Mould calculus, resurgence and combinatorial Hopf algebras.

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Table des matières

During this talk, we set :

- $f(z) = z + 1 + a(z)$ where $a(z) = \sum_{n \geq 3} \frac{a_n}{z^{n-1}} \in \frac{1}{z^2}\mathbb{C}\left\{\frac{1}{z}\right\}$:

$$\exists (\mathrm{C}_0\,;\,\mathrm{C}_1) \in (\mathbb{R}_*^+)^2\ ,\ |a_n| \leq \mathrm{C}_0 {\mathrm{C}_1}^n\ .$$

- $t(z) = z + 1$.
- $g(z) = f(z) - 1$, so $f = l \circ g$.
- $\mathbb{N}_p = \{n \in \mathbb{N}\,;\, n \geq p\}$, where $p \in \mathbb{N}$.
- $\mathrm{seq}(\Omega) = \{\underline{\omega} = (\omega_1; \cdots; \omega_r)\,;\, r \in \mathbb{N}^*\ ,\ (\omega_1, \cdots, \omega_r) \in \Omega^r\}$, where Ω is an alphabet.
- $\forall \underline{\mathbf{s}} = (s_1, \cdots, s_r) \in \mathrm{seq}(\mathbb{N})$, $\begin{cases} ||\underline{\mathbf{s}}|| = s_1 + \cdots + s_r \\ l(\underline{\mathbf{s}}) = r \end{cases}$

1 Aim.

We want to proof the following result :

Theorem : Let $\mathrm{A}_\bullet$ defined by : $\forall \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}_3)$, $\mathrm{A}_{\underline{\mathbf{s}}} = a_{s_1} \cdots a_{s_r}$.
$\mathcal{A}_\bullet$ defined by : $\forall \underline{\mathbf{S}} \in \mathrm{seq}\big(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\}\big)$,
$\mathcal{A}_{\underline{\mathbf{S}}} = \mathrm{A}_{\underline{\mathbf{s}}^1} \cdots \mathrm{A}_{\underline{\mathbf{s}}^r}$.
then :

1. there exists an explicit mould $\tau^\bullet$ defined on $\mathrm{seq}\big(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\}\big)$ such that :

$$\pi^+ = \sum_{\underline{\mathbf{S}} \in \mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\})} \mathrm{sg}^{\underline{\mathbf{S}}} \, \tau^{\underline{\mathbf{S}}} \, \mathcal{A}_{\underline{\mathbf{S}}} \ .$$

2. for all $n \in \mathbb{Z}^*$, there exists an explicit mould $\widehat{\tau}_n^\bullet$, defined on $\mathrm{seq}\big(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\}\big)$ such that :

$$\forall n \in \mathbb{Z}^*,\ \mathrm{A}^+_{2in\pi} = \mathrm{sg}(n) \sum_{\underline{\mathbf{S}} \in \mathrm{seq}(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\}) - \{\emptyset\}} \mathrm{sg}^{\underline{\mathbf{S}}} \, \widehat{\tau}_n^{\underline{\mathbf{S}}} \, \mathcal{A}_{\underline{\mathbf{S}}} \ .$$

Remark : For all $\underline{\mathbf{S}} \in \mathrm{seq}\big(\mathrm{seq}(\mathbb{N}_3) - \{\emptyset\}\big)$:

$$\begin{cases} \tau^{\underline{\mathbf{S}}} \in \mathcal{M}\mathrm{TGF} \subset \mathcal{H}(\mathbb{C} - \mathbb{Z}) \ . \\ \\ \widehat{\tau}_n^{\underline{\mathbf{S}}} \in \mathcal{M}\mathrm{ZV} \ . \end{cases}$$

2 First expression of the horn map.

We have : $h^+ = v^{att} \circ (v^{rep})^{-1}$.

In David's talk, we have seen that :

$$v^{att}(z) = z + \sum_{\underline{\mathbf{k}} \in \mathrm{seq}(\mathbb{N}_1)} \psi^{\underline{\mathbf{k}}}(z) \quad .$$

We can do the same with $\mathrm{V}^{att}(\varphi) = \varphi \circ v^{att}$, the substitution operator associate to v^{att} . We obtain :

$$\mathrm{V}^{att} = \sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \mathrm{U}_+^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}} \quad ,$$

where : $\begin{cases} \Gamma_n = \mathrm{T}^n \circ (\mathrm{G} - \mathrm{I}d) \circ \mathrm{T}^{-n} \\ \Gamma_{\underline{\mathbf{n}}} = \Gamma_{n_r} \circ \cdots \circ \Gamma_{n_1} \\ \mathrm{U}_+^{\underline{\mathbf{n}}} = \begin{cases} 1 \text{ if } 0 \leq n_r < \cdots < n_1 \\ 0 \text{ else} \end{cases} \end{cases}$

By the same $(\mathrm{V}^{rep})^{-1} = \sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \mathrm{U}_-^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}}$ where : $\mathrm{U}_-^{\underline{\mathbf{n}}} = \begin{cases} 1 \text{ if } n_r < \cdots < n_1 < 0 \\ 0 \text{ else} \end{cases}$

$$\text{So : } \mathrm{H}^+ = \left(\sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \mathrm{U}_+^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}} \right) \circ \left(\sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \mathrm{U}_-^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}} \right)$$

$$= \sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \left(\mathrm{U}_+^{\bullet} \times \mathrm{U}_-^{\bullet} \right)^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}}$$

$$= \sum_{\underline{\mathbf{n}} \in \mathrm{seq}(\mathbb{Z})} \mathrm{U}^{\underline{\mathbf{n}}} \Gamma_{\underline{\mathbf{n}}} \text{ where } \mathrm{U}^{\underline{\mathbf{n}}} = \begin{cases} 1 \text{ if } -\infty < n_r < \cdots < n_1 < +\infty \\ 0 \text{ else} \end{cases}$$

Consequently : $h^+(z) = z + \sum_{r \geq 1} \sum_{-\infty < n_r < \cdots < n_1 < +\infty} \Gamma_{n_1, \cdots, n_r} \cdot z$

3 Evaluation of $\Gamma_\bullet \cdot z$.

Cf. transparents

4 Multitangent functions / Multizetas values.

Let us make a little break with special functions...

Definition : Let $\mathcal{S}^\star_+ = \left\{ \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) ; s_1 \geq 2 \right\}$
$\mathcal{S}^\star = \left\{ \underline{\mathbf{s}} \in \mathrm{seq}(\mathbb{N}^*) ; s_1 \geq 2 \text{ et } s_{l(\underline{\mathbf{s}})} \geq 2 \right\}$

So $\forall \underline{\mathbf{s}} \in \mathcal{S}^\star_+,\ \mathcal{Z}e^{\underline{\mathbf{s}}} = \displaystyle\sum_{1 \leq n_r < \cdots < n_1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$.

$\forall \underline{\mathbf{s}} \in \mathcal{S}^\star,\ \mathcal{T}e^{\underline{\mathbf{s}}} = \displaystyle\sum_{-\infty < n_r < \cdots < n_1 < +\infty} \frac{1}{(n_1 + z)^{s_1} \cdots (n_r + z)^{s_r}}$.

Why call we "multitangents functions" ?

Because $\mathcal{T}e^1 = \dfrac{\pi}{\tan(\pi z)}$...

So the "***multicotangents functions***" $\cdots$

Notation : $\mathcal{M}\mathrm{ZV} = \mathrm{Vect}_{\mathbb{Q}}(\mathcal{Z}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}^\star_+}$.
$\mathcal{M}\mathrm{TGF} = \mathrm{Vect}_{\mathbb{Q}}(\mathcal{T}e^{\underline{\mathbf{s}}})_{\underline{\mathbf{s}} \in \mathcal{S}^\star}$.

Remark : 1. $\mathcal{Z}e^\bullet$ and $\mathcal{T}e^\bullet$ look simillar.
2. $\mathcal{Z}e^\bullet$ generalize the zetas values ; $\mathcal{T}e^\bullet$ generalize the Eisenstein series.
3. $\mathcal{Z}e^\bullet$ and $\mathcal{T}e^\bullet$ are two symetrEl mould.

Proposition 0 : $\forall \underline{\mathbf{s}} \in \mathcal{S}^\star$, $\mathcal{T}e^\bullet$ is a 1-periodic function.

Proposition 1 : $\forall \underline{\mathbf{s}} \in \mathcal{S}^\star$, $\exists (z_k)_{2 \leq k \leq m} \in \mathcal{M}\mathrm{ZV}^{m-1}$,

$$\mathcal{T}e^\bullet = \sum_{k=2}^{m} z_k \mathcal{T}e^k$$

Proof : 1. Partial fraction decomposition of $\dfrac{1}{(n_1 + \mathrm{X})^{s_1} \cdots (n_r + \mathrm{X})^{s_r}}$. give :

$$\mathcal{T}e^\bullet = \sum_{k=1}^{m} z_k \mathcal{T}e^k$$

2. $z_1 = 0$ because of iterate integral representation of multizetas values.

Proposition 2 : For all $\underline{\mathbf{s}} \in \mathcal{S}^\star$, we are able to calculate Fourier coefficients of $\mathcal{T}e^{\underline{\mathbf{s}}}$.

Proof : From Proposition 1, it suffice to evaluate Fourier coefficient of monotangent function.

$$\mathcal{T}e^s = \frac{(-1)^{s-1}}{(s-1)!} \frac{\partial^{s-1}}{\partial z^{s-1}} \left(\frac{\pi}{\tan(\pi z)} \right)$$

We know how to do for $z \longmapsto \dfrac{\pi}{\tan(\pi z)}$

It's OK

Everything here is explicit !

5 Permutation of the two sums.

Come back to the horn map h^+ :

We have :

$$h^+(z) - z =$$

$$\sum_{r\geq 1}\sum_{\underline{\mathbf{n}}\in \mathrm{seq}(\mathbb{Z})}\sum_{\substack{\underline{\mathbf{S}}\,\in\,\mathrm{seq}(\mathrm{seq}(\mathbb{N}_3)-\{\emptyset\})\\ l(\underline{\mathbf{S}})=l(\underline{\mathbf{n}})\\ l(\underline{\mathbf{s}}^1)=1}} \mathrm{sg}^{\underline{\mathbf{S}}}\left(\sum_{\underline{\mathbf{i}}\in \mathrm{triangle}(\underline{\mathbf{S}})} \mathcal{B}^{\underline{\mathbf{S}},\underline{\mathbf{i}}}\left(\prod_{p=1}^{r}\frac{1}{(z+n_p)^{||\underline{\mathbf{s}}^p||-l(\underline{\mathbf{s}}^p)+\mathrm{diag}_p^{\underline{\mathbf{i}}}}}\right)\right)\mathcal{A}_{\underline{\mathbf{S}}}$$

So, we would like :

$$\pi^+(z) - z \;=\; \sum_{r\geq 1}\sum_{\substack{\underline{\mathbf{S}}\,\in\,\mathrm{seq}(\mathrm{seq}(\mathbb{N}_3)-\{\emptyset\})\\ l(\underline{\mathbf{S}})=r\\ l(\underline{\mathbf{s}}^1)=1}} \mathrm{sg}^{\underline{\mathbf{S}}}\left(\sum_{\underline{\mathbf{i}}\in \mathrm{triangle}(\underline{\mathbf{S}})} \mathcal{B}^{\underline{\mathbf{S}},\underline{\mathbf{i}}}\mathcal{T}e^{\sigma(\underline{\mathbf{S}};\underline{\mathbf{i}})}\right)\mathcal{A}_{\underline{\mathbf{S}}}\ ,$$

$$\text{where } \sigma(\underline{\mathbf{S}};\underline{\mathbf{i}}) = (||\underline{\mathbf{s}}^p|| - l(\underline{\mathbf{s}}^p) + \mathrm{diag}_p^{\underline{\mathbf{i}}})_{1\leq k\leq r}$$

$$= \sum_{r\geq 1}\sum_{\substack{\underline{\mathbf{S}}\,\in\,\mathrm{seq}(\mathrm{seq}(\mathbb{N}_3)-\{\emptyset\})\\ l(\underline{\mathbf{S}})=r}} \mathrm{sg}^{\underline{\mathbf{S}}}\ \tau^{\underline{\mathbf{S}}}\ \mathcal{A}_{\underline{\mathbf{S}}}\ ,$$

Here :

$$\tau^{\underline{\mathbf{S}}} \;=\; \begin{cases} id_{\mathbb{C}} & ,\ \text{if } \underline{\mathbf{S}} = \emptyset \\ \displaystyle\sum_{\underline{\mathbf{i}}\in \mathrm{triangle}(\underline{\mathbf{S}})} \mathcal{B}^{\underline{\mathbf{S}},\underline{\mathbf{i}}}\ \mathcal{T}e^{\sigma(\underline{\mathbf{S}}\,;\,\underline{\mathbf{i}})} & ,\ \text{if } l(\underline{\mathbf{s}}^1) = 1 \\ 0 & ,\ \text{else} \end{cases} \quad ,$$

6 Calculation of Fourier coefficients of $h^+ - id$.

Since $h^+(z) - z = \sum_{\mathrm{seq}(\mathbb{Z})} \mathrm{U}^\bullet \Gamma_\bullet \cdot z$ is normally convergent on every compact