangent functions of weight $||\underline{\mathbf{s}}^i|| + 2$: there exist an integer n_i and some sequences $\underline{\mathbf{s}}^{i,1}\ , \cdots\ , \underline{\mathbf{s}}^{i,n_i}$ in $\mathcal{S}_{b,e}^\star$ and rational numbers $c_{i,1}\ , \cdots\ , c_{i,n_i}$ satisfying:

$$c_i \mathcal{Z}e^{\underline{\mathbf{s}}^i} \mathcal{T}e^2 = \sum_{j=1}^{n_i} c_{i,j} \mathcal{T}e^{\underline{\mathbf{s}}^{i,j}} \ .$$

We therefore obtain a non-trivial $\mathbb{Q}$-linear relation between multitangent functions (otherwise, all the c_i would be zero) not all of the same weight:

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} c_{i,j} \mathcal{T}e^{\underline{\mathbf{s}}^{i,j}} = 0 \ .$$

This is a contradiction with the absence of $\mathbb{Q}$-linear relation between multitangent functions of different weights. Therefore, we have shown:

$$\left\{\begin{array}{l}\displaystyle\sum_{k\in\mathbb{N}^*}\mathcal{T}_k=\bigoplus_{k\in\mathbb{N}^*}\mathcal{T}_k\\[2ex] \forall\underline{\boldsymbol{\sigma}}\in\mathcal{S}_b^\star,\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2\in\mathcal{T}_{||\underline{\boldsymbol{\sigma}}||+2}\end{array}\right.\Longrightarrow\sum_{k\in\mathbb{N}^*}\mathcal{Z}_k=\bigoplus_{k\in\mathbb{N}^*}\mathcal{Z}_k\ .$$

5.1.2. Transcendence of multitangent functions which are not identically zero

As an example of the absence of the existence of the $\mathbb{Q}$-linear combination between multitangent of different weight, we can of course think of Lemma 3, that is to the linear independence of monotangent functions. Another example can concern a transcendence property.

In order to use a transcendence method, it may be useful to know if a function is transcendent or not. Here, a transcendent function is defined as a function from a set $\Omega\subset\mathbb{C}$ valued in $\mathbb{C}$, which is transcendent over $\mathbb{C}[X]$. If we find a nonzero multitangent function which is not transcendent, then we would have a $\mathbb{Q}$-linear relation between multitangents of different weights. Fortunately for Conjecture 10, we can state that:

Lemma 4. *Any nonzero multitangent function is transcendent.*

Let us remark that if we want to be able to use this lemma in a transcendence argument, it will be necessary to characterize the null multitangent functions. This will be hypothetically done in a forthcoming section (see Section 8.3).

Proof. Let us consider $\underline{\mathbf{s}}\in\mathcal{S}_{b,e}^\star$ such that $\mathcal{T}e^{\underline{\mathbf{s}}}\not\equiv0$.

If we suppose that $\mathcal{T}e^{\underline{\mathbf{s}}}$ is an algebraic function, there exists a polynomial $P\in\mathbb{C}[X;Y]$ such that $P\big(z;\mathcal{T}e^{\underline{\mathbf{s}}}(z)\big)\equiv0$. Let us consider the smallest possible degree in X of such a polynomial, which will be denoted by d . Writing P in an expanded form, there exists a non-trivial familly of polynoms $(P_i)_{i\in[\![0\,;\,d]\!]}$ such that we would have:

$$\sum_{i=0}^{d}P_i(z)\left(\mathcal{T}e^{\underline{\mathbf{s}}}(z)\right)^i=0\ ,\ \text{where } z\in\mathbb{C}-\mathbb{Z}\ .$$

Thanks to the exponentially flat character of convergent multitangent functions, when z goes to the infinity outside of the real axis, we would obtain for all polynomial P and sequence $\underline{\mathbf{s}}\in\mathcal{S}_{b,e}^\star$:

$$P(z)\mathcal{T}e^{\underline{\mathbf{s}}}(z)\longrightarrow0\ .$$

Thus:

$$\sum_{i=1}^{d} P_i(z)\left(\mathcal{T}e^{\mathbf{s}}(z)\right)^i \longrightarrow 0 \ .$$

Because the function $\displaystyle\sum_{i=0}^{d} P_i\left(\mathcal{T}e^{\mathbf{s}}\right)^i$ is supposed to be null, we therefore would have $P_0(z) \longrightarrow 0$, that is P_0 would be null. From the hypothesis, $\mathcal{T}e^{\mathbf{s}}$ is not the null function. So:

$$\sum_{i=1}^{d} P_i(z)\left(\mathcal{T}e^{\mathbf{s}}(z)\right)^{i-1} = \sum_{i=0}^{d-1} P_{i+1}(z)\left(\mathcal{T}e^{\mathbf{s}}(z)\right)^i \equiv 0 \ .$$

This contradicts the fact that d is the smallest possible degree in X for such a polynomial P. Consequently, we have shown that every nonzero multitangent function is transcendent. $\square$

5.2. On a hypothetical basis of $\mathcal{M}TGF_{CV,p}$

In this paragraph, we will study the analogue of Zagier's conjecture on the dimension of $\mathcal{M}ZV_{CV,p}$. For this, we will use the reduction/projection process to translate it in $\mathcal{M}TGF_{CV}$. Recall that the Zagier conjecture states that $(\dim \mathcal{M}ZV_{CV,n})_{n\in\mathbb{N}}$ would satisfy the recurrence relation:

$$\begin{cases} c_0 = 1 \ , \ c_1 = 0 \ , \ c_2 = 1 \ . \\ c_{n+3} = c_{n+1} + c_n \ , \text{ where } n \in \mathbb{N} \ . \end{cases}$$

The translation into $\mathcal{M}TGF_{CV}$ is:

Conjecture 11. *$(\dim \mathcal{M}TGF_{CV,n+2})_{n\in\mathbb{N}}$ would satisfy the recurrence relation:*

$$\begin{cases} d_0 = d_1 = 1 \ , \ d_2 = 2 \ , \ d_3 = 3 \ . \\ d_{n+4} = d_{n+3} + d_{n+2} - d_n \ , \textit{ where } n \in \mathbb{N} \ . \end{cases}$$

Of course, this conjecture is related to that of Zagier:

Property 7. *Let us suppose that Conjecture 3 holds, i.e. that $\mathcal{M}TGF_{CV}$ is a $\mathcal{M}ZV_{CV}$-module.*
Then, Conjecture 11 is equivalent to the Zagier's conjecture.

The proof is based on Property 8. We have the following formal power series which are respectively the hypotheticaly Hilbert-Poincaré series of the hypotheticaly graded $\mathbb{Q}$-algebras $\mathcal{M}ZV_{CV}$ and $\mathcal{M}TGF_{CV}$:

$$\begin{aligned}
\mathcal{H}_{\mathcal{M}ZV_{CV}} &= \sum_{p\in\mathbb{N}} \dim\, \mathcal{M}ZV_{CV,p}X^p &\overset{?}{=}& \frac{1}{1-X^2-X^3} \ . \\
\mathcal{H}_{\mathcal{M}TGF_{CV}} &= \sum_{p\in\mathbb{N}} \dim\, \mathcal{M}TGF_{CV,p+2}X^p &\overset{?}{=}& \frac{1}{(1-X^2-X^3)(1-X)} \ .
\end{aligned}$$

So, it will to sufficient to prove that $(1-X)\mathcal{H}_{\mathcal{M}TGF_{CV}} = \mathcal{H}_{\mathcal{M}ZV_{CV}}$, which is done in the following:

Property 8. *Let us suppose that Conjecture 3 holds, i.e. that $\mathcal{M}TGF_{CV}$ is a $\mathcal{M}ZV_{CV}$-module.*
1. *If $\left(\mathcal{Z}e^{\mathbf{s}_p^k}\right)_{k=1,\cdots,dim\ \mathcal{M}ZV_{CV,p}}$ denotes a basis of $\mathcal{M}VZ_{CV,p}$ for all $p\in\mathbb{N}$, then $\left(\mathcal{Z}e^{\mathbf{s}_u^k}\mathcal{T}e^v\right)_{\substack{k=1,\cdots,dim\ \mathcal{M}ZV_u \\ u+v=p+2 \\ v\geq 2}}$ is a basis of $\mathcal{M}TGF_{CV,p+2}$ for all $p\in\mathbb{N}$.*
2. *We have: $\forall p\in\mathbb{N}$, $dim\ \mathcal{M}TGF_{CV,p+2} = \sum_{k=0}^{p} dim\ \mathcal{M}ZV_{CV,k}$.*

Proof. Because the second point follows directly from the first one, we will only prove that a basis of $\mathcal{M}TGF_{CV,p+2}$ is given by the family:

$$\left(\mathcal{Z}e^{\mathbf{s}_u^k}\mathcal{T}e^v\right)_{\substack{k=1,\cdots,\dim\ \mathcal{M}ZV_u \\ u+v=p+2 \\ v\geq 2}} \ .$$

Step 0:

Because we have supposed that $\mathcal{M}TGF_{CV}$ is a $\mathcal{M}ZV_{CV}$-module, each term of the hypothetical basis is indeed an element of $\mathcal{M}TGF_{CV,p+2}$.

Step 1: the linear independence property.

If we suppose the existence of scalars $(\lambda_{v,k})$ such that:

$$\sum_{v=2}^{p+2}\ \sum_{k=1}^{\dim\ \mathcal{M}ZV_{CV,n+2-v}} \lambda_{v,k}\mathcal{Z}e^{\mathbf{s}_u^k}\mathcal{T}e^v = 0 \ ,$$

by the linear independence of monotangent functions, we obtain:

$$\forall v \in [\![\, 2\, ;\, p+2 \,]\!] \ , \qquad \sum_{k=1}^{\dim \mathcal{MZV}_{CV,n+2-v}} \lambda_{v,k} \mathcal{Z}e^{\underline{\mathbf{s}}_u^k} = 0 \ .$$

Consequently, from the linear independence of the $\left(\mathcal{Z}e^{\underline{\mathbf{s}}_u^k}\right)_{k=1,\cdots,\dim \mathcal{MZV}_u}$, we conclude that all the scalars $\lambda_{v,k}$ are null, which concludes this step.

Step 2: the spanning property.

By the reduction into monotangent functions, one writes each multitangent function in term of $\mathcal{Z}e^{\underline{\mathbf{s}}^1}\mathcal{Z}e^{\underline{\mathbf{s}}^2}\mathcal{T}e^n$. Consequently, using the symmetr$\underline{\boldsymbol{e}}$lity of the mould $\mathcal{Z}e^\bullet$, we now only have to express each multizeta value of weight k which appears in such a relation in terms of the basis of $\mathcal{MZV}_{CV,k}$.

By this process, we would have expressed each convergent multitangent function in term of $\mathcal{Z}e^{\underline{\mathbf{s}}_u^k}\mathcal{T}e^v$, where $k \in [\![\, 1\, ;\, \dim \mathcal{MZV}_u \,]\!]$, $u+v=p+2$ and $v \geq 2$ because the reduction into monotangent functions, as well as the symmetr$\underline{\boldsymbol{e}}$lity , preserves the weight. □

To conclude this paragraph, we give in the following figure the first hypothetical dimensions of the space of multitangent functions of weight $p+2$. We can recognize the sequences $A000931$ and $A023434$ from the On-Line Encyclopedia of Integer Sequences (see [34])

p	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{MZV}_{CV,p}$	1	0	1	1	1	2	2	3	4	5	7	9	12
$\dim \mathcal{MTGF}_{CV,p+2}$	1	1	2	3	4	6	8	11	15	20	27	36	48

Figure 3: The first hypothetical dimensions of the space of multitangent functions of weight $p+2$.

5.3. The $\mathbb{Q}$-linear relations between multitangent functions

We know that multizeta values have two encodings. The first one is the one we have used since the beginning of this article, resulting from the specialization $x_n = n^{-1}$ (up to a convention of choosing the summation sequence to an increasing or a decreasing sequence) of the monomial basis of quasi-symmetric functions: it is exactly the symmetr$\underline{\boldsymbol{e}}$l mould $\mathcal{Z}e^{\bullet}$. The second one comes from an iterated integral representation (which has been mentioned in the introduction, see **??**): it is the symmetr$\underline{\boldsymbol{a}}$l mould $\mathcal{W}a^{\bullet}$.

Because of the dual process reduction/projection, we can imagine that the symmetr$\underline{\boldsymbol{a}}$l coding has a translation in the algebra $\mathcal{MTGF}$. One can think that a quadratic relation in $\mathcal{MZV}$ will be translated in $\mathcal{MTGF}$ into another quadratic relation, but this does not secure to be true (essentially because Hurwitz multizeta functions have one and only one encoding) . In fact, there are lots of null $\mathbb{Q}$-linear relations between multitangent functions; these will replace the symmetr$\underline{\boldsymbol{a}}$lity relations. Of course, the existence of these is a natural fact, but, as we will see, they all have an odd look. So, the situation is not yet completely understood.

We will see in the following that sometimes, it is possible to prove easily a relation using multitangent properties (because it is the derivative of a well-know relation between multitangent function or using a parity argument). Most of the time, these relations remain completely mysterious.

Now, let us explain what happens for small weights.

Up to weight 5. The only $\mathbb{Q}$-linear relation up to weight 5 is the surprising existence of a null multitangent function, $\mathcal{T}e^{2,1,2}$. Although it is an interesting fact, we are going to postpone the study of null multitangent functions in section 8.3) . Let us just mention that many multitangent functions are the null function and that we have a conjectural characterization of those.

Weight 6. Unless it is an easy case, this is the first interesting weight. According to dim $\mathcal{MZV}_2 =$ dim $\mathcal{MZV}_3 =$ dim $\mathcal{MZV}_4 = 1$, we deduce[8] from Conjecture 2 that dim $\mathcal{MTGF}_6 = 4$. Consequently, there exist exactly four independent $\mathbb{Q}$-linear relations between multitangent functions of weight 6. Actually, it is not difficult to find them, using the known values of multizeta values of weight 4 .

These four independent relations are given in the table 5.

Weight 7. Concerning the weight 7, we obtain table 6. Because it is conjectured that dim $\mathcal{MTGF}_7 = 6$ and because $\mathcal{MTGF}_7$ is spanned by sixteen functions, one is tempted to find exactly ten independent relations. That is

[8]Because Conjecture 4 is true for the global weight 6 as we have seen this with the table 2 .

what we obtain, but the remaining question is "are there any other relations between multitangent functions of weight 7?" Consequently, this is the first weight where we can only speak hypothetically.

5.4. On the possibility of finding relations between multizeta values from the multitangent functions

5.4.1. Two different process for multiplying two multitangent functions.

It is clear that we have two possibilities to compute a product of two multitangent functions, according to the symmetr$\underline{e}$lity of $\mathcal{T}e^\bullet$ and to the relations of reduction into monotangent functions. This is summed up in the following diagram, where "reduction" (resp. "symmetr$\underline{e}$l multiplication") indicates the linear extension to $\mathcal{M}TGF_{CV}$ (resp. $\mathcal{M}TGF_{CV} \otimes \mathcal{M}TGF_{CV}$) of the reduction process (resp. the multiplication by the symmetr$\underline{e}$lity of $\mathcal{T}e^\bullet$):

$$\begin{tikzcd}
\mathcal{M}TGF \otimes \mathcal{M}TGF \arrow[r, "\textit{symmetr}\underline{\textit{el}}", "\text{multiplication}"'] \arrow[d, "\text{reduction}\otimes\text{reduction}"'] & \mathcal{M}TGF \arrow[dd, "\text{reduction}"] \\
\mathcal{M}TGF \otimes \mathcal{M}TGF \arrow[d, "\textit{symmetr}\underline{\textit{el}}"', "\text{multiplication}"] & \\
\mathcal{M}TGF \arrow[r, "\text{reduction}"] & \mathcal{M}TGF
\end{tikzcd}$$

Obviously, these two process give the same result in $\mathcal{M}TGF_{CV}$, but the expression is different. This gives us the opportunity to find out some relations between multizeta values.

First, we see that the previous commutative diagram gives us a way to find out all the relations of symmetr$\underline{e}$lity . Let us emphasize that the following property is not the best way to prove the symmetr$\underline{e}$lity of $\mathcal{Z}e^\bullet$, but its aim is just to begin to describe the relations between multizeta values obtained from those of multitangent functions.

Property 9.
1. *The relations of symmetr$\underline{e}$lity of $\mathcal{T}e^\bullet$ and the previous commutative diagram imply all the relations of symmetr$\underline{e}$lity of $\mathcal{Z}e^\bullet$.*
2. *The previous commutative diagram gives us more relations than those of symmetr$\underline{e}$lity .*

PROOF. Let us consider two sequences $\underline{\mathbf{s}}^1$ and $\underline{\mathbf{s}}^2$ in $\mathcal{S}^\star_{b,e}$ and denote by M_0, the largest integer which appears in these two sequences, and finally set

$M = M_0 + 1$.

Using the symmetr$\underline{\boldsymbol{e}}$lity of $\mathcal{T}e^{\bullet}$ and then from the recursive definition of the set sh$\underline{\boldsymbol{e}}$, we obtain:

$$\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M} = \sum_{\underline{\mathbf{s}}\in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2\cdot M)\cup sh\underline{e}(\underline{\mathbf{s}}^1\cdot M;\underline{\mathbf{s}}^2)} \mathcal{T}e^{\underline{\mathbf{s}}\cdot M} + \sum_{\underline{\mathbf{s}}\in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} \mathcal{T}e^{\underline{\mathbf{s}}\cdot 2M} \ .$$

Thus, the coefficient of $\mathcal{T}e^{2M}$ in the product $\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}$, obtained first by the symmetr$\underline{\boldsymbol{e}}$l multiplication and then by reduction into monotangent functions, is:

$$\sum_{\underline{\mathbf{s}}\in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} \mathcal{Z}e^{\underline{\mathbf{s}}} \ .$$

On an other hand, in the reduction into monotangent functions of $\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}$ and $\mathcal{T}e^{\underline{\mathbf{s}}^2\cdot M}$, the only monotangents which may contribute to $\mathcal{T}e^{2M}$ are the terms $\mathcal{T}e^{M}$ coming from the reduction into monotangent functions of $\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}$ and $\mathcal{T}e^{\underline{\mathbf{s}}^2\cdot M}$. These are respectively equal to $\mathcal{Z}e^{\underline{\mathbf{s}}^1}\mathcal{T}e^{M}$ and $\mathcal{Z}e^{\underline{\mathbf{s}}^2}\mathcal{T}e^{M}$. Consequently, the coefficient of $\mathcal{T}e^{2M}$ in the product $\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}\mathcal{T}e^{\underline{\mathbf{s}}^1\cdot M}$, obtained first by reduction into monotangent functions, then by the symmetr$\underline{\boldsymbol{e}}$l multiplication and finally by reduction one more time is: $\mathcal{Z}e^{\underline{\mathbf{s}}^1}\mathcal{Z}e^{\underline{\mathbf{s}}^2}$.

As a consequence, we obtain the relation we are looking for:

$$\mathcal{Z}e^{\underline{\mathbf{s}}^1}\mathcal{Z}e^{\underline{\mathbf{s}}^2} = \sum_{\underline{\mathbf{s}}\in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} \mathcal{Z}e^{\underline{\mathbf{s}}} \ .$$

2. As well, the previous diagram gives us more than the symmetr$\underline{\boldsymbol{e}}$lity relations of the multizeta values. To see this, we can make the calculation in the simplest case where the diagram gives a result:

$$\begin{aligned} 1.\quad \mathcal{T}e^2 \times \mathcal{T}e^{2,2} &= 2\mathcal{Z}e^2\big(\mathcal{T}e^2\big)^2 = 4\mathcal{Z}e^2\mathcal{T}e^{2,2} + 2\mathcal{Z}e^2\mathcal{T}e^4 \\ &= 8\big(\mathcal{Z}e^2\big)^2\mathcal{T}e^2 + 2\mathcal{Z}e^2\mathcal{T}e^4 \ . \end{aligned}$$

$$\begin{aligned} 2.\quad \mathcal{T}e^2 \times \mathcal{T}e^{2,2} &= 3\mathcal{T}e^{2,2,2} + \mathcal{T}e^{4,2} + \mathcal{T}e^{2,4} \\ &= \Big(3\big(\mathcal{Z}e^2\big)^2 + 6\mathcal{Z}e^{2,2} + 8\mathcal{Z}e^4\Big)\,\mathcal{T}e^2 + 2\mathcal{Z}e^2\mathcal{T}e^4 \ . \end{aligned}$$

Thus, by the linear independence of monotangent functions, we obtain:

$$6\mathcal{Z}e^{2,2} + 8\mathcal{Z}e^{4} = 5\big(\mathcal{Z}e^{2}\big)^2 \ . \tag{12}$$

Using the symmetr$\underline{e}$lity of multizeta values, (12) can be written:

$$3\mathcal{Z}e^{4} = 4\mathcal{Z}e^{2,2} \ . \tag{13}$$

But, the only relation of symmetr$\underline{e}$lity of weight 4 is:

$$\big(\mathcal{Z}e^{2}\big)^2 = 2\mathcal{Z}e^{2,2} + \mathcal{Z}e^{4} \ . \tag{14}$$

If (13) could be proven from (14), using uniquely the symmetr$\underline{e}$lity of $\mathcal{Z}e^{\bullet}$, we would know the following values $\left\{\begin{array}{rcl} \mathcal{Z}e^{4} & = & \frac{2}{5}\big(\mathcal{Z}e^{2}\big)^2. \\ \mathcal{Z}e^{2,2} & = & \frac{3}{10}\big(\mathcal{Z}e^{2}\big)^2. \end{array}\right.$

To prove these relations, we actually need the following three equations which come from the three types of relations describing hypothetically the kernel of the map ζ:

$$\begin{array}{rcl} \big(\mathcal{Z}e^{2}\big)^2 & = & 2\mathcal{Z}e^{2,2} + \mathcal{Z}e^{4} \ . \\ \big(\mathcal{Z}e^{2}\big)^2 & = & 2\mathcal{Z}e^{2,2} + 4\mathcal{Z}e^{3,1} \ . \\ \mathcal{Z}e^{4} & = & \mathcal{Z}e^{2,2} + \mathcal{Z}e^{3,1} \ . \end{array}$$

This proves that the diagram gives us more relations than those of symmetr$\underline{e}$lity . $\square$

Let us also remark that we can write another commutative diagram for the derivative of a multitangent function, but this does not give us a way to find relations between multizeta values.

5.5. Back to the absence of the monotangent $\mathcal{T}e^1$ in the relations of reduction

We have seen that the convergent multitangent functions are exponentially flat, near infinity (see §6.3). This implies the absence of the monotangent $\mathcal{T}e^1$ in the relations of reduction. From this, we can deduce some relations between multizeta values. For all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$, we have:

$$\sum_{i=1}^{l(\underline{\mathbf{s}})} \sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r=s_i-1}} {}^iB^{\underline{\mathbf{s}}}_{\underline{\mathbf{k}}} \mathcal{Z}e^{s_1+k_1,\cdots,s_{i-1}+k_{i-1}} \mathcal{Z}e^{s_r+k_r,\cdots,s_{i+1}+k_{i+1}} = 0 \ .$$

On the other hand, we can prove these relations between multizeta values in an independent way. Consequently, this will immediately show the exponentially flat character as a corollary and also will answer the question "*Are these relations consequences of the quadratic relations?*"

Indeed, these relations are consequences of symmetr$\underline{\boldsymbol{a}}$lity relations of multizetas values. Let us denote by $S(r)$ the left hand side of the previous equality, that is, for all positive integer r and all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ of length r:

$$S(r) = \sum_{i=1}^{l(\underline{\mathbf{s}})} \sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r=s_i-1}} {}^iB^{\underline{\mathbf{s}}}_{\underline{\mathbf{k}}} \mathcal{Z}e^{s_1+k_1,\cdots,s_{i-1}+k_{i-1}} \mathcal{Z}e^{s_r+k_r,\cdots,s_{i+1}+k_{i+1}} \ .$$

We will show that $S(r) = 0$ for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$, by linearization of the product of multizeta values coming from the relations of symmetr$\underline{\boldsymbol{a}}$lity. Let us explain this in detail.

Each multizeta value can be written as an iterated integral. Such an expression can be for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b}$:

$$\mathcal{Z}e^{\underline{\mathbf{s}}} \ = \ \int_{0<u_1<\cdots<u_r<+\infty} \frac{{u_1}^{s_1-1}(u_2-u_1)^{s_2-1}\cdots(u_r-u_{r-1})^{s_r}}{(e^{u_1}-1)\cdots(e^{u_r}-1)} \frac{du_1\cdots du_r}{\displaystyle\prod_{i=1}^{r}(s_i-1)!} \ .$$

Thus, for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ of length r and all $i \in [\![\, 1 \,;\, r \,]\!]$, we have if we setting $u_0 = 0$ and $u_{r+1} = 0$:

$$\sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r=s_i-1}} {}^iB^{\underline{\mathbf{s}}}_{\underline{\mathbf{k}}} \mathcal{Z}e^{s_1+k_1,\cdots,s_{i-1}+k_{i-1}} \mathcal{Z}e^{s_r+k_r,\cdots,s_{i+1}+k_{i+1}}$$

$$= \sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r=s_i-1}} \frac{(-1)^{s_{i+1}+\cdots+s_r}}{\prod_{p=1}^{r}(s_p-1)!}\cdot\frac{(s_i-1)!}{k_1!\cdots\widehat{k_i!}\cdots k_r!}$$

$$\left(\int_{0<u_1<\cdots<u_{i-1}<+\infty}\prod_{p=0}^{i-2}\frac{(u_{p+1}-u_p)^{s_{p+1}+k_{p+1}-1}}{e^{u_{p+1}}-1}du_1\cdots du_{i-1}\right)$$

$$\left(\int_{0<u_r<\cdots<u_{i+1}<+\infty}\prod_{p=i+1}^{r}\frac{(u_p-u_{p+1})^{s_p+k_p-1}}{e^{u_p}-1}du_{i+1}\cdots du_r\right),$$

$$= \frac{(-1)^{s_{i+1}+\cdots+s_r}}{\prod_{p=1}^{r}(s_p-1)!}\int_{\{0<u_1<\cdots<u_{i-1}<+\infty\}\times\{0<u_r<\cdots<u_{i+1}<+\infty\}} g_i(u_1;\cdots;u_r)du_1\cdots\widehat{du_i}\dots du_r\ ,$$

where $g_i(u_1;\cdots;u_r) = \left(\prod_{p\in[\![1\,;r]\!]-\{i\}}\frac{(u_p-u_{p-1})^{s_p-1}}{e^{u_p}-1}\right)(u_{i+1}-u_{i-1})^{s_i-1}$, always with $u_0=0$ and $u_{r+1}=0$.

In order to conclude that $S(r)=0$, we need the shuffle product

$$\{0<u_1<\cdots<u_{i-1}<+\infty\}\times\{0<u_r<\cdots<u_{i+1}<+\infty\}\ .$$

To do it, let us introduce a few notations coming from the syntatic point of view. We now consider the alphabet $\Omega_r=\{u_1;\cdots;u_r\}$, the non-commutative polynomial $e_r=\sum_{i=1}^{r}(u_1\cdots u_{i-1})\sqcup\!\sqcup(u_r\cdots u_{i+1})$ and the set E_r of words of $\mathrm{seq}(\Omega_r)$ which appears in e_r, that is to say in the linearization of the multizeta values appearing in the game:

$$E_r=\{\omega\in\mathrm{seq}(\Omega_r)\ ;\ \langle e_r|\omega\rangle\neq 0\}$$

Finally, to each word $\omega=u_{i_1}\cdots u_{i_r}$ of $\mathrm{seq}(\Omega_r)$ which contains exactly one time all the letters of $\mathrm{seq}(\Omega_r)$ except one which will be denoted u_i, we associate an integral $I(\omega)$ defined by:

$$\begin{aligned} I(\omega) &= I(u_{i_1}\cdots u_{i_r})\\ &= (-1)^{s_{i+1}+\cdots+s_r}\int_{0<u_{i_1}<\cdots<u_{i_r}<+\infty} g_i(u_1;\cdots;u_r)du_1\cdots\widehat{du_i}\cdots du_r\ .\end{aligned}$$

Thus: $S(r) = \dfrac{1}{\prod_{p=1}^{r}(s_p-1)!}\left(\sum_{\omega\in E_r} I(\omega)\right)$.

We will evaluate the sum, in the right-hand side, by grouping pairwise the element of E_r . E_r will then be decomposed into a family of pairs of words we will call associate words.

Definition 1. *Let us consider, for all $(k;l)\in [\![\,1\,;\,r\,]\!]^2$, the morphism $\varphi_{k,l}$ from seq(Ω_r) (for the word concatenation) defined by:*

$$\begin{array}{cccc} \varphi_{k,l}: & [\![\,1\,;\,r\,]\!] & \longrightarrow & [\![\,1\,;\,r\,]\!] \\ & u_i & \longmapsto & \begin{cases} u_i \ , \ \text{if } i\neq l \ . \\ u_k \ , \ \text{if } i=l \ . \end{cases} \end{array}$$

We will say that two words ω^1 and ω^2 of seq(Ω_r) are associated when:

$$\exists i\in [\![\,1\,;\,r\,]\!], \omega^2=\varphi_{i,i+1}(\omega^1) \ \text{or}\ \omega^1=\varphi_{i,i+1}(\omega^2)\ .$$

We then write: $\omega^1 \between \omega^2$.

Let $\widetilde{\mathrm{seq}(\Omega_r)}$ be the set of words of seq(Ω_r) which contain exactly one time all the letters of Ω_r, except one. One can remark that two words of $\widetilde{\mathrm{seq}(\Omega_r)}$ can of course have a different missing letter. Finally, we associate a permutation σ_ω of $[\![\,1\,;\,r\,]\!]-\{i\}$ with each word $\omega=u_{s_1}\cdots u_{s_{r-1}}$ of $\widetilde{\mathrm{seq}(\Omega_r)}$, where i is the index of the absent letter of ω, defined by:

$$\sigma_\omega = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & r \\ s_1 & \cdots & s_{i-1} & s_i & \cdots & s_{r-1} \end{pmatrix} .$$

We have then:

Lemma 5. 1. *For all $\omega\in E_r$ and all integer $i\in [\![\,1\,;\,r-1\,]\!]$, we have:*

$$\sigma_{\varphi_{i,i+1}(\omega)} = \rho_{i,i+1}\circ\sigma_\omega\circ{\rho_{i,i+1}}^{-1} ,$$

$$\textit{with} \quad \rho_{i,i+1} : \quad [\![1\,;\, r]\!] - \{i\} \quad \longrightarrow \quad [\![1\,;\, r]\!] - \{i+1\}$$
$$k \qquad \longmapsto \qquad \begin{cases} k \ , \ \textit{if}\ k \neq i+1 \ . \\ \\ i \ , \ \textit{if}\ k = i+1 \ . \end{cases}$$

2. *For all* $\omega \in E_r$ *, there exists a unique* $\omega' \in E_r - \{\omega\}$ *such that:* $\omega \between \omega'$ *.*
3. *For all* $(\omega^1 ; \omega^2) \in {E_r}^2$ *, we have :* $\omega^1 \between \omega^2 \Longrightarrow I(\omega^1) = -I(\omega^2)$ *.*

Since E_r has 2^{r-1} elements counted with their multiplicity in E_r, we can conclude, from the second statement of the previous lemma that E_r can be cut into 2^{r-2} pairs of associated words in E_r.

Thus, according to the third statement of the lemma, we finally deduce: $S(r) = 0$. This can be written as:

Property 10. 1. *The exponentially flat character of multitangent functions implies some relations between multizeta values which are coming from the symmetr**a**lity relations of the multizeta values.*

2. *The symmetr**a**lity relations of the multizeta values impose the absence of the monotangent $\mathcal{T}e^1$ in the relation of reduction into monotangent functions, and thus force the multitangent functions to be exponentially flat.*

Consequently, one can ask the following question:
"*According to the exponentially flat character of convergent multitangent functions, are we able to find all the symmetr**a**lity relations between multizeta values? If the answer is negative, which relations do we obtain?*"

The answer is simple and comes from a rapid exploration of the table of multitangent functions. We immediately see that, in weight 5, we find all the symmetr**a**lity relations of multizeta values of weight 4, but this situation is really exceptional. For instance, in weight 6, one symmetr**a**lity relation is not obtained:

$$3\mathcal{Z}e^{2,2,1} + 6\mathcal{Z}e^{3,1,1} + \mathcal{Z}e^{2,1,2} = \mathcal{Z}e^{2,1}\mathcal{Z}e^{2}. \tag{15}$$

Nevertheless, considering all $\mathbb{Q}$-linear relations between multitangent functions, we are able to find all the symmetr**a**lity relations. To illustrate this, one can find (15) from:

$$4\mathcal{T}e^{3,1,3} - 2\mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,1,2,2} = 0 \ . \tag{16}$$

6. Analytic properties

As announced in Section 3.2, we now will see that each multitangent function will decrease to 0 when z will increase to infinity, avoiding the real axis. This is the so-called exponentially flat character of convergent multitangent functions. For studying the convergence of series involving multitangent functions, we look for a upper bound depending on the weight of the sequence $\underline{\mathbf{s}}$ and which also show us the exponentially flat character.

To obtain such an upper bound, we will have to avoid the use of the triangular inequality, which is not precise enough sharp, so we want to use directly a upper bound on the sum. First, we will focus us on Fourier coefficients of multitangents ; then, we will deal with geometric upper bounds of multitangent functions in order to obtain upper bound of the Fourier coefficients. Finally, using this, we obtain a upper bound as required.

6.1. Fourier expansion of convergent multitangent functions

It is quite obvious that all convergent multitangent functions are 1-periodic on $\mathbb{C} - \mathbb{Z}$. Thus, we are naturally interested in their Fourier expansions. We will now compute their Fourier coefficients. The result proved here is central for the explicit calculation of analytical invariants of tangent-to-identity diffeomorphisms (see [3])

Let us begin by recalling the Fourier expansion of $\mathcal{T}e^1$ (see [33]) :

$$\mathcal{T}e^1(z) = \frac{\pi}{\tan(\pi z)} = \begin{cases} i\pi + 2i\pi \sum\limits_{n<0} e^{2in\pi z} \text{ , if } \Im m\, z < 0 \ . \\ -i\pi - 2i\pi \sum\limits_{n>0} e^{2in\pi z} \text{ , if } \Im m\, z > 0 \ . \end{cases} \tag{17}$$

Since the convergence of $\mathcal{T}e^1$ is normal on $\{\zeta \in \mathbb{C}; \Im m\, \zeta < -c\}$ and $\{\zeta \in \mathbb{C}; \Im m\, \zeta > c\}$, for all $c > 0$ the expression (17) is the Fourier expansion of $\mathcal{T}e^1$. So, the differentiation property gives us for all $\sigma \in \mathbb{N} - \{0; 1\}$ and all $z \in \mathbb{C} - \mathbb{Z}$:

$$\mathcal{T}e^\sigma(z) = \begin{cases} 2i\pi \sum\limits_{n<0} \dfrac{(-2in\pi)^{\sigma-1}}{(\sigma-1)!} e^{2in\pi z} \text{ , if } \Im m\, z < 0 \ . \\ -2i\pi \sum\limits_{n>0} \dfrac{(-2in\pi)^{\sigma-1}}{(\sigma-1)!} e^{2in\pi z} \text{ , if } \Im m\, z > 0 \ . \end{cases}$$

Inserting this Fourier expansion in the expression for the reduction into monotangents (see §3) when $\underline{\mathbf{s}} \in \mathcal{S}^\star$, we obtain:

$$\begin{aligned}\mathcal{T}e^{\underline{\mathbf{s}}}(z) &= \sum_{j=1}^{r}\sum_{k=2}^{s_j} \mathcal{Z}^{\underline{\mathbf{s}}}_{j,s_j-k}\mathcal{T}e^k(z) \\ &= \left\{\begin{array}{ll} 2i\pi \displaystyle\sum_{n<0}\sum_{j=1}^{r}\left(\sum_{k=2}^{s_j}\frac{(-2in\pi)^{k-1}}{(k-1)!}\mathcal{Z}^{\underline{\mathbf{s}}}_{j,s_j-k}\right)e^{2in\pi z} & \text{, if } \Im m\, z < 0\ . \\ -2i\pi \displaystyle\sum_{n>0}\sum_{j=1}^{r}\left(\sum_{k=2}^{s_j}\frac{(-2in\pi)^{k-1}}{(k-1)!}\mathcal{Z}^{\underline{\mathbf{s}}}_{j,s_j-k}\right)e^{2in\pi z} & \text{, if } \Im m\, z > 0\ . \end{array}\right.\end{aligned}$$

Since the convergence of this series is normal on $\{\zeta \in \mathbb{C}; \Im m\, \zeta < -c\}$ and $\{\zeta \in \mathbb{C}; \Im m\, \zeta > c\}$, for all $c > 0$ we obtain the Fourier expansion of the multitangent functions:

Lemma 6. *Let us set*[9]*, for all $n \in \mathbb{Z}$ and all $\underline{\mathbf{s}} \in \mathcal{S}^\star$:*

$$\widehat{\mathcal{T}}^{\underline{\mathbf{s}}}_n = 2i\pi \sum_{j=1}^{l(\underline{\mathbf{s}})}\left(\sum_{k=2}^{s_j}\frac{(-2in\pi)^{k-1}}{(k-1)!}\mathcal{Z}^{\underline{\mathbf{s}}}_{j,s_j-k}\right)\ .$$

Then, for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star$ and all $z \in \mathbb{C} - \mathbb{Z}$, we have:

$$\mathcal{T}e^{\underline{\mathbf{s}}}(z) = \left\{\begin{array}{l} \displaystyle\sum_{n<0}\widehat{\mathcal{T}}^{\underline{\mathbf{s}}}_n q^n \ ,\ \textit{if } \Im m\, z < 0\ , \\ \displaystyle -\sum_{n>0}\widehat{\mathcal{T}}^{\underline{\mathbf{s}}}_n q^n \ ,\ \textit{if } \Im m\, z > 0\ , \end{array}\right.$$

where $q = e^{2\pi i z}$.

[9] Let us remark that the mould $\widehat{\mathcal{T}}^{\bullet}_n$ can not be a symmetr$\underline{e}$l one. For example, we have:

$$\left\{\begin{array}{l} 2\widehat{\mathcal{T}}^{2,2}_1 + \widehat{\mathcal{T}}^{4}_1 = 2\times\dfrac{4}{3}\pi^4 - \dfrac{8}{3}\pi^4 = 0\ . \\ \left(\widehat{\mathcal{T}}^{2}_1\right)^2 = 16\pi^4. \end{array}\right.$$

This explains the absence of the letter e in its name.

6.2. An upper bound for multitangent functions

In this paragraph, we will prove two geometric upper bounds (or nearly geometric ones), where the exponent will be the weight of the multitangent and then give two hypothetical upper bounds. For this, we will use elementary methods.

We begin, for a convergent multitangent function, by proving the following upper bound:

Lemma 7. *For all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star$ and all $z \in \mathbb{C} - \mathbb{R}$, we have :*

$$|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \frac{4l(\underline{\mathbf{s}})}{|\Im m\, z|^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} .$$

Proof. Let $\underline{\mathbf{s}} \in \mathcal{S}^\star$ and $z \in \mathbb{C} - \mathbb{R}$.
We will denote by $f^{\underline{\mathbf{s}}}$ the function defined on $\mathbb{R} \times \mathbb{R}_+^*$ by:

$$fe^{\underline{\mathbf{s}}}(x;y) = \sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{\big((n_1+x)^2+y^2\big)^{\frac{s_1}{2}} \cdots \big((n_r+x)^2+y^2\big)^{\frac{s_r}{2}}} .$$

We hence have: $|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq fe^{\underline{\mathbf{s}}}(\Re e\, z; |\Im m\, z|)$. Moreover, using an argument we will develop in a forthcoming section (see Section 11), we obtain the following trifactorisation:

$$fe^{\bullet}(x;y) = fe_+^{\bullet}(x;y) \times Ie^{\bullet}(x;y) \times fe_-^{\bullet}(x;y) ,$$

where the functions $fe_+^{\bullet}$, $fe_-^{\bullet}$ and $Ie^{\bullet}$ are defined on $\mathbb{R} \times \mathbb{R}_+^*$ by:

$$fe_+^{\underline{\mathbf{s}}}(x;y) = \sum_{-E(x)<n_r<\cdots<n_1<+\infty} \frac{1}{((n_1+x)^2+y^2)^{\frac{s_1}{2}} \cdots ((n_r+x)^2+y^2)^{\frac{s_r}{2}}} .$$

$$fe_-^{\underline{\mathbf{s}}}(x;y) = \sum_{-\infty<n_r<\cdots<n_1<-E(x)} \frac{1}{((n_1+x)^2+y^2)^{\frac{s_1}{2}} \cdots ((n_r+x)^2+y^2)^{\frac{s_r}{2}}} .$$

$$Ie^{\underline{\mathbf{s}}}(x;y) = \begin{cases} 0 & \text{, if } l(\underline{\mathbf{s}}) \neq 1 . \\ ((x-E(x))^2+y^2)^{-\frac{s}{2}} & \text{, if } l(\underline{\mathbf{s}}) = 1 . \end{cases}$$

Thus, we successively have:

$$fe_+^{\underline{\mathbf{s}}}(x;y) = \sum_{0<n_r<\cdots<n_1<+\infty} \frac{1}{((n_1+x-E(x))^2+y^2)^{\frac{s_1}{2}}\cdots((n_r+x-E(x))^2+y^2)^{\frac{s_r}{2}}}$$

$$\leq \frac{1}{(y^2)^{\frac{s_1-2}{2}}(y^2)^{\frac{s_2-1}{2}}\cdots(y^2)^{\frac{s_r-1}{2}}} fe_+^{2,1,\cdots,1}(x-E(x);0)$$

$$\leq \frac{1}{y^{s_1-2}y^{s_2-1}\cdots y^{s_r-1}} \sum_{0<n_r<\cdots<n_1<+\infty} \frac{1}{{n_1}^2 n_2\cdots n_r}$$

$$= \frac{\mathcal{Z}e^{2,1,\cdots,1}}{y^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} \leq \frac{2}{y^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}}\ .$$

In the same way, we have: $fe_-^{\underline{\mathbf{s}}}(x;y) \leq \dfrac{2}{y^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}}$.

Hence:

$$\begin{aligned}
|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| &\leq \sum_{k=1}^{l(\underline{\mathbf{s}})} fe_+^{\underline{\mathbf{s}}^{<k}}(\Re e\, z; |\Im m\, z|) Ie^{s_k}(\Re e\, z; |\Im m\, z|) fe_-^{\underline{\mathbf{s}}^{>k}}(\Re e\, z; |\Im m\, z|) \\
&\leq \sum_{k=1}^{l(\underline{\mathbf{s}})} \frac{2}{|\Im m\, z|^{||\underline{\mathbf{s}}^{<k}||-(k-1)-1}} \times \frac{1}{|\Im m\, z|^{s_k}} \times \frac{2}{|\Im m\, z|^{||\underline{\mathbf{s}}^{>k}||-(l(\underline{\mathbf{s}})-k)-1}} \\
&= \sum_{k=1}^{l(\underline{\mathbf{s}})} \frac{4}{|\Im m\, z|^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} = \frac{4l(\underline{\mathbf{s}})}{|\Im m\, z|^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}}\ .
\end{aligned}$$

□

Next, we present the second upper bound. It is an improvement of the first one when we restrict to nonempty sequences of $\mathrm{seq}(\mathbb{N}_2)$. The proof uses the same notations and also the same ideas as for the first upper bound.

Lemma 8. *For all $\underline{\mathbf{s}} \in seq(\mathbb{N}_2) - \{\emptyset\}$ and all $z \in \mathbb{C} - \mathbb{R}$ satisfying $|z| \geq 1$, we have:* $|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \dfrac{1}{l(\underline{\mathbf{s}})!}\left(\dfrac{2}{\sqrt{|\Im m\, z|}}\right)^{||\underline{\mathbf{s}}||}$.

Proof. Let $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_2)$ be a sequence of length r and $z \in \mathbb{C} - \mathbb{R}$ satisfying $|z| \geq 1$. Let us also consider, as in the previous proof, the notations $fe^{\underline{\mathbf{s}}}$, $fe_+^{\underline{\mathbf{s}}}$, $fe_-^{\underline{\mathbf{s}}}$ and $Ie^{\underline{\mathbf{s}}}$.

In the case where $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_2) \subset \mathcal{S}^\star$, we will improve the upper bounds which have just been found in the proof of the first upper bound by using an integral test for convergence:

$$\begin{aligned}
fe_+^{\underline{\mathbf{s}}}(x;y) &= \sum_{0<n_r<\cdots<n_1<+\infty} \prod_{i=1}^{r} \left(\frac{1}{((n_i + x - E(x))^2 + y^2)^{\frac{s_i}{2}}} \right) \\
&= \frac{1}{r!} \sum_{\substack{(n_1;\cdots;n_r)\in(\mathbb{N}^*)^r \\ i\neq j \Longrightarrow n_i\neq n_j}} \prod_{i=1}^{r} \left(\frac{1}{((n_i + x - E(x))^2 + y^2)^{\frac{s_i}{2}}} \right) \\
&\leq \frac{1}{r!} \prod_{i=1}^{r} \left(\sum_{n\in\mathbb{N}^*} \frac{1}{((n + x - E(x))^2 + y^2)^{\frac{s_i}{2}}} \right) \\
&\leq \frac{1}{r!} \prod_{i=1}^{r} \left(\int_0^{+\infty} \frac{dt}{((t + x - E(x))^2 + y^2)^{\frac{s_i}{2}}} \right) \\
&\leq \frac{1}{r!} \frac{1}{y^{||\underline{\mathbf{s}}||-2r}} \left(\int_0^{+\infty} \frac{du}{u^2+y^2} \right)^r \leq \frac{1}{r!} \frac{\left(\frac{\pi}{2}\right)^r}{y^{||\underline{\mathbf{s}}||-r}} .
\end{aligned}$$

In the same way, we have: $fe_-^{\underline{\mathbf{s}}}(x;y) \leq \dfrac{1}{r!} \dfrac{\left(\frac{\pi}{2}\right)^r}{y^{||\underline{\mathbf{s}}||-r}}$.

Hence:

$$\begin{aligned}
|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| &\leq \sum_{k=1}^{r} fe_+^{\underline{\mathbf{s}}^{<k}}(\Re e\, z; |\Im m\, z|) I e^{s_k}(\Re e\, z; |\Im m\, z|) fe_-^{\underline{\mathbf{s}}^{>k}}(\Re e\, z; |\Im m\, z|) \\
&\leq \sum_{k=1}^{r} \frac{1}{(k-1)!} \frac{\left(\dfrac{\pi}{2}\right)^{k-1}}{|\Im m\, z|^{||\underline{\mathbf{s}}^{<k}||-(k-1)}} \times \frac{1}{|\Im m\, z|^{s_k-1}} \times \\
&\qquad \frac{1}{(r-k)!} \frac{\left(\dfrac{\pi}{2}\right)^{r-k}}{|\Im m\, z|^{||\underline{\mathbf{s}}^{>k}||-(r-k)}} \\
&\leq \frac{\left(\frac{\pi}{2}\right)^{r-1}}{|\Im m\, z|^{||\underline{\mathbf{s}}||-r}} \sum_{k=1}^{r} \frac{1}{(k-1)!} \frac{1}{(r-k)!} \leq \frac{2^{r-1}}{(r-1)!} \frac{\left(\frac{\pi}{2}\right)^{r-1}}{|\Im m\, z|^{||\underline{\mathbf{s}}||-r}} \\
&\leq \frac{\pi^{r-1}}{(r-1)!} \frac{1}{|\Im m\, z|^{||\underline{\mathbf{s}}||-r}} \leq \frac{4^r}{r!} \frac{1}{|\Im m\, z|^{||\underline{\mathbf{s}}||-r}} .
\end{aligned}$$

To conclude, we only have to notice that, for $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_2)$, we have $||\underline{\mathbf{s}}|| \geq 2l(\underline{\mathbf{s}})$. Consequently, we deduce the sought upper bound:

$$|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \frac{1}{r!} \left(\frac{2}{\sqrt{|\Im m\, z|}} \right)^{||\underline{\mathbf{s}}||} .$$

$\square$

6.3. About the exponentially flat character

Now, we will focus on proving the exponentially flat character of convergent multitangent functions. Such a result is easy to understand with the Borel transform (for more details on it, see [3] , [9], [10] , [15] or [32] for an introduction of this notion and underlying ideas) . It turns out that it provides an argument allowing to justify that multitangents are exponentially flat near infinity because this transformation does not detect such functions.

The Borel transform $\mathcal{B}$ is defined for formal power series of the form $f(z) = \sum_{n\geq 0} \frac{a_n}{z^n}$ by:

$$\mathcal{B}\left(\sum_{n\geq 0} \frac{a_n}{z^{n+1}}\right)(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \ .$$

If $L : \mathcal{H}(\Omega) \longrightarrow \mathcal{H}(\Omega)$ is defined by $L(f)(z) = f(z+1)$, we thus have: $\mathcal{B}(L(f))(\zeta) = e^{-\zeta}\mathcal{B}(f)(\zeta)$; then, $\mathcal{B}\left(\sum_{p\in\mathbb{Z}} L^p(f)\right)(\zeta) = 0$.

Hence, we can deduce from this fact the exponentially flat character of convergent multitangent functions, but not in an explicit way. Although it might be possible to obtain it in an explicit way, we will not study this question here because we will use Fourier expansion to find out an explicit upper bound which shows this exponentially flat character.

6.3.1. An upper bound for Fourier coefficients of convergent multitangent functions

Let us start by finding an upper bound of the Fourier coefficients of a multitangent function with valuation at least 2 . For this purpose, let us fix a sequence $\underline{\mathbf{s}}$ of $\mathrm{seq}(\mathbb{N}_2)$ and $c > 0$.

Let $T^{\underline{\mathbf{s}}}_+$ and $T^{\underline{\mathbf{s}}}_-$ be the functions defined by $T^{\underline{\mathbf{s}}}_+(z) = \sum_{n>0} \widehat{\mathcal{T}}^{\underline{\mathbf{s}}}_n z^n$ and $T^{\underline{\mathbf{s}}}_-(z) = \sum_{n<0} \widehat{\mathcal{T}}^{\underline{\mathbf{s}}}_n z^n$. These are respectively defined and holomorphic on $\{\zeta \in \mathbb{C}; 0 < |\zeta| < e^{-2\pi c}\}$ and $\{\zeta \in \mathbb{C}; |\zeta| > e^{2\pi c}\}$.

If $z \in \mathbb{C}$ satisfies $|z| = e^{-2\pi r}$ (resp. $|z| = e^{2\pi r}$) with $r > c$, we can write: $z = e^{-2\pi r + 2i\theta\pi}$ (resp. $z = e^{2\pi r + 2i\theta\pi}$) for $\theta \in [0; 1[$. So, using the upper bound from Lemma 8, we obtain:

$$\begin{cases} |T_+^{\underline{\mathbf{s}}}(z)| = |\mathcal{T}e^{\underline{\mathbf{s}}}(\theta + ir)| \leq \left(\dfrac{2}{\sqrt{r}}\right)^{||\underline{\mathbf{s}}||} , \\ \\ |T_-^{\underline{\mathbf{s}}}(z)| = |\mathcal{T}e^{\underline{\mathbf{s}}}(\theta - ir)| \leq \left(\dfrac{2}{\sqrt{r}}\right)^{||\underline{\mathbf{s}}||} , \end{cases}$$

that is, for all $r > c$:

$$\begin{cases} \sup\limits_{|z|=e^{-2\pi r}} \left|T_+^{||\underline{\mathbf{s}}||}(z)\right| \leq \left(\dfrac{2}{\sqrt{r}}\right)^{||\underline{\mathbf{s}}||} . \\ \\ \sup\limits_{|z|=e^{2\pi r}} \left|T_-^{||\underline{\mathbf{s}}||}(z)\right| \leq \left(\dfrac{2}{\sqrt{r}}\right)^{||\underline{\mathbf{s}}||} . \end{cases}$$

Thus, for all integer n, all sequences $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_2)$ and all $r > c$, the Cauchy inequalities give us:

$$|\widehat{\mathcal{T}}_n^{\underline{\mathbf{s}}}| \leq 2\pi \left(\frac{2}{\sqrt{r}}\right)^{||\underline{\mathbf{s}}||} e^{2r|n|\pi} .$$

Hence, taking the limit as $r \longrightarrow c$ gives us:

$$\left|\widehat{\mathcal{T}}_n^{\underline{\mathbf{s}}}\right| \leq 2\pi \left(\frac{2}{\sqrt{c}}\right)^{||\underline{\mathbf{s}}||} e^{2c|n|\pi}.$$

We can repeat the same reasoning for any multitangent function, that is, we can use the upper bound from Lemma 7 instead of Lemma 8 . We obtain another upper bound of Fourier coefficients. These two inequalities are summed up in the following lemma:

Lemma 9. *Let us denote, for all $n \in \mathbb{Z}$ and all $\underline{\mathbf{s}} \in \mathcal{S}^\star$:*

$$\widehat{\mathcal{T}}_n^{\underline{\mathbf{s}}} = 2i\pi \sum_{j=1}^{l(\underline{\mathbf{s}})} \left(\sum_{k=2}^{s_j} \frac{(-2in\pi)^{k-1}}{(k-1)!} \mathcal{Z}_{j,s_j-k}^{\underline{\mathbf{s}}} \right) .$$

Then, for all integer n and $c > 0$, we have:

1. $\left|\widehat{\mathcal{T}}_n^{\underline{\mathbf{s}}}\right| \leq \dfrac{8\pi l(\underline{\mathbf{s}})}{c^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} e^{2c|n|\pi}$ *, where $\underline{\mathbf{s}} \in \mathcal{S}^\star$.*
2. $\left|\widehat{\mathcal{T}}_n^{\underline{\mathbf{s}}}\right| \leq 2\pi \left(\dfrac{2}{\sqrt{c}}\right)^{||\underline{\mathbf{s}}||} e^{2c|n|\pi}$ *, where $\underline{\mathbf{s}} \in seq(\mathbb{N}_2)$.*

6.3.2. Using the Fourier expansion

For a given $z \in \mathbb{C} - \mathbb{R}$, if we use the shorthand notation $n \lessgtr 0$ to denote $n > 0$ when $\Im m\, z > 0$, or $n < 0$ otherwise, the previous lemma and the Fourier expansion of a multitangent gives us for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star$ and all $c \in \left]0\,;|\Im m\, z|\right[$:

$$\begin{aligned}|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| &\leq \frac{8\pi l(\underline{\mathbf{s}})}{c^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} \sum_{n \lessgtr 0} e^{-2n\pi \Im m\, z} e^{2n\pi c} \\ &\leq \frac{8\pi l(\underline{\mathbf{s}})}{c^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} \frac{1}{e^{2\pi(|\Im m\, z|-c)}-1} \,.\end{aligned}$$

Moreover, we have $\dfrac{1}{e^{2x}-1} \leq \dfrac{4}{\mathrm{sh}^2(x)}$ provided $x > 0$; so this implies:

$$|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \frac{8\pi l(\underline{\mathbf{s}})}{c^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1}} \frac{4}{\mathrm{sh}^2(\pi(|\Im m\, z|-c))} \,.$$

This last inequality proves the exponentially flat character of convergent multitangent functions, which was our aim. We can also use the upper bound otbained for sequences $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_2)$. So, setting $c = \dfrac{|\Im m\, z|}{2}$, we have proved the following

Property 11. 1. *For all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star_{b,e}$ and $z \in \mathbb{C} - \mathbb{R}$, we have:*

$$|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \left(\frac{2}{|\Im m\, z|}\right)^{||\underline{\mathbf{s}}||-l(\underline{\mathbf{s}})-1} \frac{32\pi l(\underline{\mathbf{s}})}{sh^2\left(\dfrac{\pi|\Im m\, z|}{2}\right)} \,.$$

2. *For all sequences $\underline{\mathbf{s}} \in seq(\mathbb{N}_2)$ and $z \in \mathbb{C} - \mathbb{R}$, we have:*

$$|\mathcal{T}e^{\underline{\mathbf{s}}}(z)| \leq \left(\frac{2\sqrt{2}}{\sqrt{|\Im m\, z|}}\right)^{||\underline{\mathbf{s}}||} \frac{8\pi}{sh^2\left(\dfrac{\pi|\Im m\, z|}{2}\right)} \,.$$

7. Study of a symmetr$\underline{e}$l extension of multitangent functions to seq($\mathbb{N}^*$)

Our aim in this section is, if $\underline{\mathbf{s}}$ is a divergent sequence, that is when $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) - \mathcal{S}^\star$ to make a regularization of $\mathcal{T}e^{\underline{\mathbf{s}}}$. To be precise, the question is to define $\mathcal{T}e^{\underline{\mathbf{s}}}$ for $\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*)$ such that $s_1 = 1$ or $s_r = 1$, as well as when $s_1 = s_r = 1$ ...

Moreover, we want the extension satisfies the same properties as the convergent multitangent functions. So, we must keep:

1. The symmetr$\underline{\boldsymbol{e}}$lity of the extension of $\mathcal{T}e^\bullet(z)$.
2. The differentiation property (see Section 2.3).
3. The parity property (see Section 2.3).
4. The property of reduction into monotangent functions (see Section 3).

7.1. A generic method to extend the definition of a symmetr$\underline{\boldsymbol{e}}$l mould

In this section, we consider a symmetr$\underline{\boldsymbol{e}}$l mould $\mathcal{S}e^\bullet$ over the alphabet $\Omega = \mathbb{N}^*$, which is well defined for sequences in $S_b^\star = \{\underline{\mathbf{s}} \in \text{seq}(\mathbb{N}^*) \ ; \ s_1 \geq 2\}$. We want to define an extension of $\mathcal{S}e^\bullet$ for all the sequences of seq($\mathbb{N}^*$) such that the 'new' mould $\mathcal{S}e^\bullet$ is again a symmetr$\underline{\boldsymbol{e}}$l one.

The following lemma is due to Jean Ecalle. The first part is now well-known while the second was not published. To be exhaustive, we shall prove both points.

Lemma 10. 1. *For all $\theta \in \mathbb{C}$, there exists a unique symmetr$\underline{\boldsymbol{e}}$l extension of $\mathcal{S}e^\bullet$ to seq($\mathbb{N}^*$), denoted by $\mathcal{S}e_\theta^\bullet$, such that $\mathcal{S}e_\theta^1 = \theta$.*

2. *For all $\gamma \in \mathbb{C}$, let $\mathcal{N}e_\gamma^\bullet$ be the symmetr$\underline{\boldsymbol{e}}$l mould defined on sequences of seq($\mathbb{N}^*$) by:* $\mathcal{N}e_\gamma^{\underline{\mathbf{s}}} = \begin{cases} \dfrac{\gamma^r}{r!} & \text{, if } \underline{\mathbf{s}} = 1^{[r]} \ . \\ 0 & \text{, otherwise.} \end{cases}$

   Hence, for all $(\theta_1 \,;\, \theta_2) \in \mathbb{C}^2$, we have:

$$\mathcal{S}e_{\theta_1}^\bullet = \mathcal{N}e_{\theta_1 - \theta_2}^\bullet \times \mathcal{S}e_{\theta_2}^\bullet \ .$$

Proof. 1. If such an extension $\mathcal{S}e^\bullet$ exists, it may satisfy the algorithmic right move of the ones which begin an evaluation sequence. In other words, the following identities must be valid for all $k \in \mathbb{N}$ and all sequences $\underline{\mathbf{s}} \in \mathcal{S}_b^\star$:

$$(k+1)\mathcal{S}e^{1^{[k+1]}\cdot\underline{\mathbf{s}}} \ = \ \mathcal{S}e^1\mathcal{S}e^{1^{[k]}\cdot\underline{\mathbf{s}}} \ - \sum_{\underline{\mathbf{u}} \in sh\underline{\boldsymbol{e}}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}) - \{1^{[k+1]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e^{\underline{\mathbf{u}}} \ . \tag{18}$$

By induction on the number of ones which begin the sequences, these identities impose the uniqueness of the extension $\mathcal{S}e^\bullet$ to seq($\mathbb{N}^*$) .

To prove the existence of such an extension of $\mathcal{S}e^\bullet$, we define $\mathcal{S}e^\bullet$ recursively by (18). If $k \in \mathbb{N}$ and $\underline{\mathbf{s}} \in \mathcal{S}_b^\star$, we define $\mathcal{S}e_\theta^{1^{[k]}\cdot\underline{\mathbf{s}}}$ by:

$$(k+1)\mathcal{S}e_\theta^{1^{[k+1]}\cdot\underline{\mathbf{s}}} \;=\; \theta\mathcal{S}e_\theta^{1^{[k]}\cdot\underline{\mathbf{s}}} \;-\; \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}})-\{1^{[k+1]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e_\theta^{\underline{\mathbf{u}}}\;.$$

The only thing we have to verify is the symmetr$\underline{e}$lity of $\mathcal{S}e_\theta^\bullet$. This may be done by induction on $k+l$, where k and l denote the number of ones which begin the first and second sequences in the product:

$$\begin{aligned}
&(k+1)\mathcal{S}e_\theta^{1^{[k+1]}\cdot\underline{\mathbf{s}}^1}\mathcal{S}e_\theta^{1^{[l]}\cdot\underline{\mathbf{s}}^2}\\
&\quad= \theta\mathcal{S}e_\theta^{1^{[k]}\cdot\underline{\mathbf{s}}^1}\mathcal{S}e_\theta^{1^{[l]}\cdot\underline{\mathbf{s}}^2} - \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}^1)-\{1^{[k+1]}\cdot\underline{\mathbf{s}}^1\}} \mathcal{S}e_\theta^{\underline{\mathbf{u}}}\mathcal{S}e_\theta^{1^{[l]}\cdot\underline{\mathbf{s}}^2}\\
&\quad= \theta\sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k]}\cdot\underline{\mathbf{s}}^1,1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}} - \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}^1)-\{1^{[k+1]}\cdot\underline{\mathbf{s}}^1\}}\;\sum_{\underline{\mathbf{u}}'\in sh\underline{e}(\underline{\mathbf{u}};1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\quad= \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)}\;\sum_{\underline{\mathbf{u}}'\in sh\underline{e}(\underline{\mathbf{u}};1)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'} - \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}^1)}\;\sum_{\underline{\mathbf{u}}'\in sh\underline{e}(\underline{\mathbf{u}};1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\qquad +(k+1)\sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k+1]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\quad= \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2;1)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}} - \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}^1)}\;\sum_{\underline{\mathbf{u}}'\in sh\underline{e}(\underline{\mathbf{u}};1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\qquad +(k+1)\sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k+1]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\quad= \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2;1)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}} - \sum_{\underline{\mathbf{u}}\in sh\underline{e}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\qquad +(k+1)\sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k+1]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\\
&\quad= (k+1)\sum_{\underline{\mathbf{u}}\in sh\underline{e}(1^{[k+1]}\cdot\underline{\mathbf{s}}^1;1^{[l]}\cdot\underline{\mathbf{s}}^2)} \mathcal{S}e_\theta^{\underline{\mathbf{u}}'}\,,
\end{aligned}$$

where we have used the recursive definition of $\mathcal{S}e_\theta^\bullet$ in the first and third equality, the inductive step in the second one and finally the associativity and commutativity of the stuffle product in the last three equalities.

This concludes the proof of the first point and allows us to consider $\mathcal{S}e^{\bullet}_{\theta}$ for all $\theta \in \mathbb{C}$.

2. In order to prove full generality the formula expressing $\mathcal{S}e^{\bullet}_{\theta_1}$ in terms of $\mathcal{S}e^{\bullet}_{\theta_2}$, it is sufficient to prove that for all $\theta \in \mathbb{C}$, we have: $\mathcal{S}e^{\bullet}_{\theta} = \mathcal{N}e^{\bullet}_{\theta} \times \mathcal{S}e^{\bullet}_{0}$. Indeed, $\gamma \longmapsto \mathcal{N}e^{\bullet}_{\gamma}$ is a morphism from $(\mathbb{C} \,;+)$ to $(\mathcal{M}^{\bullet}_{\mathbb{C}}(\Omega) \,; \times)$.

Once again, we will prove it by induction on the number of ones which begin the evaluation sequence. It is clear that the formula is valid when $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b}$. Let us suppose that the formula holds for all sequences of $\mathrm{seq}(\mathbb{N}^*)$ which begin with at most k ones.

The algorithmic right move to the right of the ones gives, using the induction step:

$$\begin{aligned}
(k+1)\mathcal{S}e^{1^{[k+1]}\cdot\underline{\mathbf{s}}}_{\theta} &= \mathcal{S}e^{1}_{\theta}\mathcal{S}e^{1^{[k]}\cdot\underline{\mathbf{s}}}_{\theta} - \sum_{\underline{\mathbf{u}} \in sh\underline{\boldsymbol{e}}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}) - \{1^{[k+1]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e^{\underline{\mathbf{u}}}_{\theta} \\
&= \sum_{p=0}^{k} \frac{\theta^{p+1}}{p!}\mathcal{S}e^{1^{[k-p]}\cdot\underline{\mathbf{s}}}_{0} - \sum_{\underline{\mathbf{u}} \in sh\underline{\boldsymbol{e}}(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}}) - \{1^{[k+1]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e^{\underline{\mathbf{u}}}_{\theta} \\
&= \sum_{p=0}^{k} \frac{\theta^{p+1}}{p!}\mathcal{S}e^{1^{[k-p]}\cdot\underline{\mathbf{s}}}_{0} - \sum_{\underline{\mathbf{u}} \in A} \mathcal{S}e^{\underline{\mathbf{u}}}_{\theta} - \sum_{\underline{\mathbf{u}} \in B} \mathcal{S}e^{\underline{\mathbf{u}}}_{\theta} \ ,
\end{aligned}$$

where:

$$\begin{cases}
A = \left\{ 1^{[i]} \cdot 2 \cdot 1^{[k-i-1]} \cdot \underline{\mathbf{s}} \,; i \in [\![\, 0 \,;\, k-1 \,]\!] \right\} \ . \\
B = 1^{[k]} \cdot \left(sh\underline{\boldsymbol{e}}\,(1\,;\,\underline{\mathbf{s}}) - \{1 \cdot \underline{\mathbf{s}}\} \right) \ .
\end{cases}$$

We have:

$$\begin{aligned}
\sum_{\underline{\mathbf{u}}\in A} \mathcal{S}e_\theta^{\mathbf{u}} &= \sum_{i=0}^{k-1} \mathcal{S}e_\theta^{1^{[i]}\cdot 2\cdot 1^{[k-i-1]}\cdot\underline{\mathbf{s}}} = \sum_{i=0}^{k-1}\sum_{p=0}^{i} \frac{\theta^p}{p!}\mathcal{S}e_0^{1^{[i-p]}\cdot 2\cdot 1^{[k-i-1]}\cdot\underline{\mathbf{s}}} \\
&= \sum_{p=0}^{k-1}\left(\frac{\theta^p}{p!}\sum_{i=0}^{k-p-1} \mathcal{S}e_0^{1^{[i]}\cdot 2\cdot 1^{[k-p-i-1]}\cdot\underline{\mathbf{s}}}\right) \\
\sum_{\underline{\mathbf{u}}\in B} \mathcal{S}e_\theta^{\mathbf{u}} &= \sum_{p=0}^{k}\sum_{\underline{\mathbf{u}}\in sh\underline{e}\,(1\,;\,\underline{\mathbf{s}})-\{1\cdot\underline{\mathbf{s}}\}} \frac{\theta^p}{p!}\mathcal{S}e_0^{1^{[k-p]}\cdot\,\underline{\mathbf{u}}} \\
&= -\frac{\theta^k}{k!}\mathcal{S}e_0^{1\cdot\underline{\mathbf{s}}} + \sum_{p=0}^{k-1}\left(\frac{\theta^p}{p!}\sum_{\underline{\mathbf{u}}\in 1^{[k-p]}\cdot(sh\underline{e}\,(1\,;\,\cdot\underline{\mathbf{s}})-\{1\cdot\underline{\mathbf{s}}\})} \mathcal{S}e_0^{\mathbf{u}}\right)
\end{aligned}$$

Hence:

$$\begin{aligned}
\sum_{\underline{\mathbf{u}}\in sh\underline{e}\,(1\,;\,1^{[k]}\cdot\underline{\mathbf{s}})-\{1^{[k+1]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e_\theta^{\mathbf{u}} &= -\frac{\theta^k}{k!}\mathcal{S}e_0^{1\cdot\underline{\mathbf{s}}} + \sum_{p=0}^{k-1}\left(\frac{\theta^p}{p!}\sum_{\underline{\mathbf{u}}\in sh\underline{e}\,(1\,;\,1^{[k-p]}\cdot\underline{\mathbf{s}})-\{1^{[k+1-p]}\cdot\underline{\mathbf{s}}\}} \mathcal{S}e_0^{u}\right) \\
&= -\frac{\theta^k}{k!}\mathcal{S}e_0^{1\cdot\underline{\mathbf{s}}} - \sum_{p=0}^{k-1}\left(\frac{\theta^p}{p!}(k+1-p)\mathcal{S}e_0^{1^{[k+1-p]}\cdot\underline{\mathbf{s}}}\right) \\
&= -\sum_{p=0}^{k}\left(\frac{\theta^p}{p!}(k+1-p)\mathcal{S}e_0^{1^{[k+1-p]}\cdot\underline{\mathbf{s}}}\right)
\end{aligned}$$

We can now conclude:

$$\begin{aligned}
(k+1)\mathcal{S}e_\theta^{1^{[k+1]}\cdot\underline{\mathbf{s}}} &= \sum_{p=0}^{k}\frac{\theta^{p+1}}{p!}\mathcal{S}e_0^{1^{[k-p]}\cdot\underline{\mathbf{s}}} + \sum_{p=0}^{k}\left(\frac{\theta^p}{p!}(k+1-p)\mathcal{S}e_0^{1^{[k+1-p]}\cdot\underline{\mathbf{s}}}\right) \\
&= \sum_{p=0}^{k}\frac{\theta^{k+1-p}}{(k-p)!}\mathcal{S}e_0^{1^{[p]}\cdot\underline{\mathbf{s}}} + \sum_{p=1}^{k+1} p\frac{\theta^{k+1-p}}{(k+1-p)!}\mathcal{S}e_0^{1^{[p]}\cdot\underline{\mathbf{s}}} \\
&= \frac{\theta^{k+1}}{k!} + \sum_{n=1}^{k}\left((k+1)\frac{\theta^n}{n!}\mathcal{S}e_0^{1^{[k+1-n]}\cdot\underline{\mathbf{s}}}\right) + (k+1)\mathcal{S}e_0^{1^{[k+1]}\cdot\underline{\mathbf{s}}} \\
&= (k+1)\left(\mathcal{N}e_\theta^{\bullet}\times\mathcal{S}e_0^{\bullet}\right)^{1^{[k+1]}\cdot\underline{\mathbf{s}}} \quad .
\end{aligned}$$

By induction, for all sequences of type $1^{[k]}\cdot\underline{\mathbf{s}}$, with $\underline{\mathbf{s}}\in\mathcal{S}_b^\star$, we have $\mathcal{S}e_\theta^{\bullet}=\mathcal{N}e_\theta^{\bullet}\times\mathcal{S}e_0^{\bullet}$, which ends the proof of the lemma. $\square$

7.2. Trifactorization of $\mathcal{T}e^\bullet$ and consequences

The extension of multitangent functions to the divergent case is more complicated than the case exposed in the previous section. Actually, even if we want to impose the knowledge of $\mathcal{T}e^1$, one cannot apply the algorithmic right move of the ones which begin an evaluation sequence because the ones which begin or end the sequences will be sent respectively at the end or the beginning of the sequences in the game. To picture this, we have:

$$\mathcal{T}e^{1,2}(z) = \underbrace{\mathcal{T}e^1(z)}_{\substack{\text{known by}\\ \text{hypothesis}}} \times \underbrace{\mathcal{T}e^2(z)}_{\substack{\text{convergent}\\ \text{multitangent}\\ \text{function}}} - \underbrace{\mathcal{T}e^{2,1}(z)}_{\substack{\text{problem}\\ = \text{unknown}}} - \underbrace{\mathcal{T}e^3(z)}_{\substack{\text{convergent}\\ \text{multitangent}\\ \text{function}}}$$

The difficulty, here, comes from the joint management of the two sources of divergence created at $-\infty$ and $+\infty$. To overcome it, we would separate the divergence at $-\infty$ from that at $+\infty$. To this end, we will use a mould factorization in which each term has only one source of divergence. Let us remind that we have already used and proved such a factorization when we have proved the convergence rule for the multitangent functions (see p. 16) , but we give here a separate statement because of its importance:

Lemma 11. *Let $\mathcal{S}^\star_{b,e} = \{\underline{\mathbf{s}} \in seq(\mathbb{N}^*) ; s_1 \geq 2 \text{ and } s_r \geq 2\}$.*
Let us consider the symmetr$\underline{e}$l moulds $\mathcal{H}e^\bullet_+$, $\mathcal{H}e^\bullet_-$ and $\mathcal{C}e^\bullet$ valued in holomorphic functions over $\mathbb{C} - \mathbb{Z}$ and defined for all sequences $\underline{\mathbf{s}} \in \mathcal{S}^\star_b$ (resp. $\underline{\mathbf{s}} \in \mathcal{S}^\star_e$ and $\underline{\mathbf{s}} \in seq(\mathbb{N}^*)$) by:

$$\mathcal{H}e^{\underline{\mathbf{s}}}_+(z) = \sum_{0<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \text{ and } \mathcal{H}e^{\emptyset}_+(z) = 1 \ .$$

$$\mathcal{H}e^{\underline{\mathbf{s}}}_-(z) = \sum_{-\infty<n_r<\cdots<n_1<0} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \text{ and } \mathcal{H}e^{\emptyset}_-(z) = 1 \ .$$

$$\mathcal{C}e^{\underline{\mathbf{s}}}(z) = \begin{cases} 1 & \text{if } \underline{\mathbf{s}} = \emptyset \ . \\ \dfrac{1}{z^s} & \text{if } l(\underline{\mathbf{s}}) = 1 \ . \\ 0 & \text{if } l(\underline{\mathbf{s}}) > 1 \ . \end{cases}$$

Then:

$$\mathcal{T}e^\bullet = \mathcal{H}e^\bullet_+ \times \mathcal{C}e^\bullet \times \mathcal{H}e^\bullet_- \ .$$

The moulds $\mathcal{H}e_+^\bullet$ and $\mathcal{H}e_+^\bullet$ are named Hurwitz multizeta functions; $\mathcal{C}e^\bullet$ is going to play the role of a correction because it enters in game only for null summation indexes in the expression of $\mathcal{T}e^\bullet$. Let us remark that it is clear from their definition that these three moulds are symmetr$\underline{\boldsymbol{e}}$l , which explains the letter $\underline{\boldsymbol{e}}$ in their names.

First of all, we mention that this trifactorization acts as we wanted: it separates the divergence sources. Secondly, it is now clear that it is sufficient to extend (with the symmetr$\underline{\boldsymbol{e}}$lity property) the definition of the two moulds of Hurwitz multizeta functions to seq($\mathbb{N}^*$) in order to obtain an extension of $\mathcal{T}e^\bullet$ to seq($\mathbb{N}^*$) which is also symmetr$\underline{\boldsymbol{e}}$l . For this purpose, Lemma 10 can be applied: given $(\Phi_+;\Phi_-) \in \mathcal{H}(\mathbb{C}-\mathbb{Z})^2$, the moulds $\mathcal{H}e_+^\bullet$ and $\mathcal{H}e_-^\bullet$ admit a unique extension to seq($\mathbb{N}^*$) such that $\mathcal{H}e_+^1 = \Phi_+$ and $\mathcal{H}e_-^1 = \Phi_-$.

Finally, the following figure completes the figures 1 and 2. In the following diagram, we denote by

$$\mathcal{H}MZV_{CV,\pm} = \mathrm{Vect}_{\mathcal{M}ZV_{CV}(z)} \left(\mathcal{H}e_+^{\underline{\mathbf{s}}^1} \mathcal{H}e_-^{\underline{\mathbf{s}}^2} \right)_{\substack{\underline{\mathbf{s}}^1 \in \mathcal{S}_b^\star \\ \underline{\mathbf{s}}^2 \in \mathcal{S}_e^\star}} ,$$

where $\mathcal{S}_e^\star = \{(s_1;\cdots;s_r) \in \mathrm{seq}(\mathbb{N}^*)\,;\, s_r \geq 2\}$. From the trifactorisation, we see that $\mathcal{M}TGF_{CV}$ can be embedded in $\mathcal{H}MZV_{CV,\pm}$, which will be indicated by a curly arrow in the diagram. Recall that an arrow indicates a link between two algebras, while an arrow in dotted lines indicates a hypothetical link.

$$\begin{tikzcd}
\mathcal{M}ZV_{CV} \arrow[d, shift right, dotted, "\text{projection}"'] & & \mathcal{H}MZF_{+,CV} \arrow[ll, "\text{evaluation at } 0"'] \arrow[d, hook] \\
\mathcal{M}TGF_{CV} \arrow[u, shift right, "\textit{reduction}"'] \arrow[rr, hook, "\text{trifactorization}"] & & \mathcal{H}MZF_{\pm,CV}
\end{tikzcd}$$

Figure 4: Links between multizeta values and multitangent functions

As a consequence, we obtain the following

Corollary 2. *Let Φ_+ and Φ_- be two holomorphic functions over $\mathbb{C}-\mathbb{Z}$.*

The mould $\mathcal{T}e^{\bullet}$ admits a symmetr$\underline{\boldsymbol{e}}$l extension to seq$(\mathbb{N}^*)$ such that:

$$\begin{cases} \forall z \in \mathbb{C} - \mathbb{Z} \ , \ \mathcal{T}e^{1}(z) = \Phi_{+}(z) + \dfrac{1}{z} + \Phi_{-}(z) \ . \\ \\ \forall \underline{\mathbf{s}} \in seq(\mathbb{N}^*) \ , \ \mathcal{T}e^{\underline{\mathbf{s}}} = (\mathcal{H}e_{+}^{\bullet} \times \mathcal{C}e^{\bullet} \times \mathcal{H}e_{-}^{\bullet})^{\underline{\mathbf{s}}} \ . \end{cases}$$

7.3. Formal Hurwitz multizeta functions and formal multitangent functions

In order to simplify the following proof, we will work in the ring of formal power series by introducing the notion of formal Hurwitz multizeta functions and formal multitangent functions. To distinguish whether we are analytically or formally working without specifying it, we use two different notations. The formal character will be denoted by a straight capital letter while the analytic character will be denoted by a cursive capital letter (as we have always done from the begining) .

7.3.1. The mould $He_{+}^{\bullet}(X)$

We define formal Hurwitz multizeta functions as the Taylor expansions near 0 of the Hurwitz multizeta functions. Using the generalised product rule for the successive derivatives of a product as well as the formal Taylor formula, we defined the formal Hurwitz multizeta functions by $He_{+}^{\emptyset}(X) = 1$ and for all sequences $\underline{\mathbf{s}} \in \mathcal{S}_b^{\star}$ of length $r \in \mathbb{N}^*$ by:

$$He_{+}^{\underline{\mathbf{s}}}(X) = \sum_{k \geq 0} \sum_{k_1 + \cdots + k_r = k} \prod_{i=1}^{r} \binom{s_i + k_i - 1}{k_i} \mathcal{Z}e^{s_1 + k_1, \cdots, s_r + k_r}(-X)^k \ .$$

Thus, $He_{+}^{\bullet}(X)$ is a symmetr$\underline{\boldsymbol{e}}$l mould defined over $\mathcal{S}_b^{\star}$, valued in $\mathbb{C}[\![X]\!]$. According to Lemma 10, for all $S \in \mathbb{C}[\![X]\!]$, $He^{\bullet}$ has a unique symmetr$\underline{\boldsymbol{e}}$l extension to seq$(\mathbb{N}^*)$ such that $He_{+}^{1}(X) = S(X)$. We have now to define in a suitable manner $He_{+}^{1}(X)$. We can set:

$$He_{+}^{1}(X) = \sum_{k=1}^{+\infty} \mathcal{Z}e^{k+1}(-X)^k \ .$$

This definition is natural because we want the differentiation property to be satisfied by the extension of $\mathcal{H}e_{+}^{\bullet}$, as well as its formal analogue. Consequently, the analytic analogue of He_{+}^{1} is

$$\mathcal{H}e_{+}^{1}(z) = \sum_{n \geq 1} \left(\frac{1}{n+z} - \frac{1}{n} \right) \ .$$

Then, we obtain:

Property 12. *For all sequences* $\underline{\mathbf{s}} \in seq(\mathbb{N}^*)$*, we have:*

$$He_+^{\underline{\mathbf{s}}}(X) = \sum_{k\geq 0} \sum_{k_1+\cdots+k_r=k} \prod_{i=1}^{r} \binom{s_i+k_i-1}{k_i} \mathcal{Z}e^{s_1+k_1,\cdots,s_r+k_r}(-X)^k \ .$$

Proof. By uniqueness of the moulds satisfying these properties, it is sufficient to prove that the mould $\widetilde{He}_+^{\underline{\mathbf{s}}}(X)$ defined by the right hand side satisfies:

1. $\widetilde{He}_+^{\underline{\mathbf{s}}}(X)$ extends the definition of $He_+^{\underline{\mathbf{s}}}(X)$ to $\mathrm{seq}(\mathbb{N}^*)$.
2. $\widetilde{He}_+^{1}(X) = He_+^{1}(X)$.
3. $\widetilde{He}_+^{\bullet}(X)$ is a symmetr$\underline{\boldsymbol{e}}$l mould.

The third point is the only one which requires some explanations. Let us denote by $M_k^{\bullet}$ the k^{th} coefficient of $\widetilde{He}_+^{\underline{\mathbf{s}}}(X)$. In order to prove the symmetr$\underline{\boldsymbol{e}}$lity of $\widetilde{He}_+^{\underline{\mathbf{s}}}(X)$, we will show that:

$$\forall(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2) \in \big(\mathrm{seq}(\mathbb{N}^*)\big)^2, \forall p\in\mathbb{N}, \sum_{k=0}^{p} M_k^{\underline{\mathbf{s}}^1} M_{p-k}^{\underline{\mathbf{s}}^2} = \sum_{\underline{\gamma}\in sh\underline{e}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)} M_p^{\underline{\gamma}}.$$

For $(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2) \in \big(\mathrm{seq}(\mathbb{N}^*)\big)^2$ and $p\in\mathbb{N}$, we have if we denote $l(\underline{\mathbf{s}}^1)=r$ and $l(\underline{\mathbf{s}}^2)=r'$:

$$\begin{aligned}\sum_{k=0}^{p} M_k^{\underline{\mathbf{s}}^1} M_{p-k}^{\underline{\mathbf{s}}^2} &= \sum_{k_1+\cdots+k_{r+r'}=k} \left(\prod_{i=1}^{r}\binom{s_i^1+k_i-1}{k_i}\right)\left(\prod_{i=1}^{r'}\binom{s_i^2+k_{i+r}-1}{k_{i+r}}\right) \\ &\qquad \times \mathcal{Z}e^{s_1^1+k_1,\cdots,s_r^1+k_r}\,\mathcal{Z}e^{s_1^2+k_{r+1},\cdots,s_{r'}^2+k_{r'}} \\ &= \sum_{k_1+\cdots+k_{r+r'}=k} \left(\prod_{i=1}^{r}\binom{s_i^1+k_i-1}{k_i}\right)\left(\prod_{i=1}^{r'}\binom{s_i^2+k_{i+r}-1}{k_{i+r}}\right) \\ &\qquad \times\left(\sum_{\underline{\boldsymbol{\gamma}}\in sh\underline{e}(\underline{\mathbf{s}}^1+\underline{\mathbf{k}}^{\leq r};\underline{\mathbf{s}}^2+\underline{\mathbf{k}}^{>r})} \mathcal{Z}e^{\underline{\gamma}}\right),\end{aligned}$$

where $\underline{\mathbf{s}}^1+\underline{\mathbf{k}}^{\leq r}$ and $\underline{\mathbf{s}}^2+\underline{\mathbf{k}}^{>r}$ respectively denote $(s_1^1+k_1;\cdots;s_r^1+k_r)$ and $(s_1^2+k_{r+1};\cdots;s_{r'}^2+k_{r+r'})$. Two cases are possible:

1. $\underline{\boldsymbol{\gamma}}$ is a shuffle of $\underline{\mathbf{s}}^1+\underline{\mathbf{k}}^{\leq r}$ and $\underline{\mathbf{s}}^2+\underline{\mathbf{k}}^{>r}$:
   Then, we can reorder if necessary the k_i's such that the resulting term is $M_p^{\widetilde{\underline{\boldsymbol{\gamma}}}}$, where $\widetilde{\underline{\boldsymbol{\gamma}}}$ is deduced from $\underline{\boldsymbol{\gamma}}$ by setting $k_i=0$ for all i.

2. $\underline{\boldsymbol{\gamma}}$ contains one or more contractions of $\underline{\mathbf{s}}^1+\underline{\mathbf{k}}^{\leq r}$ and $\underline{\mathbf{s}}^2+\underline{\mathbf{k}}^{>r}$:
We can separate indexes which do not act on contractions from the other ones. Let us denote them by $\underline{\mathbf{s}}_i^1$ and $\underline{\mathbf{s}}_j^2$. This gives some sums of binomial coefficients:
$$\sum_{k_i+k_j=K}\binom{s_i^1+k_i-1}{k_i}\binom{s_j^1+k_j-1}{k_j}=\binom{s_i^1+s_j^2+K-1}{K},$$
which is exactly the expected binomial coefficient.
Thus, the term expected is again $M_p^{\widetilde{\underline{\boldsymbol{\gamma}}}}$, where $\widetilde{\underline{\boldsymbol{\gamma}}}$ is deduced from $\underline{\boldsymbol{\gamma}}$ by cancelling all the k_i's.

Let us remark that $\widetilde{\underline{\boldsymbol{\gamma}}}$ runs over the set $sh\underline{\boldsymbol{e}}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)$ when $\underline{\boldsymbol{\gamma}}$ runs over the set $sh\underline{\boldsymbol{e}}(\underline{\mathbf{s}}^1+\underline{\mathbf{k}}^{\leq r};\underline{\mathbf{s}}^2+\underline{\mathbf{k}}^{>r})$. We deduce from this that:
$$\sum_{k=0}^{p}M_k^{\underline{\mathbf{s}}^1}M_{p-k}^{\underline{\mathbf{s}}^2}=\sum_{\underline{\boldsymbol{\gamma}}\in sh\underline{\boldsymbol{e}}(\underline{\mathbf{s}}^1;\underline{\mathbf{s}}^2)}M_p^{\underline{\boldsymbol{\gamma}}}.$$
Consequently, $\widetilde{He}_+^{\underline{\mathbf{s}}}(X)$ is a symmetr$\underline{\boldsymbol{e}}$l mould and is equal to $He_+^{\bullet}(X)$.

$\square$

7.3.2. The mould $He_-^{\bullet}(X)$

The same can be done for the mould $He_-^{\bullet}(X)$. Thus, we can define this mould for all sequences $\underline{\mathbf{s}}\in\mathrm{seq}(\mathbb{N}^*)$ by:
$$\begin{aligned}He_-^{\underline{\mathbf{s}}}(X) &= (-1)^{||\underline{\mathbf{s}}||}He_+^{\overleftarrow{\underline{s}}}(-X)\\ &= \sum_{k\geq 0}\sum_{k_1+\cdots+k_r=k}\prod_{i=1}^{r}\binom{s_i+k_i-1}{k_i}\mathcal{Z}e_-^{s_1+k_1,\cdots,s_r+k_r}(-X)^k,\end{aligned}$$
where $\mathcal{Z}e_-^{\bullet}$ is defined from $\mathcal{Z}e^{\bullet}$ by a pseudo-parity relation:
$$\begin{aligned}\mathcal{Z}e_-^{s_1,\cdots,s_r} &= (-1)^{||\underline{\mathbf{s}}||}\mathcal{Z}e^{s_r,\cdots,s_1}=\sum_{0<n_1<\cdots<n_r}\frac{(-1)^{||\underline{\mathbf{s}}||}}{{n_1}^{s_1}\cdots{n_r}^{s_r}}\\ &= \sum_{p_r<\cdots<p_1<0}\frac{1}{{p_r}^{s_1}\cdots{p_1}^{s_r}}.\end{aligned}$$
Implicitly, this imposes:
$$\mathcal{H}e_-^1(z)=\sum_{n<0}\left(\frac{1}{n+z}-\frac{1}{n}\right).$$

7.3.3. The mould $Te^\bullet(X)$

We have just seen that for Φ_+ and Φ_- two holomorphic function over $\mathbb{C}-\mathbb{Z}$, there exists a symmetr$\underline{\boldsymbol{e}}$l extension of $\mathcal{T}e^\bullet$ to seq$(\mathbb{N}^*)$, defined by $\mathcal{T}e^\bullet = \mathcal{H}e_+^\bullet \times \mathcal{C}e^\bullet \times \mathcal{H}e_-^\bullet$, such that $\mathcal{T}e^1(z) = \mathcal{H}e_+^1(z) + \dfrac{1}{z} + \mathcal{H}e_-^1(z)$ for all $z \in \mathbb{C}-\mathbb{Z}$.

With the definition of the formal Hurwitz multizeta functions, the formal analogue $Te^\bullet(X)$ should be defined by:

$$Te^\bullet(X) = He_+^\bullet(X) \times Ce^\bullet(X) \times He_-^\bullet(X) \ ,$$

where $Ce^{\underline{\mathbf{s}}}(X) = \begin{cases} 1 & , \text{ if } l(\underline{\mathbf{s}}) = 0 \ . \\ X^{-s} & , \text{ if } l(\underline{\mathbf{s}}) = 1 \ . \\ 0 & , \text{ if } l(\underline{\mathbf{s}}) \geq 2 \ . \end{cases}$

Consequently, $Te^\bullet(X)$ is a symmetr$\underline{\boldsymbol{e}}$l mould defined on seq$(\mathbb{N}^*)$ and valued in $\mathbb{C}(\!(X)\!)$.

7.4. Properties of the extension of the mould $\mathcal{T}e^\bullet$ to seq$(\mathbb{N}^*)$

The convergent Hurwitz multizeta functions satisfies the differentiation and parity properties, as the convergent multitangent functions. We want that their extensions to seq$(\mathbb{N}^*)$ satisfy the same properties. These depend on the choice of the functions $\mathcal{H}e_+^1$ and $\mathcal{H}e_-^1$. From now on, we always define these functions by:

$$\mathcal{H}e_+^1(z) = \sum_{n>0}\left(\frac{1}{n+z} - \frac{1}{n}\right) \quad , \quad \mathcal{H}e_-^1(z) = \sum_{n<0}\left(\frac{1}{n+z} - \frac{1}{n}\right) .$$

With these definitions, these moulds are symmetr$\underline{\boldsymbol{e}}$l and satisfy:

Lemma 12. *For all sequences $\underline{\mathbf{s}} \in$ seq$(\mathbb{N}^*)$, we have:*

1. $\dfrac{\partial \mathcal{H}e_\pm^{\underline{\mathbf{s}}}}{\partial z} = -\displaystyle\sum_{i=1}^{l(\underline{\mathbf{s}})} s_i \mathcal{H}e_\pm^{\underline{\mathbf{s}}+e_i}$.
2. $\mathcal{H}e_+^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||}\mathcal{H}e_-^{\overleftarrow{\underline{s}}}(z)$ *, where $z \in \mathbb{C}-\mathbb{Z}$.*

Proof. 1. Let us begin by proving this for formal Hurwitz multizeta functions.

We just have to prove the first point because the second one is the definition of $He_{-}^{\underline{\mathbf{s}}}(X)$. For positive integers s and k, the key point of the following calculation is:

$$k\binom{s+k-1}{k} = s\binom{s+k-1}{k-1} \ .$$

Thus, for all $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)$, if the derivation of $\mathbb{C}[\![X]\!]$ is denoted by D, $D(He_{+}^{\underline{\mathbf{s}}})(X)$ is succelssively equals to:

$$\begin{aligned}
&-\sum_{k\geq 1}\sum_{\substack{k_1,\cdots,k_r\geq 0\\ k_1+\cdots+k_r=k}}\sum_{p=1}^{r} k_p\left(\prod_{i=1}^{r}\binom{s_i+k_i-1}{k_i}\right)\mathcal{Z}e^{s_1+k_1,\cdots,s_r+k_r}(-X)^{k-1}\\
&=-\sum_{p=1}^{r}\sum_{k\geq 0}\sum_{\substack{k_1,\cdots,\widehat{k_p},\cdots,k_r\geq 0\\ k_p\geq 1\\ k_1+\cdots k_r=k+1}} s_p\left(\prod_{i\in[\![\,1\,;\,r\,]\!]-\{p\}}\binom{s_i+k_i-1}{k_i}\right)\binom{s_p+k_p-1}{k_p-1}\\
&\qquad\qquad\times\mathcal{Z}e^{s_1+k_1,\cdots,s_r+k_r}(-X)^{k}\\
&=-\sum_{p=1}^{r}\sum_{k\geq 0}\sum_{\substack{k_1,\cdots,k_r\geq 0\\ k_1+\cdots+k_r=k}} s_p\left(\prod_{i\in[\![\,1\,;\,r\,]\!]-\{p\}}\binom{s_i+k_i-1}{k_i}\right)\binom{s_p+k_p}{k_p}\\
&\qquad\qquad\times\mathcal{Z}e^{s_1+k_1,\cdots,s_{p-1}+k_{p-1},s_p+1+k_p,s_{p+1}+k_{p+1},\cdots,s_r+k_r}(-X)^{k}\\
&=-\sum_{p=1}^{r} He_{+}^{s_1,\cdots,s_{p-1},s_p+1,s_{p+1},\cdots,s_r}(X) \ .
\end{aligned}$$

2. From the formal case to the analytic one.

It is well known that $0 \leq \mathcal{Z}e^{\underline{\mathbf{s}}} \leq 2$ (resp. $0 \leq \mathcal{Z}e_{-}^{\underline{\mathbf{s}}} \leq 2$) for all sequences $\underline{\mathbf{s}} \in \mathcal{S}_b^\star$ (resp. $\underline{\mathbf{s}} \in \mathcal{S}_e^\star$). Thus, the formal power series $He_{+}^{\underline{\mathbf{s}}}(X)$ and $He_{-}^{\underline{\mathbf{s}}}(X)$ are actually Taylor expansions. Consequently, the previous equalities are valid in the analytical case, first on the disc centered in 0 and with radius $\dfrac{1}{2}$ and then on $\mathbb{C}-\mathbb{Z}$, according to the identity theorem for holomorphic functions. $\square$

These properties have immediate consequences on $\mathcal{T}e^{\bullet}$ extended to seq($\mathbb{N}^*$). These, as well as Corollary 2 and the definition of $\mathcal{H}e_+^1$ and $\mathcal{H}_-^{\bullet}$ are summed up in the following theorem:

Theorem 4. *There exists a symmetr$\underline{\mathbf{e}}$l extension of $\mathcal{T}e^{\bullet}$ to seq($\mathbb{N}^*$), valued in holomorphic functions over $\mathbb{C} - \mathbb{Z}$, such that:*

$$\begin{cases} \mathcal{T}e^{\bullet} = \mathcal{H}e_+^{\bullet} \times \mathcal{C}e^{\bullet} \times \mathcal{H}e_-^{\bullet} \ . \\ \mathcal{T}e^1(z) = \dfrac{\pi}{\tan(\pi z)} \ , \text{ for all } z \in \mathbb{C} - \mathbb{Z} \ . \end{cases}$$

Moreover, the following properties hold for all sequences $\underline{\mathbf{s}} \in$ seq($\mathbb{N}^*$):

1. $\dfrac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z} = -\displaystyle\sum_{i=1}^{l(\underline{\mathbf{s}})} s_i \mathcal{T}e^{\underline{\mathbf{s}}+e_i}$.
2. $\mathcal{T}e^{\underline{\mathbf{s}}}(-z) = (-1)^{||\underline{\mathbf{s}}||} \mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z)$, *where* $z \in \mathbb{C} - \mathbb{Z}$.

Proof. From the definition of $\mathcal{H}e_+^1$ and $\mathcal{H}e_-^1$, we only have to prove the differentiation properties as well as the parity property. According to the trifactorisation, for $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)$ and if we denote $\underline{\mathbf{e}}_i = (0^{[i-1]}; 1; 0^{[l(\underline{\mathbf{s}})-i]})$, we have successively:

1.
$$\begin{aligned} \frac{\partial \mathcal{T}e^{\underline{\mathbf{s}}}}{\partial z} &= \sum_{\underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 \cdot \underline{\mathbf{s}}^3 = \underline{\mathbf{s}}} \left(\frac{\partial \mathcal{H}e_+^{\underline{\mathbf{s}}^1}}{\partial z} \mathcal{C}e^{\underline{\mathbf{s}}^2} \mathcal{H}e_-^{\underline{\mathbf{s}}^3} + \mathcal{H}e_+^{\underline{\mathbf{s}}^1} \frac{\partial \mathcal{C}e^{\underline{\mathbf{s}}^2}}{\partial z} \mathcal{H}e_-^{\underline{\mathbf{s}}^3} + \mathcal{H}e_+^{\underline{\mathbf{s}}^1} \mathcal{C}e^{\underline{\mathbf{s}}^2} \frac{\partial \mathcal{H}e_-^{\underline{\mathbf{s}}^3}}{\partial z} \right) \\ &= -\sum_{i=1}^{l(\underline{\mathbf{s}})} \sum_{\substack{\underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 \cdot \underline{\mathbf{s}}^3 = \underline{\mathbf{s}} \\ l(\underline{\mathbf{s}}^2)=1}} s_i \mathcal{H}e_+^{\underline{\mathbf{s}}^1 + \underline{\mathbf{e}}_i^{\leq l(\underline{\mathbf{s}}^1)}} \mathcal{C}e^{\underline{\mathbf{s}}^2 + \underline{\mathbf{e}}_i^{l(\underline{\mathbf{s}}^1) < \cdot \leq l(\underline{\mathbf{s}}^1) + l(\underline{\mathbf{s}}^2)}} \mathcal{H}e_-^{\underline{\mathbf{s}}^3 + \underline{\mathbf{e}}_i^{> l(\underline{\mathbf{s}}^1) + l(\underline{\mathbf{s}}^2)}} \ , \\ &= -\sum_{i=1}^{l(\underline{\mathbf{s}})} \left(s_i \sum_{\substack{\underline{\mathbf{s}}^1 \cdot \underline{\mathbf{s}}^2 \cdot \underline{\mathbf{s}}^3 = \underline{\mathbf{s}} + \underline{\mathbf{e}}_i \\ l(\underline{\mathbf{s}}^2)=1}} \mathcal{H}e_+^{\underline{\mathbf{s}}^1} \mathcal{C}e^{\underline{\mathbf{s}}^2} \mathcal{H}e_-^{\underline{\mathbf{s}}^3} \right) \\ &= -\sum_{i=1}^{l(\underline{\mathbf{s}})} s_i \mathcal{T}e^{\underline{\mathbf{s}} + \underline{\mathbf{e}}_i} \ . \end{aligned}$$

2. $\mathcal{T}e^{\underline{\mathbf{s}}}(-z) = \sum_{\underline{\mathbf{s}}^1\cdot\underline{\mathbf{s}}^2\cdot\underline{\mathbf{s}}^3=\underline{\mathbf{s}}} \mathcal{H}e_+^{\underline{\mathbf{s}}^1}(-z)\mathcal{C}e^{\underline{\mathbf{s}}^2}(-z)\mathcal{H}e_-^{\underline{\mathbf{s}}^3}(-z)$ $\square$

$$= (-1)^{||\underline{\mathbf{s}}||}\sum_{\underline{\mathbf{s}}^1\underline{\mathbf{s}}^2\underline{\mathbf{s}}^3=\underline{\mathbf{s}}} \mathcal{H}e_+^{\overleftarrow{\underline{\mathbf{s}}^3}}(z)\mathcal{C}e^{\overleftarrow{\underline{\mathbf{s}}^2}}(z)\mathcal{H}e_-^{\overleftarrow{\underline{\mathbf{s}}^1}}(z)$$

$$= (-1)^{||\underline{\mathbf{s}}||}\mathcal{T}e^{\overleftarrow{\underline{\mathbf{s}}}}(z) \ .$$

7.5. Reduction into monotangent functions

The only property of divergent multitangent which remains to be proved is the reduction into monotangent functions. From now on and until the end of this section, our aim is to find and prove such a property for convergent and divergent multitangent functions. For this, we will adapt the proof given in the convergent case. To simplify the calculation, we will apply the same technique, that is to say a partial fraction expansion, but for the generating series $\mathcal{T}ig^\bullet$. Then, we will see that the reduction into monotangent functions can not have exactly the same expression as in the convergent case.

7.5.1. A first expression of $Tig^\bullet(X)$

The moulds $\mathcal{Z}e^\bullet$, $Te^\bullet(X)$, $He_+^\bullet(X)$, $He_-^\bullet(X)$ and $Ce^\bullet(X)$ are symmetr$\underline{e}$l . We will consider their generating functions, respectively denoted by $\mathcal{Z}ig^\bullet$, $Tig^\bullet(X)$, $Hig_+^\bullet(X)$, $Hig_-^\bullet(X)$ and $Cig^\bullet(X)$. Let us remark that these moulds are valued in $\mathbb{C}[\![X]\!]$ or $\mathbb{C}(\!(X)\!)$.

We can begin with the calculation of the generating functions of $He_+^\bullet(X)$:

Lemma 13. *The generating function of the mould $He_+^\bullet(X)$, denoted by $Hig_+^\bullet(X)$ and valued in $\mathbb{C}[\![X]\!][\![(Y_r)_{r\in\mathbb{N}^*}]\!] \simeq \mathbb{C}[\![X;Y_1;Y_2;\cdots]\!]$, is :*

$$Hig_+^{Y_1,\cdots,Y_r}(X) = \mathcal{Z}ig^{Y_1-X,\cdots,Y_r-X} \ .$$

Such a result could be expected, because Hurwitz multizeta functions $\mathcal{H}e_+^\bullet(z)$ are nothing else than translations of multizeta values. Consequently, this should have a translation readable on the generating function.

Proof. Let $r \in \mathbb{N}^*$. Let us denote D_{Y_i} the derivation with respect to Y_i and $S(F)$ the constant term of $F \in \mathbb{A}[\![Y_1;\cdots;Y_r]\!]$.
For all $(k_1;\cdots;k_r) \in (\mathbb{N}^*)^r$, we have:

$$\begin{aligned}
&\frac{1}{k_1!\cdots k_r!}S\left(D_{Y_1}^{k_1}\circ\cdots\circ D_{Y_r}^{k_r}(\mathcal{Z}ig^{Y_1-X,\cdots,Y_r-X})\right)\\
&\quad= S\left(\sum_{p_1,\cdots,p_r\geq 0}\mathcal{Z}e^{p_1+k_1+1,\cdots,p_r+k_r+1}\left(\prod_{i=1}^{r}\binom{p_i+k_i}{k_i}(Y_i-X)^{p_i}\right)\right)\\
&\quad= \sum_{p_1,\cdots,p_r\geq 0}\mathcal{Z}e^{p_1+k_1+1,\cdots,p_r+k_r+1}\binom{p_1+k_1}{k_1}\cdots\binom{p_r+k_r}{k_r}(-X)^{p_1+\cdots+p_r}\\
&\quad= \sum_{p\geq 0}\sum_{p_1+\cdots+p_r=p}\mathcal{Z}e^{p_1+k_1+1,\cdots,p_r+k_r+1}\binom{p_1+k_1}{k_1}\cdots\binom{p_r+k_r}{k_r}(-X)^{p}\\
&\quad= He_+^{k_1+1,\cdots,k_r+1}(X)\ .
\end{aligned}$$

Thus, the formal Taylor formula for formal power series in several indeterminates gives: $\mathcal{Z}ig^{Y_1-X,\cdots,Y_r-X} = \sum_{k_1,\cdots,k_r\geq 0} He_+^{k_1+1,\cdots,k_r+1}(X)Y_1^{k_1}\cdots Y_r^{k_r}$

$$= Hig_+^{Y_1,\cdots,Y_r}(X)\ .$$

□

Moreover, according to the parity property, the generating function $\mathcal{Z}ig_-^{Y_1,\cdots,Y_r}$ of $\mathcal{Z}e_-^{\bullet}$ is defined by:

$$\mathcal{Z}ig_-^{Y_1,\cdots,Y_r} = (-1)^r\mathcal{Z}ig^{-Y_r,\cdots,-Y_1}\ .$$

The same calculation as in the previous proof gives:

Lemma 14. *The generating function of the mould $He_-^{\bullet}(X)$, denoted by $Hig_-^{\bullet}(X)$ and valued in $\mathbb{C}[\![X]\!][\![(Y_r)_{r\in\mathbb{N}^*}]\!] \simeq \mathbb{C}[\![X;Y_1;Y_2;\cdots]\!]$, is:*

$$Hig_-^{Y_1,\cdots,Y_r}(X) = \mathcal{Z}ig_-^{Y_1-X,\cdots,Y_r-X}.$$

These two lemmas, together with the trifactorisation, imply immediately the first expression of $Tig^{\bullet}(X)$. Even if it is a standard calculation in mould calculus, the following expression of $Tig^{\bullet}(X)$ deserves to be singled out because it is the first step of the computation of $Tig^{\bullet}(X)$.

Property 13. *Let us denote by $\mathcal{Z}ig^{\bullet}(X)$ and $\mathcal{Z}ig_-^{\bullet}(X)$ the moulds valued in $\mathbb{C}[\![X]\!][\![(Y_r)_{r\in\mathbb{N}^*}]\!] \simeq \mathbb{C}[\![X;Y_1;Y_2;\cdots]\!]$ respectively defined by:*

$$\mathcal{Z}ig^{Y_1,\cdots,Y_r}(X) = \mathcal{Z}ig^{Y_1-X,\cdots,Y_r-X}.$$

$$\mathcal{Z}ig_-^{Y_1,\cdots,Y_r}(X) = \mathcal{Z}ig_-^{Y_1-X,\cdots,Y_r-X}.$$

Then, in $\mathbb{C}(\!(X)\!)[\![(Y_r)_{r\in\mathbb{N}^*}]\!]$*, we have:*

$$Tig^\bullet(X) = \mathcal{Z}ig^\bullet(X) \times Cig^\bullet(X) \times \mathcal{Z}ig_-^\bullet(X).$$

7.5.2. Second expression of $Tig^\bullet$ *and flexion markers*

The second expression of $Tig^\bullet(X)$ will use some notations and notions introduced by Jean Ecalle for his study of flexion structures (see. [16], [17], [18] or [19]). Let us introduce them before stating the result.

Flexion markers. The four flexion markers $\lfloor$, $\rfloor$, $\lceil$ and $\rceil$ act on factorisation of (bi)sequences. So, let us consider two alphabets Ω_1 , Ω_2 and then their product $\Omega = \Omega_1 \times \Omega_2$; let us also consider a sequence $\underline{\mathbf{w}} \in \mathrm{seq}(\Omega)$ which can be factorize:

$$\underline{\mathbf{w}} = \underline{\mathbf{w}}^1 \cdots \underline{\mathbf{w}}^r \in \mathrm{seq}(\Omega) .$$

The flexion marker $\lfloor$ acts on $\underline{\mathbf{w}}^i$ by subtracting the right inferior element of $\underline{\mathbf{w}}^{i-1}$ to each inferior element of $\underline{\mathbf{w}}^i$ while the flexion marker $\lceil$ acts on $\underline{\mathbf{w}}^i$ by adding the sum of superior elements of $\underline{\mathbf{w}}^{i-1}$ to the left superior element of $\underline{\mathbf{w}}^i$. In the same way, the flexion marker $\rfloor$ acts on $\underline{\mathbf{w}}^i$ by subtracting the left inferior element of $\underline{\mathbf{w}}^{i+1}$ to each inferior element of $\underline{\mathbf{w}}^i$ while the flexion marker $\rceil$ acts on $\underline{\mathbf{w}}^i$ by adding the sum of superior elements of $\underline{\mathbf{w}}^{i+1}$ to the right superior element of $\underline{\mathbf{w}}^i$.

By the use of these flexion markers, elements of Ω_1 will be added each other while element of Ω_2 will be subtracted each other.

To clarify the definitions and the actions of the different markers, here is an example. If $\underline{\mathbf{w}} = \cdots \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} \cdots = \cdots \begin{pmatrix} u_6, \cdots, u_9 \\ v_6, \cdots, v_9 \end{pmatrix} \begin{pmatrix} u_{10}, \cdots, u_{15} \\ v_{10}, \cdots, v_{15} \end{pmatrix} \cdots,$

then we have:

$$\underline{\mathbf{a}}\rfloor = \begin{pmatrix} u_6 & ,\cdots, & u_9 \\ \\ v_{6:10} & ,\cdots, & v_{9:10} \end{pmatrix} \quad , \quad \underline{\mathbf{a}}\rceil = \begin{pmatrix} u_6 & ,\cdots, & u_8, & u_{9\cdots15} \\ \\ v_6 & ,\cdots, & v_8, & v_9 \end{pmatrix} ,$$

$$\lfloor\underline{\mathbf{b}} = \begin{pmatrix} u_{11} & ,\cdots, & u_{15} \\ \\ v_{11:9} & ,\cdots, & v_{15:9} \end{pmatrix} \quad , \quad \lceil\underline{\mathbf{b}} = \begin{pmatrix} u_{6\cdots10} & , u_{11\cdots15} & & \\ \\ v_{10} & , v_{11} & ,\cdots, & v_{15} \end{pmatrix} ,$$

where $n_{i\cdots j} = n_i + \cdots + n_j$ and $n_{i:j} = n_i - n_j$ in the variables n.

Colors. If we don't care, it is easy not to see flexion structures. But, when there are some addition or subtraction of the variables, flexion structures are possibly present. The use of colors is a good way to avoid passing next to them. So, we will stiffen a bit more the situation by using colors in a temporary way. This requires to redefine our moulds, that is, they become bimoulds.

First, there is the bimould of coloured multizeta values defined for sequences in $\mathrm{seq}\,(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*)$ by:

$$\mathcal{Z}e^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix}} = \sum_{1 \le n_r < \cdots < n_1} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{{n_1}^{s_1} \cdots {n_r}^{s_r}} \text{ where } e_k = e^{-2i\pi\varepsilon_k}, \text{ for } k \in [\![\, 1 \,; n \,]\!].$$

Its generating series $\mathcal{Z}ig^\bullet$ is then a symmetr$\underline{i}$l mould defined for sequences in $\mathrm{seq}\big(\mathbb{Q}/\mathbb{Z} \times (V_i)_{i\in\mathbb{N}^*}\big)$. This also gives us the definition of the bimould $\mathcal{Z}ig_-^\bullet$:

$$\mathcal{Z}ig_-^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ V_1, \cdots, V_r \end{pmatrix}} = (-1)^r \mathcal{Z}ig^{\begin{pmatrix} -\varepsilon_r, \cdots, -\varepsilon_1 \\ -V_r, \cdots, -V_1 \end{pmatrix}} .$$

In a similar way, we can define formal or analytical coloured Hurwitz multizeta functions as well as formal or analytical coloured multitangent functions. These are bimoulds valued in the algebra of holomorphic functions over $\mathbb{C} - \mathbb{Z}$ or in $\mathbb{C}[\![X]\!]$. If $\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix} \in \mathrm{seq}\left(\mathbb{Q}/\mathbb{Z} \times \mathbb{N}^*\right) - \{\emptyset\}$, with the notation $e_k = e^{-2i\pi\varepsilon_k}$, for $k \in [\![1 ; n]\!]$, these are respectively defined by:

$$\mathcal{H}e_+^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix}}(z) = \sum_{0<n_r<\cdots<n_1<+\infty} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} \ ,$$

$$He_+^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix}}(X) = \sum_{\substack{k, k_1, \cdots, k_r \geq 0 \\ k_1 + \cdots + k_r = k}} \left[\prod_{i=1}^{r} \begin{pmatrix} s_i + k_i - 1 \\ k_i \end{pmatrix} \right] \mathcal{Z}e^{\begin{pmatrix} \varepsilon_1 & , \cdots, & \varepsilon_r \\ s_1 + k_1, & \cdots & , s_r + k_r \end{pmatrix}} (-X)^k \ ,$$

$$\mathcal{T}e^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix}}(z) = \sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1+z)^{s_1} \cdots (n_r+z)^{s_r}} \ ,$$

$$Te^{\begin{pmatrix} \varepsilon_1, \cdots, \varepsilon_r \\ s_1, \cdots, s_r \end{pmatrix}}(X) = \sum_{\begin{pmatrix} \underline{\varepsilon}^1 \\ \underline{\mathbf{s}}^1 \end{pmatrix} \cdot \begin{pmatrix} \underline{\varepsilon}^2 \\ \underline{\mathbf{s}}^2 \end{pmatrix} \cdot \begin{pmatrix} \underline{\varepsilon}^3 \\ \underline{\mathbf{s}}^3 \end{pmatrix} = \begin{pmatrix} \underline{\varepsilon} \\ \underline{\mathbf{s}} \end{pmatrix}} He_+^{\begin{pmatrix} \underline{\varepsilon}^1 \\ \underline{\mathbf{s}}^1 \end{pmatrix}}(X) Ce^{\begin{pmatrix} \underline{\varepsilon}^2 \\ \underline{\mathbf{s}}^2 \end{pmatrix}}(X) He_-^{\begin{pmatrix} \underline{\varepsilon}^3 \\ \underline{\mathbf{s}}^3 \end{pmatrix}}(X) \ .$$

Obviously, these definitions contain some divergent cases for Hurwitz multizeta functions as well as for multitangent functions: in the first case, it is when $(\varepsilon_1; s_1) = (0; 1)$, while it is when $(\varepsilon_1; s_1) = (0; 1)$ or $(\varepsilon_r; s_r) = (0; 1)$ in the second case. In these exceptional cases, a regularization process is needed and is based, as we have done previously without colors, on the regularization of the generating series $\mathcal{Z}ig^\bullet$ and, so, on the following well-known lemma due to Jean Ecalle (see [17] p. 5 and [19] p. 6) :

Lemma 15. *Let* $\mu^{n_1, \cdots, n_r} = \dfrac{1}{r_1! \cdots r_n!}$ *where the non-increasing sequence* $\underline{\mathbf{n}} = (n_1; \cdots; n_r) \in$ *seq*$(\mathbb{N}^*)$ *attains* r_1 *times its highest value,* r_2 *times*

its second highest value, etc.
For $(u_p)_{p\in[\![\,1\,;\,r\,]\!]} \in \mathbb{C}^r$ *and* $k \in [\![\,1\,;\,r\,]\!]$ *, let* $e_k = e^{-2iu_k\pi}$.

Finally, for all $k \in \mathbb{N}^*$*, we consider the moulds* $do\mathcal{Z}ig_k^{\bullet}$ *and* $co\mathcal{Z}ig_k^{\bullet}$ *defined for all* $\begin{pmatrix} u_1, \cdots, u_r \\ v_1, \cdots, v_r \end{pmatrix} \in seq\big(\mathbb{Q}/\mathbb{Z} \times (V_i)_{i\in\mathbb{N}^*}\big)$ *by:*

$$do\mathcal{Z}ig_k^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = \begin{cases} \displaystyle\sum_{1\le n_r<\cdots<n_1<k} \frac{{e_1}^{n_1}\cdots {e_r}^{n_r}}{(n_1-V_1)\cdots(n_r-V_r)} & , \text{ if } r \neq 0\ . \\ 1 & , \text{ if } r = 0\ . \end{cases}$$

$$co\mathcal{Z}ig_k^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = \begin{cases} (-1)^r \displaystyle\sum_{1\le n_r\le\cdots\le n_1<k} \frac{\mu^{n_1,\cdots,n_r}}{n_1\cdots n_r} & , \text{ if } \underline{\mathbf{u}} \neq \underline{\mathbf{0}} \text{ and } r \neq 0\ . \\ 0 & , \text{ if } \underline{\mathbf{u}} = \underline{\mathbf{0}} \text{ and } r \neq 0\ . \\ 1 & , \text{ if } r = 0\ . \end{cases}$$

Then, the mould $\mathcal{Z}ig$ *admits an elementary mould "factorisation":*

$$\mathcal{Z}ig^{\bullet} = \lim_{n\longrightarrow+\infty} \left(co\mathcal{Z}ig_n^{\bullet} \times do\mathcal{Z}ig_n^{\bullet}\right)\ .$$

Let us remark that in this "factorisation", the mould $do\mathcal{Z}ig_k^{\bullet}$ gives us the dominant terms of $\mathcal{Z}ig^{\bullet}$, while the mould $co\mathcal{Z}ig_k^{\bullet}$ play the role of correcting the series to restore the convergence of the divergent series

$$\sum_{1\le n_r<\cdots<n_1<k} \frac{{e_1}^{n_1}\cdots {e_r}^{n_r}}{(n_1-v_1)\cdots(n_r-v_r)}\ .$$

Some new moulds. Let δ be the indicator function of $\{0\}$. Let us also consider the formal bimoulds $Qig^{\bullet}$ and $\delta^{\bullet}$ defined on $\mathrm{seq}\big(\mathbb{Q}/\mathbb{Z} \times (V_i)_{i\in\mathbb{N}^*}\big)$

by:

$$\begin{cases} Qig^{\emptyset} = 0 \ . \\ Qig^{\begin{pmatrix} u_1 \\ V_1 \end{pmatrix}} = -Te^{\begin{pmatrix} u_1 \\ 1 \end{pmatrix}}(V_1) \ . \\ Qig^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = 0 \ , \text{ if } r \geq 2 \ . \end{cases}$$

$$\begin{cases} \delta^{\emptyset} = 0 \ . \\ \delta^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = \begin{cases} \dfrac{(i\pi)^r}{r!}\delta(u_1)\cdots\delta(u_r) & , \text{ if } r \text{ is even.} \\ 0 & , \text{ if } r \text{ is odd.} \end{cases} \end{cases}$$

Second expression of $Tig^\bullet$. We will apply the previous lemma, which gives an expression of $\mathcal{Z}ig^\bullet$, in the first expression of $Tig^\bullet$. This will allow us to make a partial fraction expansion in the indeterminate X. We then obtain:

Theorem 5. *Let $\mathcal{Q}ig^\bullet$ be the bimould valued in $\mathcal{H}(\mathbb{C} - \mathbb{Z})$ and defined for all $z \in \mathbb{C} - \mathbb{Z}$ by:*

$$\mathcal{Q}ig^{y_1, \cdots, y_r}(z) = \begin{cases} -\mathcal{T}e^1(y_1 - z) & , \textit{ if } r = 1 \ . \\ 0 & , \textit{ otherwise.} \end{cases}$$

Then, for all $z \in \mathbb{C} - \mathbb{Z}$, we have in $\mathbb{C}[\![X]\!][\![(V_r)_{r \in \mathbb{N}^*}]\!]$:

$$\mathcal{T}ig^\bullet(z) = \delta^\bullet + \mathcal{Z}ig^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(z) \times \mathcal{Z}ig_-^{\lfloor\bullet} \ .$$

PROOF. Continuing to use the same principle for our notations, we set:

$$\begin{array}{rcl} do\mathcal{Z}ig_N^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}}(X) & = & do\mathcal{Z}ig_N^{\begin{pmatrix} u_1 & , \cdots, & u_r \\ V_1 - Y, & \cdots & ,V_r - Y \end{pmatrix}} . \\ co\mathcal{Z}ig_N^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}}(X) & = & co\mathcal{Z}ig_N^{\begin{pmatrix} u_1 & , \cdots, & u_r \\ V_1 - Y, & \cdots & ,V_r - Y \end{pmatrix}} . \\ do\mathcal{Z}ig_{-,N}^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}}(X) & = & (-1)^r do\mathcal{Z}ig_{-,N}^{\begin{pmatrix} -u_r, \cdots, -u_1 \\ -V_r, \cdots, -V_1 \end{pmatrix}}(X) . \\ co\mathcal{Z}ig_{-,N}^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}}(X) & = & (-1)^r co\mathcal{Z}ig_{-,N}^{\begin{pmatrix} -u_r, \cdots, -u_1 \\ -V_r, \cdots, -V_1 \end{pmatrix}}(X) . \end{array}$$

Let $\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix} \in \text{seq}\big(\mathbb{Q}/\mathbb{Z} \times (V_i)_{i \in \mathbb{N}^*}\big)$. Lemmas 13 and 15 give:

$$Tig^\bullet(X) = \lim_{N \longrightarrow +\infty} \left(co\mathcal{Z}ig_N^\bullet \times Tig_N^\bullet(X) \times co\mathcal{Z}ig_{-,N}^\bullet \right) ,$$

where $Tig_N^\bullet(X) = do\mathcal{Z}ig_N^\bullet(X) \times Cig^\bullet(X) \times do\mathcal{Z}ig_{-,N}^\bullet(X)$.

It is not difficult to obtain another form of the previous trifactorisation by proceeding in the same way as in the proof of the trifactorisation of $\mathcal{T}e^\bullet$:

$$\mathcal{T}ig_N^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = \sum_{-N < n_r < \cdots < n_1 < N} \frac{{e_1}^{n_1} \cdots {e_r}^{n_r}}{(n_1 - V_1 + X) \cdots (n_r - V_r + X)} . \tag{19}$$

Now, we can write down the partial fraction expansion in $\mathcal{T}ig_N^\bullet$:

$$Tig_N^{\begin{pmatrix} u_1, \cdots, u_r \\ V_1, \cdots, V_r \end{pmatrix}} = \sum_{k=1}^{r} \sum_{-N<n_r<\cdots<n_1<N} \left(\prod_{\substack{j \in [\![1\,;\,r]\!] \\ j \neq k}} \frac{{e_j}^{n_j}}{n_j - n_k + V_k - V_j} \right) \times \frac{{e_k}^{n_k}}{n_k - V_k + X} \ .$$

Plugging this expansion in (19), after some calculations, we obtain in $\mathbb{C}[\![X]\!][\![(V_r)_{r\in\mathbb{N}^*}]\!]$:

$$Tig^\bullet(X) = \delta^\bullet + \mathcal{Z}ig^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(X) \times \mathcal{Z}ig_-^{\lfloor\bullet} \ .$$

It is clear that $\mathcal{Q}ig^\bullet(z)$ is also well defined in a neighbourhood of 0 in $\mathrm{seq}(\mathbb{C})$. Moreover, the generating functions $\mathcal{Z}ig^\bullet$ and $\mathcal{Z}ig_-^\bullet$ are actually Taylor expansions defined in $\mathrm{seq}\big(D(0;1)\big)$. Here, the key point is that $|\mathcal{Z}e^{\underline{\mathbf{s}}}| \leq 4^r r!$ for all sequences $\underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*)$ of length r. Such an upper bound is far from being precise, but is sufficient for our purpose. A proof of it comes from the algorithmic right move of the ones which begin an evaluation sequence and:

$$\sharp sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}) = \sum_{k=0}^{\min(a;b)} 2^k \binom{a}{a-k} \binom{b}{b-k} \ .$$

This gives a neighbourhood of 0 in $\mathrm{seq}(\mathbb{C})$ where $\mathcal{Z}ig^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(X) \times \mathcal{Z}ig_-^{\lfloor\bullet}$ defines an analytic function. The identity theorem for holomorphic functions concludes the proof of this theorem.

$\square$

To conclude this section, let us explain why the corrective term $\delta^\bullet$ is mandatory.

Let us imagine this is not the case, that is to say that we have two functions φ and ψ such that the mould $\mathcal{T}e^\bullet$ is extended to the divergent case by $\mathcal{T}e^\bullet = \mathcal{H}e^\bullet_{+,\varphi} \times \mathcal{C}e^\bullet \times \mathcal{H}e^\bullet_{-,\psi}$, where $\mathcal{H}e^\bullet_{+,\varphi}$ and $\mathcal{H}e^\bullet_{-,\psi}$ are respectively the extension to the divergent case of the moulds $\mathcal{H}e^\bullet_+$ and $\mathcal{H}e^\bullet_-$ such that $\mathcal{H}e^1_+ = \varphi$ and $\mathcal{H}e^1_- = \psi$.

Then, we would have the following identity because of the fundamental equality proved in the previous theorem, but without the corrective term:

$$\mathcal{H}ig^{\bullet}_{+,\varphi} \times \mathcal{C}ig^{\bullet} \times \mathcal{H}ig^{\bullet}_{-,\psi} = \mathcal{T}ig^{\bullet} = \mathcal{Z}ig^{\bullet\rfloor} \times \mathcal{Q}ig^{\lceil\bullet\rceil}(z) \times \mathcal{Z}ig^{\lfloor\bullet}_{-} \ .$$

In particular, we would have equality of the constant terms of these generating functions, that is, we would have $1 = 0 \cdots$ Consequently, we can not find a choice of the functions φ and ψ that extend the Hurwitz multizeta functions such that there is no corrective term in the reduction into monotangent of divergent multitangent functions.

7.5.3. Reduction into monotangent functions for divergent multitangent functions

Theorem 5 admits the following corollary which comes from a direct formal power series expansion of $\mathcal{T}ig^{\bullet}(z)$. This corresponds exactly to the fourth point mentionned at the beginning of this section. Let us remark that, from now on, we only consider moulds and not bimoulds.

Let us recall that we have introduced the following notations (see sections 3.1 and 7.5.2):

$$\begin{aligned}
{}^{i}B^{\underline{\mathbf{s}}}_{\underline{\mathbf{k}}} &= \left(\prod_{l=1}^{i-1}(-1)^{k_l}\right)\left(\prod_{l=i+1}^{r}(-1)^{s_l}\right)\left(\prod_{\substack{l=1\\ l\neq i}}^{r}\binom{s_l+k_l-1}{s_l-1}\right) . \\
\mathcal{Z}^{\underline{\mathbf{s}}}_{i,k} &= \sum_{\substack{k_1,\cdots,\widehat{k_i},\cdots,k_r\geq 0\\ k_1+\cdots+\widehat{k_i}+\cdots+k_r=k}} {}^{i}B^{\underline{\mathbf{s}}}_{\underline{\mathbf{k}}}\mathcal{Z}e^{s_r+k_r,\cdots,s_{i+1}+k_{i+1}}\mathcal{Z}e^{s_1+k_1,\cdots,s_{i-1}+k_{i-1}} . \\
\delta^{\underline{\mathbf{s}}} &= \begin{cases} \dfrac{(i\pi)^r}{r!} & \text{, if } \underline{\mathbf{s}} = 1^{[r]} \text{ et if } r \text{ is even.} \\ 0 & \text{, otherwise.} \end{cases}
\end{aligned}$$

Then, we have:

Theorem 6. *(Reduction into monotangent functions, version* 2*)*
For all sequences $\underline{\mathbf{s}} \in seq(\mathbb{N}^*)$*, we have:*

$$\mathcal{T}e^{\underline{\mathbf{s}}}(z) = \delta^{\underline{\mathbf{s}}} + \sum_{i=1}^{r}\sum_{k=1}^{s_i}\mathcal{Z}^{\underline{\mathbf{s}}}_{i,s_i-k}\mathcal{T}e^{k}(z) \ .$$

Moreover, if $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$*, the summation of* k *begins at* 2 .

This result is computable. One can give complete tables for divergent multitangent functions up to a fixed weight, as in the convergent case (see table 1 for the convergent case and table 7 for the divergent case) .

For example, one can see that $\mathcal{T}e^{2,1}$ and $\mathcal{T}e^{1,2}$ are null. As already said in the introduction, this remarkable fact shows us that the relation of symmetr$\underline{\boldsymbol{e}}$lity $\mathcal{T}e^2\mathcal{T}e^1 = \mathcal{T}e^{2,1} + \mathcal{T}e^{1,2} + \mathcal{T}e^3 = \mathcal{T}e^3(z)$ allows us to find in a different way (more complicated, but more general) the simplest relations between Eisenstein series.

8. Some explicit calculations of multitangent functions

Before presenting some explicit calculations of multitangent functions, let us remind a few notations. If $\underline{\boldsymbol{\alpha}}$ is any sequence, then $\underline{\boldsymbol{\alpha}}^{[r]}$ denotes the sequence $\underbrace{\underline{\boldsymbol{\alpha}}\cdot\cdots\cdot\underline{\boldsymbol{\alpha}}}_{r \text{ times}}$, where the sequence $\underline{\boldsymbol{\alpha}}$ is repeated k times. In particular, $n^{[k]}$ is the sequence $(n;\cdots;n)$ where n is repeated k times.

8.1. Calculation of $\mathcal{T}e^{1^{[r]}}(z)$ *, for* $r \in \mathbb{N}$

For all $r \in \mathbb{N}$, $\mathcal{T}e^{1^{[r]}}(z)$ is the constant term of $Tig^{Y_1,\cdots,Y_r}$, so:

$$\mathcal{T}e^{1^{[r]}}(X) = \left\{\begin{array}{ll} \dfrac{(i\pi)^r}{r!}, & \text{if } r \in 2\mathbb{Z} \\ \quad 0 & , \text{if } r \notin 2\mathbb{Z} \end{array}\right\} + \left(\sum_{k=0}^{r-1} \mathcal{Z}ig^{0^{[k]}}\mathcal{Z}ig_-^{0^{[r-1-k]}}\right)\mathcal{T}e^1(X) \ .$$

We have evaluate $\displaystyle\sum_{k=0}^{n} \mathcal{Z}ig^{0^{[k]}}\mathcal{Z}ig_-^{0^{[n-k]}}$ for $n \in \mathbb{N}^*$, by considering the product $\mathcal{Z}_+\mathcal{Z}_-$, where:

$$\left\{\begin{array}{l} \mathcal{Z}_+ = \displaystyle\sum_{n\geq 0} \mathcal{Z}ig^{0^{[n]}}X^n = \sum_{n\geq 0} \mathcal{Z}e^{1^{[n]}}X^n \ . \\ \mathcal{Z}_- = \displaystyle\sum_{n\geq 0} \mathcal{Z}ig_-^{0^{[n]}}X^n = \sum_{n\geq 0} \mathcal{Z}e_-^{1^{[n]}}X^n \ . \end{array}\right.$$

The mould $\mathcal{Z}e^\bullet$ and $\mathcal{Z}e_-^\bullet$ being symmetr$\underline{\boldsymbol{e}}$l , we automatically obtain the following formal differential equations (see Property 15) :

$$\begin{cases} D\mathcal{Z}_+ = \mathcal{Z}_+ \times \left(\sum_{n\geq 0} (-1)^n \mathcal{Z}e^{n+1} X^n \right) = \mathcal{Z}_+ He^1_+ \ . \\ D\mathcal{Z}_- = \mathcal{Z}_- \times \left(\sum_{n\geq 0} (-1)^n \mathcal{Z}e^{n+1}_- X^n \right) = \mathcal{Z}_- He^1_- \ . \end{cases}$$

$$\begin{aligned} \text{So: } D(\mathcal{Z}_+\mathcal{Z}_-) &= \mathcal{Z}_+ He^1_+ \mathcal{Z}_- + \mathcal{Z}_+ He^1_- \mathcal{Z}_- \ = \ \mathcal{Z}_+\mathcal{Z}_-\big(He^1_+ + He^1_-\big) \\ &= -2\mathcal{Z}_+\mathcal{Z}_- \left(\sum_{n\geq 0} \mathcal{Z}e^{2n+2} X^{2n+1} \right) \\ &= -2\mathcal{Z}_+\mathcal{Z}_- \left(\sum_{n\geq 1} \mathcal{Z}e^{2n} X^{2n-1} \right) \ . \end{aligned}$$

Letting Exp be the formal exponential, we obtain:

$$\mathcal{Z}_+\mathcal{Z}_- = Exp\left(-\sum_{n\geq 1} \frac{\mathcal{Z}e^{2n}}{n} X^{2n} \right) .$$

On the other hand, in $\mathbb{C}(\!(X)\!)$, we have:

$$He^1_+(X) + He^1_-(X) = Te^1(X) - X^{-1} = \pi \frac{\cos(\pi X)}{\sin(\pi X)} - \frac{1}{X} \ .$$

Indeed, this relation is valid in $\mathbb{C}[\![X]\!]$, so:

$$He^1_+(X) + He^1_-(X) = D\left(Log\left(\frac{\sin(\pi X)}{\pi X} \right) \right) \ .$$

So that:

$$\mathcal{Z}_+\mathcal{Z}_- = Exp\left(-\sum_{n\geq 1} \frac{\mathcal{Z}e^{2n}}{n} X^{2n} \right) = \frac{\sin(\pi X)}{\pi X} = \sum_{n\geq 0} (-1)^n \frac{(\pi X)^{2n}}{(2n+1)!} \ .$$

Finally, we obtain:

$$Te^{1^{[r]}}(X) \ = \ \begin{Bmatrix} \dfrac{(i\pi)^r}{r!} & \text{, if } r \in 2\mathbb{Z} \\ 0 & \text{, if } r \notin 2\mathbb{Z} \end{Bmatrix} + \begin{Bmatrix} 0 & \text{, if } r \in 2\mathbb{Z} \\ \dfrac{(i\pi)^{r-1}}{(2r-1)!} & \text{, if } r \notin 2\mathbb{Z} \end{Bmatrix} \times Te^1(X) \ ,$$

and the analytic equality follows for all $z \in \mathbb{C} - \mathbb{Z}$:

$$\mathcal{T}e^{1^{[r]}}(z) = \begin{cases} (-1)^p \dfrac{\pi^{2p}}{(2p)!} & \text{, if } r = 2p \ . \\ (-1)^p \dfrac{\pi^{2p}}{(2p+1)!} \mathcal{T}e^1(z) & \text{, if } r = 2p+1 \ . \end{cases}$$

8.2. Calculation of $\mathcal{T}e^{n^{[k]}}(z)$, for $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$

We will give an explicit evaluation of all multitangent functions of the form $\mathcal{T}e^{n^{[k]}}(z)$, for $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$, in term of monotangent functions and multizeta values. This will be done by proving the following property:

Property 14. *Let $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$. Denote by E the floor function and define the functions $t_{k,n}$ for $(k; n) \in \mathbb{N} \times \mathbb{N}^*$ by:*

$$\forall x \in \mathbb{R}, t_{k,n}(x) = \begin{cases} \cos^{(n-1)}(x) & \text{, } \textit{if } k \textit{ is odd.} \\ \sin^{(n-1)}(x) & \text{, } \textit{if } k \textit{ is even.} \end{cases}$$

Consider the moulds $sg^\bullet$, $e^\bullet$ and $s^\bullet$, which are $\mathbb{C}$-valued and defined over the alphabet $\Omega = \{1; -1\}$:

$$sg^{\underline{\varepsilon}} = \prod_{k=1}^{n} \varepsilon_k \ , \qquad s^{\underline{\varepsilon}} = \sum_{k=1}^{n} \varepsilon_k \ , \qquad e^{\underline{\varepsilon}} = \sum_{k=1}^{n} \varepsilon_k e^{(2k-1)\frac{i\pi}{n}} \ .$$

Then, for all $z \in \mathbb{C} - \mathbb{Z}$, we have:

$$\mathcal{T}e^{n^{[k]}}(z) = \frac{(-1)^{n-1+E(\frac{kn+1}{2})}\pi^{kn}}{(kn)!(2\sin(\pi z))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}}(e^{\underline{\varepsilon}})^{kn} t_{kn,n}(s^{\underline{\varepsilon}}\pi z) \ .$$

In order to prove this, we will use an elementary theory of formal power series in one indeterminate. The central point is the following lemma. This gives us a formal differential equation automatically satisfied by the generating functions of the family of multitangent functions under consideration. Then, we only have to find out a formal power series expansion of solutions of this equation.

8.2.1. A property linking symmetr$\underline{e}$lity and formal differential equation

Let us begin by proving the following general property concerning symmetr$\underline{e}$l moulds:

Property 15. *Let us consider a commutative algebra $\mathbb{A}$, a semigroup $(\Omega;+)$ and a symmetr$\underline{e}$l mould $Se^\bullet \in \mathcal{M}^\bullet_{\mathbb{A}}(\Omega)$.*
For all $\omega \in \Omega$, we set:

$$F_\omega = \sum_{p=0}^{+\infty} Se^{\omega^{[p]}} X^p \quad , \quad G_\omega = \sum_{p=0}^{+\infty} (-1)^p Se^{(p+1)\omega} X^p.$$

For a given $\omega \in \Omega$, the formal power series F_ω satisfies the differential equation:

$$DY = YG_\omega \ .$$

Let us point out that this property is well known is combinatorics as the Newton relations for symmetric functions. Here, the term F_ω means to be the elementary symmetric functions while the term G_ω is then the power sums.

The proof of this property we will give here is based on the algorithmic removal to the right of the ones which begin an evaluation sequence $\omega \in \Omega$ of the mould $Se^\bullet$; so the proof is exactly based on the notion of symmetr$\underline{e}$lity . This algorithm is recursively presented by the following formula:

$$\forall p \in \mathbb{N}, \forall \omega \in \Omega, Se^{\omega^{[p]}} Se^{\omega} = (p+1) Se^{\omega^{[p+1]}} + \sum_{k=0}^{p-1} Se^{\omega^{[k]}, 2\omega, \omega^{[p-k-1]}} .$$

Proof. Let us fix $\omega \in \Omega$ and introduce the temporary notation $u_{p,l}$ for $(p;l) \in \mathbb{N} \times \mathbb{N}^*$:

$$u_{p,l} = (-1)^l \sum_{k=0}^{p} Se^{\omega^{[k]}, l\omega, \omega^{[p-k]}} .$$

Then, using the symmetr$\underline{e}$lity property, we have for $(p;l) \in (\mathbb{N}^*)^2$:

$$\begin{aligned} (-1)^l Se^{\omega^{[p]}} Se^{l\omega} &= (-1)^l \sum_{k=0}^{p} Se^{\omega^{[k]}, l\omega, \omega^{[p-k]}} - (-1)^{l+1} \sum_{k=0}^{p-1} Se^{\omega^{[k]}, (l+1)\omega, \omega^{[p-1-k]}} \\ &= u_{p,l} - u_{p-1,l+1} \ . \end{aligned}$$

This implies successively, for $p \in \mathbb{N}^*$:

$$\begin{aligned}
\sum_{l=0}^{p-1}(-1)^l Se^{\omega^{[p-l]}} Se^{(l+1)\omega} &= -\sum_{l=1}^{p}(-1)^l Se^{\omega^{[p-(l-1)]}} Se^{l\omega} \\
&= -\sum_{l=1}^{p}\left(u_{p-(l-1),l} - u_{p-l,l+1}\right) \\
&= u_{0,p+1} - u_{p,1} \\
&= (-1)^{p+1} Se^{(p+1)\omega} + (p+1) Se^{\omega^{[p+1]}} .
\end{aligned}$$

Then: $(p+1)Se^{\omega^{[p+1]}} = \sum_{l=0}^{p}(-1)^l Se^{\omega^{[p-1]}} Se^{(l+1)\omega}$, for all $p \in \mathbb{N}^*$.

Since the previous equality is also true for $p = 0$, we can state the following equality between formal power series:

$$DF_\omega = F_\omega G_\omega \ .$$

$\square$

Using the fact that two formal power series with the same formal derivative differ only by their constant term, it is not difficult to see, if $\mathbb{A}$ is a ring and if $\varphi \in \mathbb{A}[\![X]\!]$, then formal power series satisfying $DY = YD\varphi$ are defined by:

$$Y(X) = CExp(\varphi(X) - \varphi(0)), C \in \mathbb{A} \ .$$

Here, Exp refers to the formal exponential. The resolution of such a formal differential equation boils down to a problem of expressing a formal indefinite integral. The constant C is then determined by the constant term in Y.

Indeed, as already announced in the course of the evaluation of $Te^{1,\cdots,1}(X)$ (see Section 8.1), by the symmetr$\underline{e}$lity of $\mathcal{Z}e^\bullet$, the formal power series $\mathcal{Z}_+ = \sum_{r\geq 0} \mathcal{Z}e^{1^{[r]}} X^r$ satisfies the formal differential equation:

$$D\mathcal{Z}_+ = \mathcal{Z}_+ \left(\sum_{p\geq 0}(-1)^p \mathcal{Z}e^{p+1} X^p\right) = \mathcal{Z}_+ He^1_+ \ .$$

8.2.2. Application to the mould $Te^{\bullet}(X)$

Recall that the mould $Te^{\bullet}(X)$ has been extended to seq($\mathbb{N}^*$) in the previous section, in such a way as to preserve the symmetr$\underline{\boldsymbol{e}}$lity property. Hence, the previous property applies: if we set, for $n \in \mathbb{N}^*$, $T_n = \sum_{p=0}^{+\infty} Te^{n^{[p]}}(X)Y^p$ and $U_n = \sum_{p=0}^{+\infty}(-1)^p Te^{n(p+1)}(X)Y^p$, we then have, for all positive integer n, in $\mathbb{C}(\!(X)\!)[\![Y]\!]$:

$$DT_n = T_n U_n \ .$$

We just need to compute a formal indefinite integral of U_n in order to calculate $Te^{n^{[p]}}(X)$ for all $p \in \mathbb{N}$. Let us consider $V_n \in \mathbb{C}(\!(X)\!)[\![Y]\!]$ defined by $V_n(X;Y) = U_n(X;Y^n)$. A permutation of formal summation symbols (which is a priori a non-authorized operation) , followed by a partial fraction expansion, suggests we have for all positive integer n:

$$nY^{n-1}V_n(X;Y) = -\sum_{k=0}^{n-1} e^{(2k+1)\frac{i\pi}{n}} Te^1\left(X - e^{(2k+1)\frac{i\pi}{n}}Y\right) \ .$$

Recall that, here, S and $D_{(Y)}$ denote respectively the taking of the constant term and the formal derivative relative to the indeterminate Y. Indeed, the Taylor formula permits to prove this relation in the ring $\mathbb{C}(\!(X)\!)[\![Y]\!]$. For $l \in \mathbb{N}$, if we denote the right hand side of the previous equality by W_n, we have successively:

$$\begin{aligned}
\frac{1}{l!}S\left(D^l_{(Y)}W_n\right) &= S\left(-\sum_{k=0}^{n-1} e^{(2k+1)(l+1)\frac{i\pi}{n}} Te^{l+1}\left(X - e^{(2k+1)\frac{i\pi}{n}}Y\right)\right) \\
&= -\left(\sum_{k=0}^{n-1} e^{(2k+1)(l+1)\frac{i\pi}{n}}\right) Te^{l+1}(X) \\
&= \begin{cases} 0 & \text{, si } l+1 \not\equiv 0[n] \ . \\[2ex] n(-1)^{q+1}Te^{qn}(X) & \text{, si } l+1 = qn \ . \end{cases}
\end{aligned}$$

$$\text{Hence:}\quad W_n = \sum_{l=0}^{+\infty} \frac{1}{l!} S\left(D^l_{(Y)} W_n\right) Y^l = \sum_{q=1}^{+\infty} n(-1)^{q+1} Te^{qn}(X) Y^{qn-1}$$
$$= nY^{n-1} \sum_{q=0}^{+\infty} (-1)^q Te^{n(q+1)}(X) Y^{qn}$$
$$= nY^{n-1} V_n(X;Y)\ .$$

In the ring $\mathbb{C}(\!(X)\!)[\![Y]\!]$, we therefore have:

$$nY^{n-1} V_n(X;Y) = -\sum_{k=0}^{n-1} e^{(2k+1)\frac{i\pi}{n}} Te^1\left(X - e^{(2k+1)\frac{i\pi}{n}} Y\right)\ .$$

The ring morphism $\varphi_n : \mathbb{C}(\!(X)\!)[\![Y]\!] \longrightarrow \mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$ defined by

$$\varphi_n(Y) = Y^{1/n}$$

is a continuous one for the l-adic topology; we hence observe that if P is a polynomial with coefficients in $\mathbb{C}(\!(X)\!)$, then $\varphi(P(X;Y)) = P(X;Y^{1/n})$. This can be extended to formal power series of $\mathbb{C}(\!(X)\!)[\![Y]\!]$, using the continuity of φ_n and the density of polynomials.

Transposed in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$ using the morphisms φ_n, the relation expressing $V_n(X;Y)$ becomes in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$:

$$U_n(X;Y) = -\frac{1}{n} \sum_{k=0}^{n-1} e^{(2k+1)\frac{i\pi}{n}} Te^1\left(X - e^{(2k+1)\frac{i\pi}{n}} Y^{\frac{1}{n}}\right) Y^{\frac{1}{n}-1}\ .$$

A priori, this last equality is in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$, while by definition we have $U_n \in \mathbb{C}(\!(X)\!)[\![Y]\!]$. We can then proceed component by component in the ring $\mathbb{C}(\!(X)\!)[\![Y]\!]$.

To express T_n by using the general formula of solving a first order formal differential equation, it is sufficient to determine the exponential of the formal indefinite integral (in Y), without constant term, of $\omega Te^1(X+\omega Y)$ in $\mathbb{C}(\!(X)\!)[\![Y]\!]$.

To this purpose, let us remind we have proved in $\mathbb{C}(\!(X)\!)[\![Y]\!]$, the relation $Te^1(X+Y) = \dfrac{\pi}{\tan(\pi X + \pi Y)}$. Therefore, the formal indefinite integral in Y of $\omega Te^1(X+\omega Y)$, for $\omega \in \mathbb{C}$, without constant term, is given by $\mathrm{Log}\left(\dfrac{\sin(\pi(X+\omega Y))}{\sin(\pi X)}\right)$. Consequently, in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$, the formal indefinite primitive in Y without constant term of $\dfrac{\omega}{n} Te^1\left(X+\omega Y^{\frac{1}{n}}\right) Y^{\frac{1}{n}-1}$ is

$\mathrm{Log}\left(\dfrac{\sin(\pi(X+\omega Y^{\frac{1}{n}}))}{\sin(\pi X)}\right)$. Thus, by solving the formal differential equation in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$, we deduce that for all positive integer n:

$$T_n = \sum_{p=0}^{+\infty} Te^{n^{[p]}}(X) Y^p = \frac{\prod_{k=0}^{n-1} \sin\left(\pi\left(X - e^{(2k+1)\frac{i\pi}{n}} Y^{\frac{1}{n}}\right)\right)}{\sin^n(\pi X)} \ .$$

Let us insist on the fact that, although seeming to be a priori a relation in $\mathbb{C}(\!(X)\!)[\![Y^{1/n}]\!]$, this equality hold in fact in $\mathbb{C}(\!(X)\!)[\![Y]\!]$, by definition of T_n .

8.2.3. A new formal power series expansion of T_n

In order to calculate $Te^{n^{[p]}}(X)$ for $(n;p) \in (\mathbb{N}^*)^2$, we need a formal power series expansion of T_n expressed in another way than its definition. To get this new expansion, it is convenient to expand a product of many sinus terms.

It is easily seen, by induction on n, that in $\mathbb{C}[\![X_1;\cdots;X_n]\!]$:

$$\prod_{k=1}^{n} \sin(X_k) = \frac{(-1)^{n-1}}{2^n} \sum_{(\varepsilon_1;\cdots;\varepsilon_n)\in\{+1;-1\}^n} (-1)^{\sharp\{k\in[\![\,1\,;\,n\,]\!];\varepsilon_k=-1\}} \sin^{(n-1)}\left(\sum_{k=1}^{n} \varepsilon_k X_k\right) \ .$$

Let us consider the moulds $sg^\bullet$, $e^\bullet$ and $s^\bullet$, which are $\mathbb{C}$-valued and defined over the alphabet $\Omega = \{1;-1\}$ for all sequences $\underline{\varepsilon} \in \mathrm{seq}(\Omega)$ by: $sg^{\underline{\varepsilon}} = \displaystyle\prod_{k=1}^{n} \varepsilon_k = (-1)^{\sharp\{i\in[\![\,1\,;\,n\,]\!];\varepsilon_i=-1\}}$, $s^{\underline{\varepsilon}} = \displaystyle\sum_{k=1}^{n} \varepsilon_k$ and $e^{\underline{\varepsilon}} = \displaystyle\sum_{k=1}^{n} \varepsilon_k e^{(2k-1)\frac{i\pi}{n}}$.

Let E be the floor function and define all for $(k;n) \in \mathbb{N}\times\mathbb{N}^*$ the functions $t_{k,n}$ by:

$$\forall x \in \mathbb{R} \ , \ t_{k,n}(x) = \begin{cases} \cos^{(n-1)}(x) & \text{, if } k \text{ is odd.} \\ \sin^{(n-1)}(x) & \text{, if } k \text{ is even.} \end{cases}$$

It follows that for all $n \in \mathbb{N}^*$, we have successively:

$$
\begin{aligned}
T_n &= \frac{(-1)^{n-1}}{(2\sin(\pi X))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}} \sin^{(n-1)}\left(s^{\underline{\varepsilon}}\pi X + e^{\underline{\varepsilon}}\pi Y^{\frac{1}{n}}\right) \\
&= \frac{(-1)^{n-1}}{(2\sin(\pi X))^n} \sum_{\underline{\varepsilon}=(\varepsilon;\cdots;\varepsilon_n)\in\Omega^n} \left(sg^{\underline{\varepsilon}} \sum_{k=0}^{+\infty} \frac{(-1)^{E(\frac{k+1}{2})}(e^{\underline{\varepsilon}}\pi)^k}{k!} t_{k,n}(s^{\underline{\varepsilon}}\pi X) Y^{\frac{k}{n}} \right), \\
&= \sum_{k=0}^{+\infty} \left(\frac{(-1)^{n-1+E(\frac{k+1}{2})}\pi^k}{k!(2\sin(\pi X))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}}(e^{\underline{\varepsilon}})^k t_{k,n}(s^{\underline{\varepsilon}}\pi X) Y^{\frac{k}{n}} \right).
\end{aligned}
$$

Note that we have used in the second equality

$$
\sin^{(n-1)}(X+Y) = \sum_{k=0}^{+\infty} \frac{(-1)^{E(\frac{k+1}{2})}}{k!} t_{k,n}(X) Y^k \text{ in } \mathbb{C}[\![X;Y]\!] ,
$$

and that in the last sum, Ω^n is a finite set.

As already indicated before the end of the previous paragraph, we have by definition of T_n : $T_n \in \mathbb{C}(\!(X)\!)[\![Y]\!]$. This imposes the vanishing of the coefficients of $Y^{\frac{k}{n}}$ if $n \nmid k$ in the previous equality. Hence:

$$
T_n = \sum_{k=0}^{+\infty} \left(\frac{(-1)^{n-1+E(\frac{kn+1}{2})}\pi^{kn}}{(kn)!(2\sin(\pi X))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}}(e^{\underline{\varepsilon}})^{kn} t_{kn,n}(s^{\underline{\varepsilon}}\pi X) Y^k \right).
$$

It follows that we have proved, for all $(n;k) \in \mathbb{N}^* \times \mathbb{N}$, the formal equality announced in Property 14:

$$
Te^{n^{[k]}} = \frac{(-1)^{n-1+E(\frac{kn+1}{2})}\pi^{kn}}{(kn)!(2\sin(\pi X))^n} \sum_{\underline{\varepsilon}=(\varepsilon_1;\cdots;\varepsilon_n)\in\Omega^n} sg^{\underline{\varepsilon}}(e^{\underline{\varepsilon}})^{kn} t_{kn,n}(s^{\underline{\varepsilon}}\pi X) .
$$

To conclude this calculation, we only have to justify that the analytic equality follows from the formal one. This is obvious because each (convergent or divergent) multitangent function is a Laurent series at 0 which is exactly given by the expression of the associated formal multitangent function. In the componentwise equality which has just been proved, we can thus replace the straight capital letters by cursive capital letters to conclude the proof of Property 14.

8.2.4. A few examples

For $n = 1$, this result gives, for $k \in \mathbb{N}^*$ and $z \in \mathbb{C} - \mathbb{Z}$:

$$\mathcal{T}e^{1^{[k]}}(z) = \begin{cases} \dfrac{(-1)^p \pi^{2p}}{(2p)!} & , \text{ if } k = 2p \ . \\ \dfrac{(-1)^p \pi^{2p}}{(2p+1)!} \mathcal{T}e^{1}(z) & , \text{ if } k = 2p+1 \ . \end{cases}$$

Also, for $n = 2$, this result gives, for $k \in \mathbb{N}^*$ and $z \in \mathbb{C} - \mathbb{Z}$:

$$\mathcal{T}e^{2^{[k]}}(z) = \frac{2^{2k-1}\pi^{2k-2}}{(2k)!} \mathcal{T}e^{2}(z).$$

The table 8 gives some others explicit results from this property.

8.3. About odd, even or null multitangent functions

Surprisingly, there exists convergent multitangent functions which are null (see table 1). The first multitangent with this property is $\mathcal{T}e^{2,1,2}$. It is easy to see: the reduction into monotangent functions impose on $\mathcal{T}e^{2,1,2}$ to be $\mathbb{C}$-linearly dependent to $\mathcal{T}e^{2}$, hence to be an even function ; nevertheless, the parity property tells us $\mathcal{T}e^{2,1,2}$ is also an odd function. Necessary, the multitangent function $\mathcal{T}e^{2,1,2}$ is the null function.

In the same manner, we can state the following lemma:

Lemma 16. *Let* $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e} \cap seq\big(\{1;2\}\big)$ *be a symmetric sequence (i.e.* $\overleftarrow{\underline{\mathbf{s}}} = \underline{\mathbf{s}}$*), of odd weight and of length greater than* 1 *.*
Then, $\mathcal{T}e^{\underline{\mathbf{s}}}$ *is the null function.*

When we look at a table of convergent multitangent functions up to weight 18, it seems that the converse is also true:

Conjecture 12. *(Caracterisation of null multitangent functions) The null convergent multitangent functions are exactly the multitangent functions* $\mathcal{T}e^{\underline{\mathbf{s}}}$ *with symmetric sequence* $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e} \cap seq\big(\{1;2\}\big)$*, of odd weight and of length greater than* 1 *.*

We mentionned that this conjecture is true for length 3 (see [3])

Let us remark that even (resp. odd) components of an odd (resp. even) multitangent function are naturaly null. The following question is then an interesting one: " *If* $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$ *(or seq*$(\mathbb{N}^*)$*)* , *is there any component* $\mathcal{T}e^k$, $k \in [\![\, 2\,;\, max(s_1;\cdots;s_r)\,]\!]$ *which does not appear in the reduction into monotangent functions of* $\mathcal{T}e^{\underline{\mathbf{s}}}$ *?*"

It seems that the answer might be no, except when the multitangent function which does not have this component is an odd or even function.

An other question is also: " *If* $\mathcal{T}e^{\underline{\mathbf{s}}}$ *is an odd or even function, have we necessary got* $\overleftarrow{\underline{\mathbf{s}}} = \underline{\mathbf{s}}$ *?*" This time, it seems to be yes. The converse is already acquired, according to the property of parity.

All of this discussion can be summed up in the following conjecture (which obviously implies the previous one) :

Conjecture 13. *(Caracterisation of odd or even multitangent functions) Let* $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$.
1. *If the component* $\mathcal{T}e^k$, $k \in [\![\, 2\,;\, max(s_1;\cdots;s_r)\,]\!]$, *does not appear in the reduction into monotangent functions of* $\mathcal{T}e^{\underline{\mathbf{s}}}$, *then* $\mathcal{T}e^{\underline{\mathbf{s}}}$ *will be of opposite parity of* k *(and thus may be the null function)* .
2. *The multitangent function* $\mathcal{T}e^{\underline{\mathbf{s}}}$ *is an odd or even function if and only if* $\overleftarrow{\underline{\mathbf{s}}} = \underline{\mathbf{s}}$.

8.4. Explicit calculation of some multitangent functions

The reduction into monotangent functions allows us to do some explicit calculations of multitangent functions. We will give a few examples in the convergent case.

In order to apply this reduction simply, here are a few elementary remarks:

1. Only the indexes i satisfying $s_i \geq 2$ give a contribution to the expression of the reduction into monotangent functions.
2. If $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e} \cap \mathrm{seq}(\{1;2;3\})$ is a symmetric sequence (ie $\overleftarrow{\underline{\mathbf{s}}} = \underline{\mathbf{s}}$) of even weight, only the monotangent function $\mathcal{T}e^2$ has to be considered; this means that only the indexes $k = 2$ give a contribution to the reduction.
3. If $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e} \cap \mathrm{seq}(\{1;2;3\})$ is a symmetric sequence of odd weight, only the monotangent function $\mathcal{T}e^3$ has to be considered; this means that only the indexes $k = 3$ give a contribution to the reduction.

Applying these remarks, a simple calculation gives us the results of the table 9.

9. Conclusion

In this article, we have thoroughly investigated the algebra $\mathcal{MTGF}_{CV}$ of multitangent functions, spanned as a $\mathbb{Q}$-vector space by the functions:

$$\begin{array}{rcl} \mathcal{T}e^{\underline{\mathbf{s}}} : \ \mathbb{C} - \mathbb{Z} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \displaystyle\sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \ , \end{array}$$

for sequences in $\mathcal{S}^{\star}_{b,e} = \left\{ \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) ; s_1 \geq 2 \text{ and } s_{l(\underline{\mathbf{s}})} \geq 2 \right\}$.

The first properties we have proved are elementary ones and concern the symmetr$\underline{\boldsymbol{e}}$lity of the mould $\mathcal{T}e^{\bullet}$, the differentiation property and the parity property. Another seemingly easy property is in fact a deep one, namely the reduction into monotangent functions:

Theorem 7. ***(Reduction into monotangent functions)***
For all sequences $\underline{\mathbf{s}} = (s_1; \cdots ; s_r) \in seq(\mathbb{N}^*)$, there exists an explicit family $(z_k^{\underline{\mathbf{s}}})_{k \in [\![0 ; M]\!]} \in \mathcal{MZV}_{CV}^{M+1}$, with $M = \max\limits_{i \in [\![1 ; r]\!]} s_i$, such that:

$$\forall z \in \mathbb{C} - \mathbb{Z} \ , \ \mathcal{T}e^{\underline{\mathbf{s}}}(z) = z_0^{\underline{\mathbf{s}}} + \sum_{k=1}^{max(s_1; \cdots ; s_r)} z_k^{\underline{\mathbf{s}}} \mathcal{T}e^k(z) \ .$$

Moreover, if $\underline{\mathbf{s}} \in \mathcal{S}^{\star}_{b,e}$, then $z_0^{\underline{\mathbf{s}}} = z_1^{\underline{\mathbf{s}}} = 0$.

We have then immediately derived that for all $p \geq 2$, we have:

$$\mathcal{MTGF}_{CV,p} \subseteq \bigoplus_{k=0}^{p-2} \mathcal{MZV}_{CV,p-k} \cdot \mathcal{T}e^k \ .$$

Then, we have explained why the reduction into monotangent functions is such an important operation. The reason is that this process has in a certain sense a converse, namely the projection onto multitangent functions. According to Conjectures 2 , 3 and 4 and Properties 4 and 5, we have proved the following:

Theorem 8. ***(Projection onto multitangent functions)***
The following assertions are equivalent:

1. *For all non negative integer* p , $\mathcal{M}TGF_{CV,p} = \displaystyle\bigoplus_{k=0}^{p-2} \mathcal{M}ZV_{CV,p-k} \cdot \mathcal{T}e^k$.
2. $\mathcal{M}TGF_{CV}$ *is a* $\mathcal{M}ZV_{CV}$*-module.*
3. *For all sequence* $\underline{\boldsymbol{\sigma}} \in \mathcal{S}_e^\star$, $\mathcal{Z}e^{\underline{\boldsymbol{\sigma}}}\mathcal{T}e^2 \in \mathcal{M}TGF_{CV,||\underline{\boldsymbol{\sigma}}||+2}$.

By an argument of linear algebra, we have explained that the largest p is, the stronger are the reasons to believe in the previous assertions. We have verified them up to weight 18 .

The third important fact, which was used during the regularization process of divergent multitangent functions, is the trifactorisation of the multitangent functions: all multitangents can be expressed as a finite product of Hurwitz multizeta functions in such a way as to preserve the exponentially flat character of multitangent functions.

Finally, the links between the algebra of multizeta values and the algebra of multitangent functions are summed up by these three properties and the following diagram:

$$\begin{tikzcd}
\mathcal{M}ZV_{CV} \arrow[d, shift left, dotted, "\text{projection}"] & \mathcal{H}MZF_{+,CV} \arrow[l, "\text{evaluation at } 0"'] \arrow[d, hook] \\
\mathcal{M}TGF_{CV} \arrow[u, shift left, "reduction"] \arrow[r, hook, "\text{trifactorization}"] & \mathcal{H}MZF_{\pm,CV}
\end{tikzcd}$$

As an example of the "duality" multizeta values/multitangent functions, we have explained that if the hypothetical $\mathcal{M}ZV_{CV}$-module structure holds, then we have a conjecture concerning the dimension of $\mathcal{M}TGF_{CV,p}$ which is actually equivalent to Zagier's conjecture on multizeta values. This justified the following table of conjectural dimensions:

If these dimensions looks reasonable, this is because of the existence of many $\mathbb{Q}$-linear relations between multitangent functions. For instance,

$$4\mathcal{T}e^{3,1,3} - 2\mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,1,2,2} = 0 \ .$$

is an interesting relation because it implies a relation between multizeta values, discussed at the end of Section 5.

p	0	1	2	3	4	5	6	7	8	9	10	11	12
dim $\mathcal{M}ZV_{CV,p}$	1	0	1	1	1	2	2	3	4	5	7	9	12
dim $\mathcal{M}TGF_{CV,p+2}$	1	1	2	3	4	6	8	11	15	20	27	36	48

Figure 5: The first hypothetical dimensions of multitangent vector space of weight $p+2$.

Now, the remaining question is to find a new method to prove that there is no non-trivial $\mathbb{Q}$-linear relations between the multitangent functions which are supposed to span $\mathcal{M}TGF_{CV,p}$. To illustrate this, if we were able to prove the absence of non-trivial $\mathbb{Q}$-linear relations between $\mathcal{T}e^{5}$, $\mathcal{T}e^{3,2}$, $\mathcal{T}e^{2,3}$ and $\mathcal{T}e^{2}$, this would imply a well-known fact: $\zeta(3) = \mathcal{Z}e^{3} \notin \mathbb{Q}$. Such a partial result would already be an important breakthrough, because such a method would certainly be generalisable to other weights, while Apery's method is not. Nevertheless, such a method would probably not give an upper bound of the irrationality measure, while Apery's method can.

Probably, the new method would come from the study of the Hurwitz multizeta functions, and more precisely from the study of algebraic relations in the algebra $\mathcal{H}MZV_{\pm,CV}$.

A. Introduction to mould notations and calculus

For all this annex, references can be found in many text of Jean Ecalle. See for instance [15] or [17] ; see also for other presentation than these of Jean Ecalle : [12] or [31].

A.1. Notion of moulds

A **mould** is a function defined over a free monoid seq(Ω) (or sometimes over a subset of seq(Ω)), valued in an algebra $\mathbb{A}$. Concretely, this means that "a mould is a function with a variable number of variables".

Thus, moulds depend of sequences $\underline{\mathbf{w}} = (w_1; \cdots ; w_r)$ of any length r. The variables w_i are elements of Ω. We will often identify sequences of seq(Ω) and non-commutative polynomials over the alphabet Ω.

In a general way, we will use the mould notations:

1. Sequences will always be written in bold and underlined, with an upper indexation if necessary. We call length of $\underline{\mathbf{w}}$ and denote $l(\underline{\mathbf{w}})$ the number of elements of $\underline{\mathbf{w}}$. Without more precisions, we will use the letter r to indicate the length of any sequences. We also define the weight of $\underline{\mathbf{w}}$, when Ω has a semi-group structure, by:

$$||\underline{\mathbf{w}}|| = w_1 + \cdots + w_r \ .$$

2. For a given mould, traditionally denoted by M as a map from $\mathrm{seq}(\Omega)$ to $\mathbb{A}$, we will prefer the notation $M^{\underline{\boldsymbol{\omega}}}$ which indicate the evaluation of the mould $M^\bullet$ on the sequence $\underline{\boldsymbol{\omega}}$ of $\mathrm{seq}(\Omega)$.
3. We shall use the notation $\mathcal{M}^\bullet_{\mathbb{A}}(\Omega)$ to refer us to the set of all the moulds constructed over the alphabet Ω and valued in the algebra $\mathbb{A}$.

A.2. Mould operations

Moulds can be, among other operations, added, multiplied by a scalar as well as multiplied, composed, and so on. In this article, among the operations we will use, only the multiplication needs to be defined: if $(A^\bullet; B^\bullet) \in \left(\mathcal{M}^\bullet_{\mathbb{A}}(\Omega)\right)^2$, then, the mould multiplication $M^\bullet = A^\bullet \times B^\bullet$ is defined for all sequences $\underline{\boldsymbol{\omega}} \in \mathrm{seq}(\Omega)$ by:

$$M^{\underline{\boldsymbol{\omega}}} = \sum_{\substack{(\underline{\boldsymbol{\omega}}^1;\underline{\boldsymbol{\omega}}^2)\in(\Omega^\star)^2 \\ \underline{\boldsymbol{\omega}}^1\cdot\underline{\boldsymbol{\omega}}^2=\underline{\boldsymbol{\omega}}}} A^{\underline{\boldsymbol{\omega}}^1} B^{\underline{\boldsymbol{\omega}}^2} = \sum_{i=0}^{l(\underline{\boldsymbol{\omega}})} A^{\underline{\boldsymbol{\omega}}^{\leq i}} B^{\underline{\boldsymbol{\omega}}^{>i}} \ .$$

There are a few explanations relative to notations. For a sequence $\underline{\boldsymbol{\omega}} = (\omega_1; \cdots ; \omega_r) \in \mathrm{seq}(\Omega)$ and an integer $k \in [\![\, 0 \,;\, r \,]\!]$, we write:

$$\underline{\boldsymbol{\omega}}^{\leq k} = \begin{cases} \emptyset & \text{, if } k = 0 \\ \\ (\omega_1; \cdots ; \omega_k) & \text{, if } k > 0 \end{cases} \quad \text{and} \quad \underline{\boldsymbol{\omega}}^{>k} = \begin{cases} (\omega_{k+1}; \cdots ; \omega_r) & \text{, if } k < r \ . \\ \\ \emptyset & \text{, if } k = r \ . \end{cases}$$

Let us remark that the two deconcatenations $\emptyset \cdot \underline{\omega}$ and $\underline{\omega} \cdot \emptyset$ intervene in the definition of the mould multiplication and refer respectively to the index $i = 0$ and $i = l(\underline{\omega})$. We will denote such a product by:

$$(A^\bullet \times B^\bullet)^{\underline{\omega}} = \sum_{\underline{\omega}^1\cdot\underline{\omega}^2=\underline{\omega}} A^{\underline{\omega}^1} B^{\underline{\omega}^2} \ .$$

Finally, $(\mathcal{M}^\bullet_{\mathbb{A}}(\Omega),+,\cdots,\times)$ is an associative but non-commutative $\mathbb{A}$-algebra with unit, whose invertible are easily characterised:

$$(\mathcal{M}^\bullet_{\mathbb{A}}(\Omega))^\times = \{M^\bullet \in \mathcal{M}^\bullet_{\mathbb{A}}(\Omega)\,;\, M^\emptyset \in \mathbb{A}^\times\}\ .$$

We will denote by $(M^\bullet)^{\times -1}$ the multiplicative inverse of a mould $M^\bullet$, when it exists.

A.3. Symmetr$\underline{\boldsymbol{a}}$lity

Let us first remind that the shuffle product of two words $P = p_1 \cdots p_r$ and $Q = q_1 \cdots q_s$ constructed over the alphabet Ω is denoted by $\sqcup\!\sqcup$ and defined recursively by:

$$\begin{cases} P \sqcup\!\sqcup \varepsilon = \varepsilon \sqcup\!\sqcup P = P \quad , \\ \\ P \sqcup\!\sqcup Q = p_1\big(p_2 \cdots p_r \sqcup\!\sqcup Q\big) + q_1\big(P \sqcup\!\sqcup q_2 \cdots q_r\big)\ , \end{cases}$$

where ε is the empty word. As an example, if $P = a \cdot b$ and $Q = c$, we have $P \sqcup\!\sqcup Q = abc + acb + cab$.

In order to have a better understanding of the shuffle, one can have a visual representation of it. One can see a word as a desk of card, then the shuffle of two words becomes the set of all the result one can obtained by inserting *classically* one desk of cards in another one.

The multiset $sh\underline{\boldsymbol{a}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$, where $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are sequences of $\mathrm{seq}(\Omega)$, is defined to be the set of all monomials that appears in the non-commutative polynomial $\underline{\boldsymbol{\alpha}} \sqcup\!\sqcup \underline{\boldsymbol{\beta}}$, counted with its multiplicity.

When $\mathbb{A}$ an algebra, we define a symmetr$\underline{\boldsymbol{a}}$l mould $Ma^\bullet$ to be a mould of $\mathcal{M}^\bullet_{\mathbb{A}}(\Omega)$ which satisfies for all $(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}) \in \big(\mathrm{seq}(\Omega)\big)^2$:

$$Ma^{\underline{\boldsymbol{\alpha}}} Ma^{\underline{\boldsymbol{\beta}}} = \sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{a}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})}{}' \ Ma^{\underline{\boldsymbol{\gamma}}}\ .$$

Here, the sum $\displaystyle\sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{a}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})}{}' \ Ma^{\underline{\boldsymbol{\gamma}}}$ is a shorthand for $\displaystyle\sum_{\underline{\boldsymbol{\gamma}} \in \mathrm{seq}(\Omega)} \mathrm{mult}\begin{pmatrix} \underline{\boldsymbol{\alpha}}\,;\,\underline{\boldsymbol{\beta}} \\ \underline{\boldsymbol{\gamma}} \end{pmatrix} Ma^{\underline{\boldsymbol{\gamma}}}$, where $\mathrm{mult}\binom{\underline{\boldsymbol{\alpha}}\,;\,\underline{\boldsymbol{\beta}}}{\underline{\boldsymbol{\gamma}}}$ is the coefficient of the monomial $\underline{\boldsymbol{\gamma}}$ in the product $\underline{\boldsymbol{\alpha}} \sqcup\!\sqcup \underline{\boldsymbol{\beta}}$ and is equal to $\langle \underline{\boldsymbol{\alpha}} \sqcup\!\sqcup \underline{\boldsymbol{\beta}} | \underline{\boldsymbol{\gamma}} \rangle$. From now on, we shall omit the prime on the sum:

$$Ma^{\underline{\boldsymbol{\alpha}}}Ma^{\underline{\boldsymbol{\beta}}} = \sum_{\underline{\boldsymbol{\gamma}}\in\mathrm{seq}(\Omega)} \langle \underline{\boldsymbol{\alpha}} \sqcup\!\sqcup \underline{\boldsymbol{\beta}} | \underline{\boldsymbol{\gamma}} \rangle Ma^{\underline{\boldsymbol{\gamma}}} = \sum_{\gamma\in sh\underline{\boldsymbol{a}}(\boldsymbol{\alpha},\boldsymbol{\beta})} Ma^{\underline{\boldsymbol{\gamma}}} \ .$$

The symmetr$\underline{\boldsymbol{a}}$lity imposes, through a multitude of relations, a strong rigidity. For example, if $(x; y) \in \Omega^2$ and $Ma^\bullet$ denote a symmetr$\underline{\boldsymbol{a}}$l mould, then we have necessarily:

$$\begin{aligned} Ma^x Ma^y &= Ma^{x,y} + Ma^{y,x} \\ Ma^{x,y} Ma^y &= Ma^{y,x,y} + 2Ma^{x,y,y} \ . \end{aligned}$$

A.4. Symmetr$\underline{\boldsymbol{e}}$lity

Let $(\Omega, \cdot)$ be an alphabet with an semi-group structure. Let us first remind that the stuffle product of two words $P = p_1 \cdots p_r$ and $Q = q_1 \cdots q_s$ constructed over the alphabet Ω is denoted by $\star$ and defined recursively by:

$$\begin{cases} P \star \varepsilon = \varepsilon \star P = P \ . \\ P \star Q = p_1\big(p_2 \cdots p_r \star Q\big) + q_1\big(P \star q_2 \cdots q_s\big) + (p_1 \cdot q_1)\big(p_2 \cdots p_r \star q_2 \cdots q_s\big) \ , \end{cases}$$

where ε is again the empty word. As an example, in $\mathrm{seq}(N)$, if $P = 1 \cdot 2$ and $Q = 3$, then: $P \star Q = 1 \cdot 2 \cdot 3 + 1 \cdot 3 \cdot 2 + 3 \cdot 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 2$.

As well as for the shuffle product, one can imagine a visual representation of the stuffle product. Seeing one more time a word as a desk of card, then the stuffle of two words becomes the set of all the result one can obtain by inserting *magically* one desk of blue cards in a desk of red cards. By magically, we mean that some new cards may happen: these new ones are hydrid cards, that is, one of its sides is blue while the other is red. Such a hybrid card can only be obtained when two cards of different colors are situated side by side in a classic shuffle of the two desks of cards. In the previous example, the hybrid cards are 5, coming from the shuffling $1 \cdot 5$, and 4, from $4 \cdot 2$.

The multiset $sh\underline{\boldsymbol{a}}(\underline{\boldsymbol{\alpha}}; \underline{\boldsymbol{\beta}})$, where $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are sequences of $\mathrm{seq}(\Omega)$, is defined to be the set of all monomials that appears in the non-commutative polynomial $\underline{\boldsymbol{\alpha}} \sqcup\!\sqcup \underline{\boldsymbol{\beta}}$, counted with its multiplicity.

The multiset $sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$, where $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are sequences in $\mathrm{seq}(\Omega)$, is defined to be the set of all monomials that appears in the non-commutative polynomial $\underline{\boldsymbol{\alpha}} \star \underline{\boldsymbol{\beta}}$, counting with its multiplicity.

When Ω is an alphabet, which is also an additive semigroup and $\mathbb{A}$ an algebra, we define a symmetr$\underline{\boldsymbol{e}}$l mould $Me^\bullet$ to be a mould of $\mathcal{M}^\bullet_{\mathbb{A}}(\Omega)$ which satisfies for all $(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}) \in \big(\mathrm{seq}(\Omega)\big)^2$:

$$Me^{\underline{\boldsymbol{\alpha}}} Me^{\underline{\boldsymbol{\beta}}} = \sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}}\,;\underline{\boldsymbol{\beta}})}{}' \; Me^{\underline{\boldsymbol{\gamma}}} \ .$$

Here, the sum $\displaystyle\sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}}\,;\underline{\boldsymbol{\beta}})}{}' \; Me^{\underline{\boldsymbol{\gamma}}}$ is a shorthand for $\displaystyle\sum_{\underline{\boldsymbol{\gamma}} \in \mathrm{seq}(\Omega)} \mathrm{mult}\binom{\underline{\boldsymbol{\alpha}}\,;\,\underline{\boldsymbol{\beta}}}{\underline{\boldsymbol{\gamma}}} Me^{\underline{\boldsymbol{\gamma}}}$, where $\mathrm{mult}\binom{\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}}}{\underline{\boldsymbol{\gamma}}}$ is the coefficient of the monomial $\underline{\boldsymbol{\gamma}}$ in the product $\underline{\boldsymbol{\alpha}} \star \underline{\boldsymbol{\beta}}$ and is equal to $\langle \underline{\boldsymbol{\alpha}} \star \underline{\boldsymbol{\beta}} | \underline{\boldsymbol{\gamma}} \rangle$. From now on, we also omit the prime:

$$Me^{\underline{\boldsymbol{\alpha}}} Me^{\underline{\boldsymbol{\beta}}} = \sum_{\underline{\boldsymbol{\gamma}} \in \mathrm{seq}(\Omega)} \langle \underline{\boldsymbol{\alpha}} \star \underline{\boldsymbol{\beta}} | \underline{\boldsymbol{\gamma}} \rangle Me^{\underline{\boldsymbol{\gamma}}} = \sum_{\underline{\boldsymbol{\gamma}} \in sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}},\underline{\boldsymbol{\beta}})} Me^{\underline{\boldsymbol{\gamma}}} \ .$$

As well as the symmetr$\underline{\boldsymbol{a}}$lity, the symmetr$\underline{\boldsymbol{e}}$lity imposes a strong rigidity. For example, if $(x;y) \in \Omega^2$ and $Me^\bullet$ denote a symmetr$\underline{\boldsymbol{e}}$l mould, then we have necessarily:

$$\begin{aligned} Me^x Me^y &= Me^{x,y} + Me^{y,x} + Me^{x+y}. \\ Me^{x,y} Me^y &= Me^{y,x,y} + 2Me^{x,y,y} + Me^{x+y,y} + Me^{x,2y} \ . \end{aligned}$$

A.5. Symmetr$\underline{\boldsymbol{i}}$lity

If $Me^\bullet$ is a symmetr$\underline{\boldsymbol{e}}$l mould over $\mathrm{seq}(\mathbb{N}^*)$, valued in a commutative algebra $\mathbb{A}$, then its generating functions, denoted by $Mig^\bullet$, is defined by:

$$\begin{cases} Mig^{\emptyset} = 1 \ . \\ Mig^{v_1,\cdots,v_r} = \displaystyle\sum_{s_1,\cdots,s_r \geq 1} Me^{s_1,\cdots,s_r} {v_1}^{s_1-1} \cdots {v_r}^{s_r-1} \in \mathbb{A}[\![v_1;\cdots;v_r]\!] \ . \end{cases}$$

The mould $Mig^\bullet$ is then automatically a symmetr$\underline{\boldsymbol{i}}$l mould, that is to say that it satisfies the relation:

$$Mig^{\underline{\mathbf{v}}} Mig^{\underline{\mathbf{w}}} = \sum_{\underline{\mathbf{x}} \in sh\underline{\boldsymbol{i}}(\underline{\mathbf{v}};\underline{\mathbf{w}})} Mig^{\underline{\mathbf{x}}} \ .$$

The multiset $sh\underline{\boldsymbol{i}}(\underline{\mathbf{v}};\underline{\mathbf{w}})$ is also a quasi-shuffle product as defined in [22], as the stuffle. If $\underline{\mathbf{v}}$ and $\underline{\mathbf{w}}$ are sequences over an alphabet of indeterminates, this set is defined exactly in the same way as $sh\underline{\boldsymbol{e}}(\underline{\mathbf{v}};\underline{\mathbf{w}})$, but, here, the contraction of the quasi-shuffle product is an abstract contraction defined over $(\mathbb{N}^*)^2$. The evaluation of a mould $Mig^\bullet$ on a sequence which has such contraction is then done by induction and given by the formula:

$$Mig^{\underline{\mathbf{v}}\cdot(x\circledast y)\cdot\underline{\mathbf{w}}} = \frac{Mig^{\underline{\mathbf{v}}\cdot x\cdot\underline{\mathbf{w}}} - Mig^{\underline{\mathbf{v}}\cdot y\cdot\underline{\mathbf{w}}}}{x-y} \ .$$

A.6. Some examples of rules

Envisaged as a simple system of notations, the mould language already leads us to concise formulas as well as the economy of long sequences of indexes. But its real utility resids in the different mould operations and the rules which indicate how these affect (preserve or transform) basic symmetries.

For example:

1. altern$\underline{\boldsymbol{a}}$l $\circ$ altern$\underline{\boldsymbol{a}}$l = altern$\underline{\boldsymbol{a}}$l .
2. symmetr$\underline{\boldsymbol{e}}$l $\circ$ symmetr$\underline{\boldsymbol{e}}$l = symmetr$\underline{\boldsymbol{e}}$l .
3. altern$\underline{\boldsymbol{a}}$/$\underline{\boldsymbol{e}}$l conjugated by symmetr$\underline{\boldsymbol{a}}$/$\underline{\boldsymbol{e}}$l = altern$\underline{\boldsymbol{a}}$/$\underline{\boldsymbol{e}}$l .
4. exponential (altern$\underline{\boldsymbol{a}}$/$\underline{\boldsymbol{e}}$l) = symtr$\underline{\boldsymbol{a}}$/$\underline{\boldsymbol{e}}$l .

A.7. Some notations

We will always write in bold, italic and underlined the vowel which indicates not only a symmetry of the considered moulds, but also the nature of the products of sequences which will appear. Using this, it will become simpler to distinguish symmetr$\underline{\boldsymbol{a}}$l , symmetr$\underline{\boldsymbol{e}}$l and symmetr$\underline{\boldsymbol{i}}$l moulds as well as to distinguish the set $sh\underline{\boldsymbol{a}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$, $sh\underline{\boldsymbol{e}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$ and $sh\underline{\boldsymbol{i}}(\underline{\boldsymbol{\alpha}};\underline{\boldsymbol{\beta}})$.

The moulds that we consider will carry in their name the vowelic alteration whose immediately indicates their symmetry type. For example, the mould $\mathcal{T}e^\bullet(z)$ is a symmetr$\underline{\boldsymbol{e}}$l mould (see p 91), while $\mathcal{Z}ig^\bullet$ is symmetr$\underline{\boldsymbol{i}}$l (see p. 70). The absence of this vowel will also indicate that the mould verifies no symmetry.

Finally, if $\underline{\boldsymbol{\alpha}} = (\alpha_1;\cdots;\alpha_n)$ is a sequence constructed over an alphabet Ω, we respectively denote by $\overleftarrow{\underline{\boldsymbol{\alpha}}}$ and $\underline{\boldsymbol{\alpha}}^{[k]}$, the opposite sequence $\underline{\boldsymbol{\alpha}}$ and the sequence $\underline{\boldsymbol{\alpha}}$ repeated k times:

$$\overleftarrow{\underline{\boldsymbol{\alpha}}} = (\alpha_n; \cdots; \alpha_1) \ , \qquad\qquad , \qquad\qquad \underline{\boldsymbol{\alpha}}^{[k]} = \underbrace{\underline{\boldsymbol{\alpha}} \cdots \underline{\boldsymbol{\alpha}}}_{k \text{ times}} \ .$$

Weight	Multitangent functions with no multizeta linearisation	Multitangent functions with exact evaluation of multizeta values
4	$\mathcal{T}e^{2,2} = 2\mathcal{Z}e^{2}\mathcal{T}e^{2}$.	
5	$\mathcal{T}e^{2,3} = -3\mathcal{Z}e^{3}\mathcal{T}e^{2} + \mathcal{Z}e^{2}\mathcal{T}e^{3}$.	
	$\mathcal{T}e^{3,2} = 3\mathcal{Z}e^{3}\mathcal{T}e^{2} + \mathcal{Z}e^{2}\mathcal{T}e^{3}$.	
	$\mathcal{T}e^{2,1,2} = 0$.	
6	$\mathcal{T}e^{2,4} = 4\mathcal{Z}e^{4}\mathcal{T}e^{2} - 2\mathcal{Z}e^{3}\mathcal{T}e^{3} + \mathcal{Z}e^{2}\mathcal{T}e^{4}$.	$\mathcal{T}e^{2,4} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} - 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4}$.
	$\mathcal{T}e^{3,3} = -6\mathcal{Z}e^{4}\mathcal{T}e^{2}$.	$\mathcal{T}e^{3,3} = -\frac{12}{5}\zeta(2)^2\mathcal{T}e^{2}$.
	$\mathcal{T}e^{4,2} = 4\mathcal{Z}e^{4}\mathcal{T}e^{2} + 2\mathcal{Z}e^{3}\mathcal{T}e^{3} + \mathcal{Z}e^{2}\mathcal{T}e^{4}$.	$\mathcal{T}e^{4,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} + 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4}$.
	$\mathcal{T}e^{2,1,3} = -\left(\mathcal{Z}e^{2,2} + \mathcal{Z}e^{3,1}\right)\mathcal{T}e^{2} + \mathcal{Z}e^{2,1}\mathcal{T}e^{3}$.	$\mathcal{T}e^{2,1,3} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} + \zeta(3)\mathcal{T}e^{3}$.
	$\mathcal{T}e^{2,2,2} = 2\left(\mathcal{Z}e^{2,2} + \left(\mathcal{Z}e^{2}\right)^2\right)\mathcal{T}e^{2}$.	$\mathcal{T}e^{2,2,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2}$.
	$\mathcal{T}e^{3,1,2} = -\left(\mathcal{Z}e^{2,2} + \mathcal{Z}e^{3,1}\right)\mathcal{T}e^{2} - \mathcal{Z}e^{2,1}\mathcal{T}e^{3}$.	$\mathcal{T}e^{3,1,2} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} - \zeta(3)\mathcal{T}e^{3}$.
	$\mathcal{T}e^{2,1,1,2} = 2\mathcal{Z}e^{2,1,1}\mathcal{T}e^{2}$.	$\mathcal{T}e^{2,1,1,2} = \frac{4}{5}\zeta(2)^2\mathcal{T}e^{2}$.

Table 1. Tabulation of the multitangent functions of weight 4 , 5 and 6.

$\|\|\underline{s}\|\| = 4$	$\mathcal{Z}e^{2}\mathcal{T}e^{2} = \frac{1}{2}\mathcal{T}e^{2,2} \quad .$
$\|\|\underline{s}\|\| = 5$	$\mathcal{Z}e^{3}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{3,2} - \frac{1}{6}\mathcal{T}e^{2,3}$. $\mathcal{Z}e^{2,1}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{3,2} - \frac{1}{6}\mathcal{T}e^{2,3}$.
$\|\|\underline{s}\|\| = 6$	$\mathcal{Z}e^{4}\mathcal{T}e^{2} = -\frac{1}{6}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{2,2}\mathcal{T}e^{2} = -\frac{1}{8}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{3,1}\mathcal{T}e^{2} = -\frac{1}{24}\mathcal{T}e^{3,3}$. $\mathcal{Z}e^{2,1,1}\mathcal{T}e^{2} = -\frac{1}{6}\mathcal{T}e^{3,3}$.
$\|\|\underline{s}\|\| = 7$	$\mathcal{Z}e^{5}\mathcal{T}e^{2} = -\frac{1}{30}\mathcal{T}e^{5,2} - \frac{1}{15}\mathcal{T}e^{4,3} + \frac{1}{15}\mathcal{T}e^{3,4} + \frac{1}{30}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{4,1}\mathcal{T}e^{2} = \frac{1}{12}\mathcal{T}e^{2,2,3} - \frac{1}{12}\mathcal{T}e^{3,2,2} - \frac{1}{40}\mathcal{T}e^{5,2} - \frac{1}{20}\mathcal{T}e^{4,3} + \frac{1}{20}\mathcal{T}e^{3,4} + \frac{1}{40}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{3,2}\mathcal{T}e^{2} = \frac{1}{4}\mathcal{T}e^{3,2,2} - \frac{1}{4}\mathcal{T}e^{2,2,3} + \frac{7}{120}\mathcal{T}e^{5,2} + \frac{7}{60}\mathcal{T}e^{4,3} - \frac{7}{60}\mathcal{T}e^{3,4} - \frac{7}{120}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{2,3}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{2,2,3} - \frac{1}{6}\mathcal{T}e^{3,2,2} - \frac{1}{15}\mathcal{T}e^{5,2} - \frac{2}{15}\mathcal{T}e^{4,3} + \frac{2}{15}\mathcal{T}e^{3,4} + \frac{1}{15}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{3,1,1}\mathcal{T}e^{2} = \frac{1}{12}\mathcal{T}e^{2,2,3} - \frac{1}{12}\mathcal{T}e^{3,2,2} - \frac{1}{40}\mathcal{T}e^{5,2} - \frac{1}{20}\mathcal{T}e^{4,3} + \frac{1}{20}\mathcal{T}e^{3,4} + \frac{1}{40}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{2,2,1}\mathcal{T}e^{2} = \frac{1}{4}\mathcal{T}e^{3,2,2} - \frac{1}{4}\mathcal{T}e^{2,2,3} + \frac{7}{120}\mathcal{T}e^{5,2} + \frac{7}{60}\mathcal{T}e^{4,3} - \frac{7}{60}\mathcal{T}e^{3,4} - \frac{7}{120}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{2,1,2}\mathcal{T}e^{2} = \frac{1}{6}\mathcal{T}e^{2,2,3} - \frac{1}{6}\mathcal{T}e^{3,2,2} - \frac{1}{15}\mathcal{T}e^{5,2} - \frac{2}{15}\mathcal{T}e^{4,3} + \frac{2}{15}\mathcal{T}e^{3,4} + \frac{1}{15}\mathcal{T}e^{2,5}$. $\mathcal{Z}e^{2,1,1,1}\mathcal{T}e^{2} = -\frac{1}{30}\mathcal{T}e^{5,2} - \frac{1}{15}\mathcal{T}e^{4,3} + \frac{1}{15}\mathcal{T}e^{3,4} + \frac{1}{30}\mathcal{T}e^{2,5}$.

Table 2. Some examples of projection onto multitangent functions.

$p = 4$:

$$\begin{pmatrix} \frac{8}{5} \\ -\frac{12}{5} \\ \frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$p = 5$:

$$\begin{pmatrix} -5 & 0 \\ 10 & 0 \\ -10 & 0 \\ 5 & 0 \\ \frac{15}{2} & -6 \\ -\frac{15}{2} & 6 \end{pmatrix}$$

$p = 6$:

$$\begin{pmatrix} \frac{48}{35} & 0 \\ -\frac{24}{7} & 0 \\ \frac{32}{7} & 0 \\ -\frac{24}{7} & 0 \\ \frac{48}{35} & 0 \\ \frac{352}{105} & -6 \\ -\frac{88}{35} & \frac{9}{2} \\ \frac{16}{105} & 3 \\ -\frac{88}{35} & \frac{9}{2} \\ \frac{16}{7} & -9 \\ \frac{16}{105} & 3 \\ \frac{24}{35} & 0 \end{pmatrix}$$

$p = 7$:

$$\begin{pmatrix} -7 & 0 & 0 \\ 21 & 0 & 0 \\ -35 & 0 & 0 \\ 35 & 0 & 0 \\ -21 & 0 & 0 \\ 7 & 0 & 0 \\ 0 & 0 & 0 \\ 14 & -20 & \frac{16}{5} \\ -14 & 20 & \frac{16}{5} \\ 21 & -10 & \frac{16}{5} \\ -14 & 20 & \frac{16}{5} \\ 0 & 0 & 0 \\ -28 & 0 & \frac{48}{5} \\ 14 & -20 & \frac{16}{5} \\ 28 & 0 & -\frac{48}{5} \\ -21 & 10 & \frac{16}{5} \\ -\frac{441}{16} & 15 & 0 \\ \frac{441}{16} & -15 & 0 \\ -\frac{189}{16} & 15 & -\frac{24}{5} \\ \frac{189}{16} & -15 & \frac{24}{5} \end{pmatrix}$$

Table 3. Obtained matrix, for the weight $p \in [\![4\,;\,7]\!]$, applying the explained method relatively to the unit-cleansing for multitangent functions.

$\lvert\lvert \underline{\mathbf{s}} \rvert\rvert = 5$	$\mathcal{T}e^{2,1,2} = 0 \ .$
$\lvert\lvert \underline{\mathbf{s}} \rvert\rvert = 6$	$\mathcal{T}e^{3,1,2} = \frac{1}{6}\mathcal{T}e^{3,3} + \frac{1}{4}\mathcal{T}e^{2,4} - \frac{1}{4}\mathcal{T}e^{4,2} \ .$ $\mathcal{T}e^{2,1,3} = \frac{1}{6}\mathcal{T}e^{3,3} - \frac{1}{4}\mathcal{T}e^{2,4} + \frac{1}{4}\mathcal{T}e^{4,2} \ .$ $\mathcal{T}^{2,1,1,2} = -\frac{1}{3}\mathcal{T}e^{3,3} \ .$
$\lvert\lvert \underline{\mathbf{s}} \rvert\rvert = 7$	$\mathcal{T}e^{4,1,2} = \frac{1}{6}\mathcal{T}e^{2,2,3} - \frac{1}{6}\mathcal{T}e^{3,2,2} - \frac{1}{3}\mathcal{T}e^{5,2} + \frac{7}{48}\mathcal{T}e^{4,3} + \frac{23}{48}\mathcal{T}e^{3,4} + \frac{1}{3}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{3,1,3} = \frac{1}{5}\mathcal{T}e^{2,3,2} \ .$ $\mathcal{T}e^{2,1,4} = \frac{1}{3}\mathcal{T}e^{3,2,2} + \frac{1}{3}\mathcal{T}e^{5,2} + \frac{13}{24}\mathcal{T}e^{4,3} + \frac{5}{24}\mathcal{T}e^{3,4} - \frac{1}{3}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{2,1,1,3} = \frac{1}{3}\mathcal{T}e^{3,2,2} + \frac{1}{12}\mathcal{T}e^{5,2} - \frac{1}{48}\mathcal{T}e^{4,3} - \frac{17}{48}\mathcal{T}e^{3,4} - \frac{1}{12}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{2,1,2,2} = \frac{2}{3}\mathcal{T}e^{3,2,2} - \frac{1}{6}\mathcal{T}e^{5,2} - \frac{5}{24}\mathcal{T}e^{4,3} + \frac{11}{24}\mathcal{T}e^{3,4} + \frac{1}{6}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{2,2,1,2} = \frac{2}{3}\mathcal{T}e^{3,2,2} + \frac{1}{6}\mathcal{T}e^{5,2} + \frac{11}{24}\mathcal{T}e^{4,3} - \frac{5}{24}\mathcal{T}e^{3,4} - \frac{1}{6}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{3,1,1,2} = -\frac{1}{3}\mathcal{T}e^{3,2,2} - \frac{1}{12}\mathcal{T}e^{5,2} - \frac{23}{48}\mathcal{T}e^{4,3} - \frac{7}{48}\mathcal{T}e^{3,4} + \frac{1}{12}\mathcal{T}e^{2,5} \ .$ $\mathcal{T}e^{2,1,1,1,2} = 0 \ .$

Table 4. Some examples of unit cleansing for multitangent functions of weight 5, 6 and 7.

Relations in $\mathcal{M}TGF_6$	Equivalent relations in $\mathcal{M}ZV$	
	Relations	Origin
$\mathcal{T}e^{3,1,2} + \mathcal{T}e^{2,1,3} + \mathcal{T}e^{2,1,1,2} = 0$.	$\mathcal{Z}e^{2,1,1} = \mathcal{Z}e^{3,1} + \mathcal{Z}e^{2,2}$.	Double-shuffle
$2\mathcal{T}e^{3,1,2} + \mathcal{T}e^{2,2,2} + 2\mathcal{T}e^{2,1,3} = 0$.	$\big(\mathcal{Z}e^{2}\big)^2 = 4\mathcal{Z}e^{3,1} + 2\mathcal{Z}e^{2,2}$.	Shuffle
$\mathcal{T}e^{2,4} - \mathcal{T}e^{4,2} + 2\mathcal{T}e^{2,1,3} - 2\mathcal{T}e^{3,1,2} = 0$.	$\begin{cases} \mathcal{Z}e^{3}\mathcal{Z}e^{2} = 6\mathcal{Z}e^{4,1} + 3\mathcal{Z}e^{3,2} + \mathcal{Z}e^{2,3} \text{ .} \\ \mathcal{Z}e^{3} = \mathcal{Z}e^{2,1} \text{ .} \end{cases}$	Shuffle Double-shuffle
$3\mathcal{T}e^{3,1,2} + 3\mathcal{T}e^{2,1,3} - \mathcal{T}e^{3,3} = 0$.	$\mathcal{Z}e^{4} = \mathcal{Z}e^{3,1} + \mathcal{Z}e^{2,2}$	Double-shuffle

Table 5. The four independent $\mathbb{Q}$-linear relations between multitangent functions of weight 6.

Relations in $\mathcal{M}TGF_7$
$\mathcal{T}e^{2,1,1,1,2} = 0$.
$-4\mathcal{T}e^{3,1,3} + \mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,1,1,3} = 0$.
$4\mathcal{T}e^{3,1,3} - 2\mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,1,2,2} = 0$.
$-4\mathcal{T}e^{3,1,3} + 2\mathcal{T}e^{3,1,1,2} + \mathcal{T}e^{2,2,1,2} = 0$.
$\mathcal{T}e^{4,1,2} + 5\mathcal{T}e^{3,1,3} + \mathcal{T}e^{2,1,4} = 0$.
$-\mathcal{T}e^{4,3} + \mathcal{T}e^{4,1,2} + 5\mathcal{T}e^{3,1,3} + \mathcal{T}e^{2,2,3} - 4\mathcal{T}e^{3,1,1,2} = 0$.
$-5\mathcal{T}e^{3,1,3} + \mathcal{T}e^{2,3,2} = 0$.
$\mathcal{T}e^{4,3} - \mathcal{T}e^{4,1,2} + \mathcal{T}e^{3,2,2} - 8\mathcal{T}e^{3,1,3} + 4\mathcal{T}e^{3,1,1,2} = 0$.
$-\mathcal{T}e^{5,2} + \mathcal{T}e^{2,5} - 4\mathcal{T}e^{4,1,2} - 18\mathcal{T}e^{3,1,3} + 4\mathcal{T}e^{3,1,1,2} = 0$.
$\mathcal{T}e^{4,3} + \mathcal{T}e^{3,4} + 8\mathcal{T}e^{3,1,3} = 0$.

Table 6. The ten independent $\mathbb{Q}$-linear relations between multitangent functions of weight 7.

Weight	Multitangent functions
2	$\mathcal{T}e^{1,1} = -3\,\zeta(2)\ .$
3	$\mathcal{T}e^{1,2} = 0\ .$ $\mathcal{T}e^{2,1} = 0\ .$ $\mathcal{T}e^{1,1,1} = -\zeta(2)\mathcal{T}e^{1}\ .$
4	$\mathcal{T}e^{1,3} = -\zeta(2)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{3,1} = -\zeta(2)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,1,2} = -\dfrac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,2,1} = 0\ .$ $\mathcal{T}e^{2,1,1} = -\dfrac{1}{2}\zeta(2)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,1,1,1} = \dfrac{3}{2}\zeta(2)^{2}\ .$

Weight	Multitangent functions
5	$\mathcal{T}e^{1,4} = \zeta(3)\mathcal{T}e^{2} - \zeta(2)\mathcal{T}e^{3}\ .$ $\mathcal{T}e^{4,1} = -\zeta(3)\mathcal{T}e^{2} - \zeta(2)\mathcal{T}e^{3}\ .$ $\mathcal{T}e^{1,1,3} = \zeta(3)\mathcal{T}e^{2} - \dfrac{1}{2}\zeta(2)\mathcal{T}e^{3}\ .$ $\mathcal{T}e^{1,2,2} = -2\,\zeta(3)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,3,1} = 0\ .$ $\mathcal{T}e^{2,2,1} = 2\,\zeta(3)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{3,1,1} = -\zeta(3)\mathcal{T}e^{2} - \dfrac{1}{2}\zeta(2)\mathcal{T}e^{3}\ .$ $\mathcal{T}e^{1,1,1,2} = \dfrac{1}{3}\zeta(3)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,1,2,1} = 0\ .$ $\mathcal{T}e^{1,2,1,1} = 0\ .$ $\mathcal{T}e^{2,1,1,1} = -\dfrac{1}{3}\zeta(3)\mathcal{T}e^{2}\ .$ $\mathcal{T}e^{1,1,1,1,1} = \dfrac{3}{10}\zeta(2)^{2}\mathcal{T}e^{1}\ .$

Table 7. Tabulation of the divergent multitangent functions of weight 2 , 3 , 4 and 5.

$$\begin{cases} \mathcal{T}e^{1^{[2k]}} = \dfrac{(-1)^k \pi^{2k}}{(2k)!} \ . \\ \mathcal{T}e^{1^{[2k+1]}} = \dfrac{(-1)^k \pi^{2k}}{(2k+1)!} \mathcal{T}e^1 \ . \end{cases}$$

$$\mathcal{T}e^{2^{[k]}} = \frac{2^{2k-1}\pi^{2(k-1)}}{(2k)!}\mathcal{T}e^2 \ .$$

$$\begin{cases} \mathcal{T}e^{3^{[2k]}} = \dfrac{3(-1)^k 2^{6k-2}\pi^{6k-2}}{(6k)!}\mathcal{T}e^2 \ . \\ \mathcal{T}e^{3^{[2k+1]}} = \dfrac{3(-1)^k 2^{6k+1}\pi^{6k}}{(6k+3)!}\mathcal{T}e^3 \ . \end{cases}$$

$$\mathcal{T}e^{4^{[k]}} = \frac{2^{4k-1}\pi^{4(k-1)}}{(4k)!}\Big(\big(2(-1)^k + 2^{2k-1}\big)\big(\mathcal{T}e^2\big)^2 - 3(-1)^k\mathcal{T}e^4\Big) \ .$$

$$\begin{cases} \mathcal{T}e^{5^{[2k]}} = \dfrac{5}{16}\dfrac{(-1)^k 2^{9k}\pi^{10k-4}}{(10k)!}\Big(\big(2^k\cdot 3 + \alpha^k + \bar{\alpha}^k\big)\big(\mathcal{T}e^2\big)^2 - 2^{k+1}\cdot 3\mathcal{T}e^4\Big) \ , \\ \mathcal{T}e^{5^{[2k+1]}} = \dfrac{5(-1)^k 2^{9k}\pi^{10k}}{(10k+5)!}\Big(2^{k+3}\cdot 3\mathcal{T}e^5 + \alpha_n\mathcal{T}e^1\big(\mathcal{T}e^2\big)^2\Big) \ , \end{cases}$$

$$\text{where } \alpha_n = 11\big(\alpha^k + \bar{\alpha}^k - 2^{k+1}\big) + 5\sqrt{5}\big(\alpha^k - \bar{\alpha}^k\big) \text{ and } \begin{cases} \alpha = 123 + 5\sqrt{5} \ . \\ \bar{\alpha} = 123 - 5\sqrt{5} \ . \end{cases}$$

$$\mathcal{T}e^{6^{[k]}} = \frac{2^{6k-5}\pi^{6(k-1)}}{(6k)!}\Big(360\mathcal{T}e^6 - 18u_n\mathcal{T}e^2\mathcal{T}e^4 - v_n\big(\mathcal{T}e^2\big)^3\Big) \ ,$$

$$\text{where } \begin{cases} u_n & = & 26 + (-27)^k \ . \\ v_n & = & 30 - 6(-27)^k - 3\cdot 2^k \ . \end{cases}$$

Table 8. Examples of calculations of multitangent functions with repeated argument.

$$
\begin{array}{rrcl}
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{2,1^{[p]},2} & = & \big(1+(-1)^p\big)\mathcal{Z}e^{p+2}\mathcal{T}e^2\ . \\
\forall (p;q) \in \mathbb{N}^2\ , & \mathcal{T}e^{2,1^{[p]},2,1^{[q]},2} & = & \big((-1)^{p+q}\mathcal{Z}e^{p+2,q+2} + (-1)^q\mathcal{Z}e^{p+2}\mathcal{Z}e^{q+2} + \mathcal{Z}e^{q+2,p+2}\big)\mathcal{T}e^2\ . \\
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{2,1^{[p]},3,1^{[p]},2} & = & (-1)^p\big(\mathcal{Z}e^{p+2}\big)^2\mathcal{T}e^3\ . \\
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{3,1^{[2p+1]},3} & = & 2\mathcal{Z}e^{3,1^{[2p+1]}}\mathcal{T}e^3 = 2\mathcal{Z}e^{2p+3,1}\mathcal{T}e^3\ . \\
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{\{3,1\}^{[p]},3} & = & \big(2^{4p+4} - 2(-4)^{p+1}\big)\dfrac{\pi^{4p}}{(4p+4)!}\mathcal{T}e^3\ . \\
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{2,1^{[p]},3} & = & -\Big(\big((-1)^p+1\big)\mathcal{Z}e^{p+2,1} + \mathcal{Z}e^{p+3}\Big)\mathcal{T}e^2 + \mathcal{Z}e^{p+2}\mathcal{T}e^3\ . \\
\forall p \in \mathbb{N}\ , & \mathcal{T}e^{3,1^{[p]},2} & = & \Big(\big((-1)^p+1\big)\mathcal{Z}e^{p+2,1} + (-1)^p\mathcal{Z}e^{p+3}\Big)\mathcal{T}e^2 + (-1)^p\mathcal{Z}e^{p+2}\mathcal{T}e^3\ .
\end{array}
$$

Table 9. Examples of explicit calculations of multitangent functions.

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