On Cayley graphs of algebraic structures

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Abstract

We present simple graph-theoretic characterizations of Cayley graphs for left-cancellative and cancellative monoids, groups, left-quasigroups and quasigroups. We show that these characterizations are effective for the suffix graphs of word rewriting systems.

To describe the structure of a group, Cayley introduced in 1878 [3] the concept of graph for any group $(G,\cdot)$ according to any generating subset S. This is simply the set of labeled oriented edges $g \xrightarrow{s} g{\cdot}s$ for every g of G and s of S. Such a graph, called Cayley graph, is directed and labeled in S (or an encoding of S by symbols called letters or colors). The study of groups by their Cayley graphs is a main topic of algebraic graph theory [2, 4, 1]. A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [7]: an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. However, this algebraic characterization is not well suited for deciding whether a possibly infinite graph is a Cayley graph. It is pertinent to look for characterizations by graph-theoretic conditions. This approach was clearly stated by Hamkins in 2010: Which graphs are Cayley graphs? [5]. In this paper, we present simple graph-theoretic characterizations of Cayley graphs for firstly left-cancellative and cancellative monoids, and then for groups. These characterizations are then extended to any subset S of left-cancellative magmas, left-quasigroups, quasigroups, and groups. Finally, we show that these characterizations are effective for the suffix transition graphs of labeled word rewriting systems.

Generalized Cayley graphs of left-cancellative magmas

Cayley graphs are directed labeled graphs without isolated vertex. Precisely let A be an arbitrary (finite or infinite) set. A directed *A-graph* (V,G) is defined by a set V of *vertices* and a subset $G \subseteq V{\times}A{\times}V$ of *edges*. Any edge $(s,a,t) \in G$ is from the *source* s to the *target* t with *label* a, and is also written by the *transition* $s \xrightarrow{a}_G t$ or directly $s \xrightarrow{a} t$ if G is clear from the context. The sources and targets of edges form the set $V_G = \{\ s \mid \exists\ a,t\ \ (s \xrightarrow{a} t \vee t \xrightarrow{a} s)\ \}$ of *non-isolated vertices* of G, and $A_G = \{\ a \mid \exists\ s,t\ \ (s \xrightarrow{a} t)\ \}$ is the set of its edge labels. We assume that any graph (V,G) is without isolated vertex: $V = V_G$ hence the graph can be identified with its edge set G. For instance $\Upsilon = \{\ s \xrightarrow{n} s-n \mid s \in \mathbb{R} \wedge n \in \mathbb{Z}\ \}$ is a graph of vertex set $\mathbb{R}$ and of label set $\mathbb{Z}$. For any graph G, we denote by $G_{|P} = \{\ (s,a,t) \in G \mid s,t \in P\ \}$ its *vertex-restriction* to $P \subseteq V_G$, and by $G^{|Q} = \{\ (s,a,t) \in G \mid a \in Q\ \}$ its *label-restriction* to $Q \subseteq A$. Let $\longrightarrow_G$ be the unlabeled edge relation *i.e.* $s \longrightarrow_G t$ if $s \xrightarrow{a}_G t$ for some $a \in A$. The image of a vertex s by $\longrightarrow_G$ is the set $\longrightarrow_G(s) = \{\ t \mid s \longrightarrow_G t\ \}$ of *successors* of s. The *accessibility* relation $\longrightarrow_G^*$ is the reflexive and transitive closure under composition of $\longrightarrow_G$. We denote by $G_{\downarrow s}$ the restriction of G to the set $\longrightarrow_G^*(s)$ of vertices accessible from a vertex s. For instance $\Upsilon_{\downarrow 0} = \{\ m \xrightarrow{n} m-n \mid m,n \in \mathbb{Z}\ \}$. A *root* s is a vertex from which any vertex is accessible: $G_{\downarrow s} = G$.

Recall that a *magma* (or groupoid) is a set M equipped with a binary operation $\cdot : M{\times}M \longrightarrow M$ that sends any two elements $p,q \in M$ to the element $p{\cdot}q$. Given a subset $Q \subseteq M$ and an injective mapping $[\![\ ]\!] : Q \longrightarrow A$, we define the following *generalized Cayley graph*:

$$\mathcal{C}[\![M,Q]\!] \;=\; \{\; p \xrightarrow{[\![q]\!]} p\cdot q \mid p \in M \,\wedge\, q \in Q \;\}.$$

It is of vertex set M and of label set $[\![Q]\!] \;=\; \{\; [\![q]\!] \mid q \in Q \;\}$. We denote $\mathcal{C}[\![M,Q]\!]$ by $\mathcal{C}(M,Q)$ when $[\![\;]\!]$ is the identity. For instance $\Upsilon \;=\; \mathcal{C}(\mathbb{R},\mathbb{Z})$ for the magma $(\mathbb{R},-)$.

Among many properties of these graphs, we retain only three basic ones. First and by definition, any generalized Cayley graph is *deterministic*: there are no two edges of the same source and label *i.e.* $(r \xrightarrow{a} s \,\wedge\, r \xrightarrow{a} t) \implies s = t$. Furthermore any generalized Cayley graph G is *source-complete*: for all vertex s and label a, there is an a-edge from s *i.e.* $\forall\, s \in V_G \;\; \forall\, a \in A_G \;\; \exists\, t\; (s \xrightarrow{a} t)$.

Recall that a magma $(M,\cdot)$ is *left-cancellative* if $r\cdot p = r\cdot q \implies p = q$ for any $p,q,r \in M$. Any generalized Cayley graph of a left-cancellative magma is *simple*: there are no two edges with the same source and target: $(s \xrightarrow{a} t \wedge s \xrightarrow{b} t) \implies a = b$. Under the assumption of the axiom of choice, these three properties characterize the generalized Cayley graphs of left-cancellative magmas.

Theorem 1. *In ZFC set theory, a graph is a generalized Cayley graph of a left-cancellative magma if and only if it is simple, deterministic and source-complete.*

We can remove the assumption of the axiom of choice by restricting to finitely labeled graphs.

Cayley graphs of left-cancellative and cancellative monoids

Recall that a magma $(M,\cdot)$ is a *semigroup* if $\cdot$ is associative: $(p\cdot q)\cdot r \;=\; p\cdot(q\cdot r)$ for any $p,q,r \in M$. A *monoid* $(M,\cdot)$ is a semigroup with an *identity* element 1: $1{\cdot}p \;=\; p{\cdot}1 \;=\; p$ for all $p \in M$. The *submonoid generated* by $Q \subseteq M$ is $Q^* \;=\; \{\; q_1{\cdot}\ldots{\cdot}q_n \mid n \geq 0 \,\wedge\, q_1,\ldots,q_n \in Q \;\}$ the least submonoid containing Q. A *monoid Cayley graph* is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ for some monoid M generated by Q which means that 1 is a root of $\mathcal{C}[\![M,Q]\!]$.

Let us strengthen Theorem 1 to get a graph-theoretic characterization of the Cayley graphs of left-cancellative monoids. We need to introduce a structural property to describe their symmetry. Recall that an *isomorphism* from a graph G to a graph H (an *automorphism* of G for $G = H$) is a bijection h from V_G to V_H such that $s \xrightarrow{a}_G t \iff h(s) \xrightarrow{a}_H h(t)$. Two vertices s,t of a graph G are *accessible-isomorphic* and we write $s \downarrow_G t$ if $t = h(s)$ for some isomorphism h from $G_{\downarrow s}$ to $G_{\downarrow t}$. A graph G is *arc-symmetric* if all its vertices are accessible-isomorphic: $s \downarrow_G t$ for every $s,t \in V_G$. For instance $\Upsilon^{|\{-1\}}_{|\mathbb{N}} \;=\; \{\; n \xrightarrow{-1} n+1 \mid n \in \mathbb{N} \;\}$ is arc-symmetric but $\Upsilon^{|\{1\}}_{|\mathbb{N}} \;=\; \{\; n \xrightarrow{1} n{-}1 \mid n \in \mathbb{N} \;\}$ is not arc-symmetric. Any arc-symmetric graph is source-complete. By adding in Theorem 1 the arc-symmetry and the existence of a root, we get a graph-theoretic characterization of the Cayley graphs of left-cancellative monoids.

Theorem 2. *A graph is a Cayley graph of a left-cancellative monoid if and only if it is simple, deterministic, rooted and arc-symmetric.*

We can adapt Theorem 2 to characterize the Cayley graphs of cancellative monoids. Recall that a magma M is *cancellative* if it is left-cancellative, and *right-cancellative*: $p\cdot r = q\cdot r \implies p = q$ for all $p,q,r \in M$. Any generalized Cayley graph of a right-cancellative magma is *co-deterministic* meaning that the *inverse* $G^{-1} \;=\; \{\; (t,a,s) \mid (s,a,t) \in G \;\}$ of G is deterministic: there are no two edges with the same target and label *i.e.* $(s \xrightarrow{a} r \wedge t \xrightarrow{a} r) \implies s = t$. By adding in Theorem 2 the co-determinism, we get a characterization of the Cayley graphs of cancellative monoids.

Theorem 3. *A graph is a Cayley graph of a cancellative monoid if and only if it is simple, deterministic, co-deterministic, rooted and arc-symmetric.*

Cayley graphs of groups

Recall that a *group* $(M, \cdot)$ is a monoid whose each element $p \in M$ has an inverse p^{-1}: $p{\cdot}p^{-1} = 1 = p^{-1}{\cdot}p$. Any Cayley graph $\mathcal{C}[\![M, Q]\!]$ of a group $M = Q^*$ is *strongly connected*: any vertex is a root. We get a graph-theoretic characterization of these monoid Cayley graphs of groups just by strengthening in Theorem 2 the existence of a root by the strong connectivity.

Theorem 4. *A graph is a monoid Cayley graph of a group if and only if it simple, deterministic, strongly connected and arc-symmetric.*

We can now consider a *group Cayley graph* as a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ such that M is a group equal to the *subgroup generated* by Q which is the least subgroup $(Q \cup Q^{-1})^*$ containing Q where $Q^{-1} = \{\ q^{-1} \mid q \in Q\ \}$ is the set of inverses of the elements in Q. Any monoid Cayley graph of a group M is a (group) Cayley graph of M. Note that the unrooted graph $\Upsilon^{|\{-1\}}_{|\mathbb{Z}} = \{\ n \xrightarrow{-1} n+1 \mid n \in \mathbb{Z}\ \}$ is equal to $\mathcal{C}[\![\mathbb{Z}, \{1\}]\!]$ for the group $(\mathbb{Z}, +)$ with $[\![1]\!] = -1$. To characterize the Cayley graphs of groups, we need to extend the arc-symmetry by no longer restricting by accessibility. Two vertices s, t of a graph G are *isomorphic* and we write $s \simeq_G t$ if $t = h(s)$ for some automorphism h of G. A graph G is *symmetric* (or vertex-transitive) if all its vertices are isomorphic: $s \simeq_G t$ for every $s, t \in V_G$. Any symmetric graph is arc-symmetric, and $\Upsilon^{|\{-1\}}_{|\mathbb{N}}$ is arc-symmetric but not symmetric. We can present a graph-theoretic characterization of the Cayley graphs (of groups).

Theorem 5. *A graph is a Cayley graph of a group if and only if it is simple, deterministic, co-deterministic, connected and symmetric.*

By removing the connectivity, we get all the generalized Cayley graphs of groups.

Theorem 6. *In ZFC set theory, a graph is a generalized Cayley graph of a group if and only if it is simple, deterministic, co-deterministic, symmetric.*

Generalized Cayley graphs of left-quasigroups

A magma $(M, \cdot)$ is a *left-quasigroup* if for each $p, q \in \mathsf{M}$, there is a unique $r \in M$ such that $p{\cdot}r = q$. This property ensures that each element of M occurs exactly once in each row of the Cayley table. Any simple, deterministic and source-complete graph G is an *out-regular graph*: all its vertices have the same out-degree *i.e.* $|{\longrightarrow_G}(s)| = |{\longrightarrow_G}(t)|$ for any $s, t \in V_G$. This remains true with respect to non-successor vertices for any generalized Cayley graph G of a left-quasigroup: it is *co-out-regular* in the sense that $|V_G - {\longrightarrow_G}(s)| = |V_G - {\longrightarrow_G}(t)|$ for any $s, t \in V_G$. It suffices to add this condition to Theorem 1 to characterize the generalized Cayley graphs of left-quasigroups.

Theorem 7. *In ZFC set theory, a graph is a generalized Cayley graph of a left-quasigroup if and only if it is simple, deterministic, source-complete and co-out regular.*

For the graphs having only a finite number of labels, we can remove the assumption of the axiom of choice, and also the co-out-regularity which then corresponds to the characterization of Theorem 1.

Theorem 8. *A finitely labeled graph is a generalized Cayley graph of a left-quasigroup if and only if it is simple, deterministic, source-complete if and only if it is a generalized Cayley graph of a left-cancellative magma.*

Generalized Cayley graphs of quasigroups

A magma $(M, \cdot)$ is a *quasigroup* if $\cdot$ obeys the *Latin square* property: for each $p, q \in \mathsf{M}$, there is a unique $r \in \mathsf{M}$ such that $p{\cdot}r = q$ and there is a unique $s \in \mathsf{M}$ such that $s{\cdot}p = q$. This property ensures that each element of M occurs exactly once in each row and exactly once in each column of the Cayley table. Any generalized Cayley graph G of a quasigroup is simple, deterministic and source-complete, co-deterministic and *target-complete* meaning that G^{-1} is source-complete: for all vertex t and label a, there is an a-edge of target t *i.e.* $\forall\, t \in V_G\ \ \forall\, a \in A_G\ \ \exists\, s\ (s \xrightarrow{a}_G t)$. With these five properties, G is a *regular graph*: $|{\longrightarrow_G}(s)| = |{\longrightarrow_{G^{-1}}}(t)|$ for any $s, t \in V_G$. We also get that G is *co-regular*: $|V_G - {\longrightarrow_G}(s)| = |V_G - {\longrightarrow_{G^{-1}}}(t)|$ for any $s, t \in V_G$. With the axiom of choice, these properties are sufficient to characterize the generalized Cayley graphs of quasigroups.

Theorem 9. *In ZFC set theory, a graph is a generalized Cayley graph of a quasigroup if and only if it is simple, deterministic, co-deterministic, source-complete, target-complete and co-regular.*

We can remove the co-regularity for the finitely labeled graphs.

Decidability results

We show the effectiveness of the previous characterizations for the family of suffix-recognizable graphs of finite degree which includes the finite graphs and the transition graphs of pushdown automata [6]. A *suffix graph* over an alphabet N is of the form $\bigcup_{i=1}^{n} W_i(u_i \xrightarrow{a_i} v_i)$ where $n \geq 0$, $u_1, v_1, \ldots, u_n, v_n \in N^*$ and $W_1, \ldots, W_n$ are regular languages over N. Such a graph has a decidable isomorphism problem and a decidable monadic theory.

Theorem 10. *We can decide whether a suffix graph G is a Cayley graph of a left-cancellative monoid, of a cancellative monoid, of a group, and whether G is a generalized Cayley graph of a left-quasigroup, of a quasigroup, of a group.*
In the affirmative, $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ where $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$ and with a computable suitable binary operation on V_G and vertex r.

We can consider its generalization to all the suffix-recognizable graphs which form the first level of a stack hierarchy for which any graph has a decidable monadic second-order theory.

This is only a first approach in the structural description and the effectiveness of Cayley graphs of algebraic structures. A full version with proofs and examples is available in arxiv.

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