On Cayley graphs of algebraic structures

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Abstract

We present simple graph-theoretic characterizations of Cayley graphs for left-cancellative monoids, groups, left-quasigroups and quasigroups. We show that these characterizations are effective for the end-regular graphs of finite degree.

1 Introduction

To describe the structure of a group, Cayley introduced in 1878 [7] the concept of graph for any group $(G,\cdot)$ according to any generating subset S. This is simply the set of labeled oriented edges $g \xrightarrow{s} g{\cdot}s$ for every g of G and s of S. Such a graph, called Cayley graph, is directed and labeled in S (or an encoding of S by symbols called letters or colors). The study of groups by their Cayley graphs is a main topic of algebraic graph theory [3, 8, 2]. A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [15] : an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. However, this algebraic characterization is not well suited for deciding whether a possibly infinite graph is a Cayley graph. It is pertinent to look for characterizations by graph-theoretic conditions. This approach was clearly stated by Hamkins in 2010: Which graphs are Cayley graphs? [10]. In this paper, we present simple graph-theoretic characterizations of Cayley graphs for firstly left-cancellative and cancellative monoids, and then for groups. These characterizations are then extended to any subset S of left-cancellative magmas, left-quasigroups, quasigroups, and groups. Finally, we show that these characterizations are effective for the end-regular graphs of finite degree [13] which are the graphs finitely decomposable by distance from a(ny) vertex or equivalently are isomorphic to the suffix transition graphs of labeled word rewriting systems.

Let us present the main structural characterizations starting with the Cayley graphs of left-cancellative monoids. Among many properties of these graphs, we retain only three basic ones. First and by definition, any Cayley graph is deterministic: there are no two arcs of the same source and label. Furthermore, the left-cancellative condition implies that any Cayley graph is simple: there are no two arcs of the same source and goal. Finally, any Cayley graph is rooted: there is a path from the identity element to any vertex. To these three necessary basic conditions is added a structural property, called forward vertex-transitive: all the vertices are accessible-isomorphic *i.e.* the induced subgraphs by vertex accessibility are isomorphic. These four properties characterize the Cayley graphs of left-cancellative monoids. To describe exactly the Cayley graphs of cancellative monoids, we just have to add the co-determinism: there are no two arcs of the same target and label. This characterization is strengthened for the Cayley graphs of groups using the same properties but expressed in both arc directions: these are the graphs that are connected, deterministic, co-deterministic, and vertex-transitive: all the vertices are isomorphic.

We also consider the Cayley graph of a magma G according to any subset S and that we called generalized. The characterizations obtained require the assumption of the axiom of choice. First, a graph is a generalized Cayley graph of a left-cancellative magma if and only if it is deterministic, simple, source-complete: for any label of the graph and from any vertex, there is at least one edge. This equivalence does not require the axiom of choice for finitely labeled graphs, and in this case, these graphs are also the generalized Cayley graphs of left-quasigroups. Moreover, a finitely labeled graph is a generalized Cayley graph of a quasigroup if and only if it is also co-deterministic and target-complete: for any label of the graph and to any vertex, there is at least one edge. We also characterize all the generalized Cayley graphs of left-quasigroups, and of quasigroups. Finally, a graph is a generalized Cayley graph of a group if anf only if it is simple, vertex-transitive, deterministic and co-deterministic.

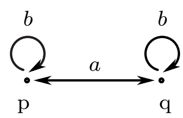
2 Directed labeled graphs

We consider directed labeled graphs without isolated vertex. We recall some basic concepts such as determinism, completeness and vertex-transitivity. We introduce the notions of accessible-isomorphic vertices and forward vertex-transitive graph.

Let A be an arbitrary (finite or infinite) set. A directed *A-graph* (V,G) is defined by a set V of *vertices* and a subset $G \subseteq V{\times}A{\times}V$ of *edges*. Any edge $(s,a,t) \in G$ is from the *source* s to the *target* t with *label* a, and is also written by the *transition* $s \xrightarrow{a}_G t$ or directly $s \xrightarrow{a} t$ if G is clear from the context. The sources and targets of edges form the set V_G of *non-isolated vertices* of G and we denote by A_G the set of edge labels:

$$V_G \;=\; \{\, s \mid \exists\, a,t \;\; (s \xrightarrow{a} t \,\vee\, t \xrightarrow{a} s) \,\} \quad \text{and} \quad A_G \;=\; \{\, a \mid \exists\, s,t \;\; (s \xrightarrow{a} t) \,\}.$$

Thus $V - V_G$ is the set of *isolated vertices*. From now on, we assume that any graph (V,G) is without isolated vertex (*i.e.* $V = V_G$), hence the graph can be identified with its edge set G. We also exclude the empty graph $\emptyset$: every graph is a non-empty set of labeled edges. For instance $\Upsilon \;=\; \{\, s \xrightarrow{n} s{+}n \mid s \in \mathbb{R} \wedge n \in \mathbb{Z} \,\}$ is a graph of vertex set $\mathbb{R}$ and of label set $\mathbb{Z}$. As any graph G is a set, there are no two edges with the same source, target and label. We say that a graph is *simple* if there are no two edges with the same source and target: $(s \xrightarrow{a} t \wedge s \xrightarrow{b} t) \implies a = b$. We say that G is *finitely labeled* if A_G is finite. We denote by $G^{-1} \;=\; \{\, (t,a,s) \mid (s,a,t) \in G \,\}$ the *inverse* of G. A graph is *deterministic* if there are no two edges with the same source and label: $(r \xrightarrow{a} s \wedge r \xrightarrow{a} t) \implies s = t$. A graph is *co-deterministic* if its inverse is deterministic: there are no two edges with the same target and label: $(s \xrightarrow{a} r \wedge t \xrightarrow{a} r) \implies s = t$. For instance, the graph Υ is simple, not finitely labeled, deterministic and co-deterministic. A graph G is *complete* if there is an edge between any couple of vertices: $\forall\, s,t \in V_G \;\; \exists\, a \in A_G \; (s \xrightarrow{a}_G t)$. A graph G is *source-complete* if for all vertex s and label a, there is an a-edge from s: $\forall\, s \in V_G \;\; \forall\, a \in A_G \;\; \exists\, t \; (s \xrightarrow{a}_G t)$. A graph is *target-complete* if its inverse is source-complete: $\forall\, t \in V_G \;\; \forall\, a \in A_G \;\; \exists\, s \; (s \xrightarrow{a}_G t)$. For instance, Υ is source-complete, target-complete but not complete. Another example is given by the graph $\text{Even} \;=\; \{(p,a,q)\;,\;(p,b,p)\;,\;(q,a,p)\;,\;(q,b,q)\}$ represented as follows:



It is simple, deterministic, co-deterministic, complete, source-complete and target-complete. The *vertex-restriction* $G_{|P}$ of G to a set P is the induced subgraph of G by $P \cap V_G$:

$$G_{|P} \;=\; \{\, (s,a,t) \in G \mid s,t \in P \,\}.$$

The *label-restriction* $G^{|P}$ of G to a set P is the subset of all its edges labeled in P:

$$G^{|P} \;=\; \{\, (s,a,t) \in G \mid a \in P \,\}.$$

Let $\longrightarrow_G$ be the unlabeled edge relation *i.e.* $s \longrightarrow_G t$ if $s \xrightarrow{a}_G t$ for some $a \in A$. We denote by $\longrightarrow_G(s) = \{\, t \mid s \longrightarrow_G t \,\}$ the set of *successors* of $s \in V_G$. We write $s \not\longrightarrow_G t$ if there is no edge in G from s to t *i.e.* $G \cap \{s\}{\times}A{\times}\{t\} = \emptyset$. The *accessibility* relation $\longrightarrow_G^* = \bigcup_{n\geq 0} \longrightarrow_G^n$ is the reflexive and transitive closure under composition of $\longrightarrow_G$. A graph G is *accessible* from $P \subseteq V_G$ if for any $s \in V_G$, there is $r \in P$ such that $r \longrightarrow_G^* s$. We denote by $G_{\downarrow P}$ the induced subgraph of G to the vertices accessible from P which is the greatest subgraph of G accessible from P. For instance $\Upsilon_{\downarrow\{0\}} = \{\, m \xrightarrow{n} m{+}n \mid m, n \in \mathbb{Z} \,\}$ is a complete subgraph of Υ. A *root* r is a vertex from which G is accessible *i.e.* $G_{\downarrow\{r\}}$ also denoted by $G_{\downarrow r}$ is equal to G. A graph G is *strongly connected* if every vertex is a root: $s \longrightarrow_G^* t$ for all $s, t \in V_G$. A graph G is *co-accessible* from $P \subseteq V_G$ if G^{-1} is accessible from P. We denote by $d_G(s,t) = \min\{\, n \mid s \longrightarrow^n_{G\,\cup\,G^{-1}} t \,\}$ the *distance* between $s, t \in V_G$ with $\min(\emptyset) = \omega$. A graph G is *connected* if $G \cup G^{-1}$ is strongly connected *i.e.* $d_G(s,t) \in \mathbb{N}$ for any $s, t \in V_G$. Recall that a *connected component* of a graph G is a maximal connected subset of G; we denote by $\mathrm{Comp}(G)$ the set of connected components of G. A *representative set* of $\mathrm{Comp}(G)$ is a vertex subset $P \subseteq V_G$ having exactly one vertex in each connected component: $|P \cap V_C| = 1$ for any $C \in \mathrm{Comp}(G)$; it induces the *canonical mapping* $\pi_P : V_G \longrightarrow P$ associating with each vertex s the vertex of P in the same connected component: $s \longrightarrow^*_{G\,\cup\,G^{-1}} \pi_P(s)$ for any $s \in V_G$. For instance, $[0,1[$ is a representative set of $\mathrm{Comp}(\Upsilon)$ and its canonical mapping is defined by $\pi_{[0,1[}(x) = x - \lfloor x \rfloor$ for any $x \in \mathbb{R}$.
A *path* $(s_0, a_1, s_1, \ldots, a_n, s_n)$ of *length* $n \geq 0$ in a graph G is a sequence $s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_n} s_n$ of n consecutive edges, and we write $s_0 \xrightarrow{a_1\ldots a_n} s_n$ for indicating the source s_0, the target s_n and the label word $a_1 \ldots a_n \in A_G^*$ of the path where A_G^* is the set of words over A_G (the free monoid generated by A_G) and ε is the empty word (the identity element).
Recall that a *morphism* from a graph G into a graph H is a mapping h from V_G into V_H such that $s \xrightarrow{a}_G t \implies h(s) \xrightarrow{a}_H h(t)$. If, in addition h is bijective and h^{-1} is a morphism, h is called an *isomorphism* from G to H; we write $G \equiv_h H$ or directly $G \equiv H$ if we do not specify an isomorphism, and we say that G and H are *isomorphic*. An *automorphism* of G is an isomorphism from G to G. Two vertices s, t of a graph G are *isomorphic* and we write $s \simeq_G t$ if $t = h(s)$ for some automorphism h of G.
A graph G is *vertex-transitive* if all its vertices are isomorphic: $s \simeq_G t$ for every $s, t \in V_G$. For instance, the previous graphs Υ and Even are vertex-transitive.
Two vertices s, t of a graph G are *accessible-isomorphic* and we write $s \downarrow_G t$ if $t = h(s)$ for some isomorphism h from $G_{\downarrow s}$ to $G_{\downarrow t}$. A graph G is *forward vertex-transitive* if all its vertices are accessible-isomorphic: $s \downarrow_G t$ for every $s, t \in V_G$.

▶ **Fact 1.** Any vertex-transitive graph is forward vertex-transitive which is source-complete.

For instance $\Upsilon^{|\{1\}}_{|\mathbb{N}} = \{\, n \xrightarrow{1} n+1 \mid n \in \mathbb{N} \,\}$ is forward vertex-transitive but not vertex-transitive. On the other hand $\Upsilon^{|\{-1\}}_{|\mathbb{N}} = \{\, n \xrightarrow{-1} n-1 \mid n \in \mathbb{N} \,\}$ is not forward vertex-transitive: two distinct vertices are not accessible-isomorphic.

3 Cayley graphs of left-cancellative and cancellative monoids

We present graph-theoretic characterizations for the Cayley graphs of left-cancellative monoids (Theorem 7), of cancellative monoids (Theorem 8), of cancellative semigroups (Theorem 10).

A *magma* (or groupoid) is a set M equipped with a binary operation $\cdot : M{\times}M \longrightarrow M$ that sends any two elements $p, q \in M$ to the element $p \cdot q$.

Given a subset $Q \subseteq M$ and an injective mapping $[\![\ ]\!] : Q \longrightarrow A$, we define the graph

$$\mathcal{C}[\![M,Q]\!] \;=\; \{\ p \xrightarrow{[\![q]\!]} p \cdot q \mid p \in M \ \wedge\ q \in Q\ \}$$

which is called a *generalized Cayley graph* of M. It is of vertex set M and of label set $[\![Q]\!] = \{\ [\![q]\!] \mid q \in Q\ \}$. We denote $\mathcal{C}[\![M,Q]\!]$ by $\mathcal{C}(M,Q)$ when $[\![\ ]\!]$ is the identity. For instance $\Upsilon = \mathcal{C}(\mathbb{R},\mathbb{Z})$ for the magma $(\mathbb{R},+)$. We also write $\mathcal{C}[\![M]\!]$ instead of $\mathcal{C}[\![M,M]\!]$ and $\mathcal{C}(M) = \mathcal{C}(M,M) = \{\ p \xrightarrow{q} p \cdot q \mid p,q \in M\ \}$.

▶ **Fact 2.** Any generalized Cayley graph is deterministic and source-complete.

For instance taking the magma $(\mathbb{Z},-)$ and $[\![-1]\!] = a$, $\mathcal{C}[\![\mathbb{Z},\{-1\}]\!] \;=\; \{\ n \xrightarrow{a} n{+}1 \mid n \in \mathbb{Z}\ \}$.
By adding $[\![1]\!] = b$, $\mathcal{C}[\![\mathbb{Z},\{1,-1\}]\!] \;=\; \{\ n \xrightarrow{a} n+1 \mid n \in \mathbb{Z}\ \} \cup \{\ n \xrightarrow{b} n-1 \mid n \in \mathbb{Z}\ \}$.
We say that a magma $(M,\cdot)$ is *left-cancellative* if $r{\cdot}p = r{\cdot}q \implies p = q$ for any $p,q,r \in M$.
Similarly $(M,\cdot)$ is *right-cancellative* if $p \cdot r = q \cdot r \implies p = q$ for any $p,q,r \in M$.
A magma is *cancellative* if it is both left-cancellative and right-cancellative.

▶ **Fact 3.** Any generalized Cayley graph of a left-cancellative magma is simple.
Any generalized Cayley graph of a right-cancellative magma is co-deterministic.

Recall also that $(M,\cdot)$ is a *semigroup* if $\cdot$ is associative: $(p \cdot q) \cdot r \;=\; p \cdot (q \cdot r)$ for any $p,q,r \in M$. A *monoid* $(M,\cdot)$ is a semigroup with an *identity* element 1: $1{\cdot}p \;=\; p{\cdot}1 \;=\; p$ for all $p \in M$. The *submonoid generated* by $Q \subseteq M$ is the least submonoid $Q^* \;=\; \{\ q_1{\cdot} \ldots {\cdot}q_n \mid n \geq 0 \ \wedge\ q_1,\ldots,q_n \in Q\ \}$ containing Q.
When a monoid is left-cancellative, its generalized Cayley graphs are forward vertex-transitive.

▶ **Proposition 4.** *Any generalized Cayley graph of a left-cancellative monoid is forward vertex-transitive.*

Proof.
Let $G \;=\; \mathcal{C}[\![M,Q]\!]$ for some left-cancellative monoid $(M,\cdot)$ and some $Q \subseteq M$.
Let $r \in M$. We have to check that $1 \downarrow_G r$.
By induction on $n \geq 0$ and for any $q_1,\ldots,q_n \in Q$ and $s \in M$, we have

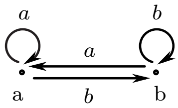
$$r \xrightarrow{[\![q_1]\!]\ldots[\![q_n]\!]}_G s \iff s \;=\; (\ldots(r{\cdot}q_1)\ldots){\cdot}q_n\,.$$

As $\cdot$ is associative, we get $V_{G_{\downarrow r}} \;=\; \{\ s \mid r \longrightarrow^*_G s\ \} \;=\; r{\cdot}Q^*$. In particular $V_{G_{\downarrow 1}} \;=\; Q^*$.
We consider the mapping $f_r : M \longrightarrow M$ defined by $f_r(p) = r{\cdot}p$ for any $p \in M$.
As $\cdot$ is left-cancellative, f_r is injective.
Furthermore f_r is an isomorphism on its image: for any $p,q,p' \in M$,

$$p \xrightarrow{[\![q]\!]}_G p' \iff f_r(p) \xrightarrow{[\![q]\!]}_G f_r(p').$$

The associativity of $\cdot$ gives the necessary condition.
The associativity and the left-cancellative property of $\cdot$ gives the sufficient condition.
Thus f_r restricted to Q^* is an isomorphism from $G_{\downarrow 1}$ to $G_{\downarrow r}$ hence $1 \downarrow_G r$. ◀

We can not generalize Proposition 4 to the left-cancellative semigroups. For instance the semigroup $M = \{a,b\}$ with $x{\cdot}y = y$ for any $x,y \in M$ is left-cancellative but the graph $\mathcal{C}(M)$ represented below is not forward vertex-transitive.



A *monoid Cayley graph* is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ of a monoid M generated by Q which means that the identity element 1 is a root of $\mathcal{C}[\![M,Q]\!]$.

▶ **Fact 5.** A monoid M is generated by $Q \iff 1$ is a root of $\mathcal{C}[\![M,Q]\!]$.

Under additional simple conditions, let us establish the converse of Proposition 4.
For any graph G and any vertex r, we introduce the *path-relation* $\mathrm{Path}_G(r)$ as the ternary relation on V_G defined by

$$(s,t,x) \in \mathrm{Path}_G(r) \text{ if there exists } u \in A_G^* \text{ such that } r \xrightarrow{u}_G t \text{ and } s \xrightarrow{u}_G x.$$

If for any $s,t \in V_G$ there exists a unique x such that $(s,t,x) \in \mathrm{Path}_G(r)$, we denote by $*_r : V_G{\times}V_G \longrightarrow V_G$ the binary *path-operation* on V_G defined by $(s,t,s *_r t) \in \mathrm{Path}_G(r)$ for any $s,t \in V_G$. This is illustrated as follows:

and we also write ${}_G*_r$ when we need to specify G. Let us give conditions so that this path-operation exists and is associative and left-cancellative.

▶ **Proposition 6.** *Let r be a root of a deterministic and forward vertex-transitive graph G. Then $(V_G, *_r)$ is a left-cancellative monoid of identity r and generated by $\longrightarrow_G(r)$.*
If G is co-deterministic then $*_r$ is cancellative.
If G is simple then $G = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$.

Proof.
i) Let $s,t \in V_G$. Let us check that there is a unique x such that $(s,t,x) \in \mathrm{Path}_G(r)$.
As r is a root, there exists u such that $r \xrightarrow{u}_G t$.
As G is source-complete, there exists x such that $s \xrightarrow{u}_G x$. Hence $(s,t,x) \in \mathrm{Path}_G(r)$.
Let $(s,t,y) \in \mathrm{Path}_G(r)$. There exists $v \in A_G^*$ such that $r \xrightarrow{v}_G t$ and $s \xrightarrow{v}_G y$.
As G is forward vertex-transitive, we have $r \downarrow_G s$.
As G is deterministic, we get $s \xrightarrow{v}_G x$ hence $x = y$.
Thus $*_r$ exists and is denoted by $\cdot$ in the rest of this proof.
Let us show that $(V_G, \cdot)$ is a left-cancellative monoid.

ii) Let us show that $\cdot$ is associative.
Let $x,y,z \in V_G$. We have to check that $(x{\cdot}y){\cdot}z = x{\cdot}(y{\cdot}z)$.
As r is a root, there exists $v,w \in A_G^*$ such that $r \xrightarrow{v} y$ and $r \xrightarrow{w} z$.
By (i), $x \xrightarrow{v} x{\cdot}y \xrightarrow{w} (x{\cdot}y){\cdot}z$ and $y \xrightarrow{w} y{\cdot}z$
So $r \xrightarrow{vw} y{\cdot}z$ hence $x \xrightarrow{vw} x{\cdot}(y{\cdot}z)$.
As G is deterministic, we get $(x{\cdot}y){\cdot}z = x{\cdot}(y{\cdot}z)$.

iii) Let us check that r is an identity element.
Let $s \in V_G$. As $r \xrightarrow{\varepsilon} r$, we get $s \xrightarrow{\varepsilon} s{\cdot}r$ *i.e.* $s{\cdot}r = s$.
For $r \xrightarrow{u} s$, we have $r \xrightarrow{u} r{\cdot}s$. As G is deterministic, we get $r{\cdot}s = s$.

iv) Let us check that $\cdot$ is left-cancellative. Let $s,t,t' \in V_G$ such that $s{\cdot}t = s{\cdot}t'$.
There exists $u,v \in A_G^*$ such that $r \xrightarrow{u} t$ and $r \xrightarrow{v} t'$.
So $s \xrightarrow{u} s{\cdot}t$ and $s \xrightarrow{v} s{\cdot}t'$. As $s{\cdot}t = s{\cdot}t'$ and $r \downarrow_G s$, we get $r \xrightarrow{v} t$.
As G is deterministic, we have $t = t'$.

v) Let us check that $Q = \longrightarrow_G(r)$ is a generating subset of V_G. Let $s \in V_G$.
There exists $n \geq 0$, $a_1,\ldots,a_n \in A_G$ and $s_0,\ldots,s_n$ such that $r = s_0 \xrightarrow{a_1} s_1 \ldots s_{n-1} \xrightarrow{a_n} s_n = s$.
By Fact 1, there exists $r_1,\ldots,r_n$ such that $r \xrightarrow{a_1} r_1, \ldots, r \xrightarrow{a_n} r_n$.
For every $1 \leq i \leq n$, $s_i = s_{i-1}{\cdot}r_i$ hence $s = r{\cdot}r_1 \cdot \ldots {\cdot}r_n = r_1 \cdot \ldots {\cdot}r_n \in Q^*$.

vi) Assume that G is co-deterministic. Let us check that $\cdot$ is right-cancellative.
Let $s,s',t \in V_G$ such that $s{\cdot}t = s'{\cdot}t$.

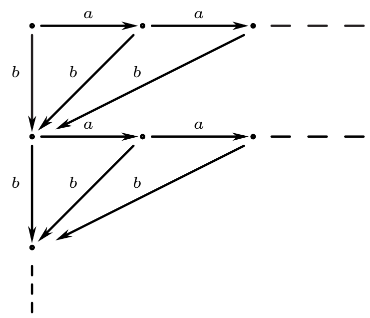
There exists $u \in A_G^*$ such that $r \xrightarrow{u} t$. So $s \xrightarrow{u} s{\cdot}t$ and $s' \xrightarrow{u} s'{\cdot}t = s{\cdot}t$.
As G is co-deterministic, we get $s = s'$.

vii) Assume that G is simple. Let $Q = \{ s \mid r \longrightarrow_G s \}$.
As G is simple and deterministic, we define the following injection $[\![\ ]\!]$ from Q into A_G by

$$[\![s]\!] = a \quad \text{for } r \xrightarrow{a}_G s.$$

Let $K = \mathcal{C}[\![V_G, Q]\!]$. Let us show that $G = K$.
$\subseteq$: Let $s \xrightarrow{a}_G t$. As $r \downarrow_G s$, there exists r' such that $r \xrightarrow{a}_G r'$.
So $s \xrightarrow{a}_G s{\cdot}r'$. As G is deterministic, $s{\cdot}r' = t$.
Furthermore $r' \in Q$ and $[\![r']\!] = a$. So $s \xrightarrow{a}_K s{\cdot}r' = t$.
$\supseteq$: Let $s \xrightarrow{a}_K t$. So $a = [\![r']\!]$ for some $r' \in Q$.
Thus $t = s{\cdot}r'$ and $r \xrightarrow{a}_G r'$. So $s \xrightarrow{a}_G s{\cdot}r' = t$. ◀

For instance let us consider a graph G of the following representation:



It is a *skeleton* of the graph of ω^2 where a is the successor and b goes to the next limit ordinal: $(V_G, \longrightarrow_G^*)$ is isomorphic to $(\omega^2, \leq)$. By Proposition 6, it is a Cayley graph of a left-cancellative monoid. Precisely to each word $u \in b^*a^*$, we associate the unique vertex $<u> \in V_G$ accessible from the root by the path labeled by u. Thus

$$G = \{ <b^m a^n> \xrightarrow{a} <b^m a^{n+1}> \mid m,n \geq 0 \} \cup \{ <b^m a^n> \xrightarrow{b} <b^{m+1}> \mid m,n \geq 0 \}.$$

By Proposition 6, $(V_G, *_{<\varepsilon>})$ is a left-cancellative monoid where for any $m,n,p,q \geq 0$,

$$<b^m a^n> *_{<\varepsilon>} <b^p a^q> = \begin{cases} <b^m a^{n+q}> & \text{if } p = 0 \\ <b^{m+p} a^q> & \text{if } p \neq 0 \end{cases}$$

and we have $G = \mathcal{C}[\![V_G, \{<a>, <b>\}]\!]$ with $[\![<a>]\!] = a$ and $[\![<b>]\!] = b$.

Propositions 4 and 6 give a graph-theoretic characterization of the Cayley graphs of left-cancellative monoids.

▶ **Theorem 7.** *A graph is a Cayley graph of a left-cancellative monoid if and only if it is rooted, simple, deterministic and forward vertex-transitive.*

Proof.
We obtain the necessary condition by Proposition 4 with Facts 2, 3, 5.
The sufficient condition is given by Proposition 6. ◀

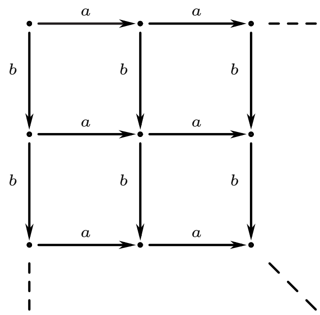
We can restrict Theorem 7 to cancellative monoids.

▶ **Theorem 8.** *A graph is a Cayley graph of a cancellative monoid if and only if it is rooted, simple, deterministic, co-deterministic, forward vertex-transitive.*

Proof.
$\Longrightarrow$: By Theorem 7 and Fact 3.

$\Longleftarrow$: By Proposition 6. ◀

The previous graph is not co-deterministic hence, by Theorem 8 or Fact 3, is not a Cayley graph of a cancellative monoid. On the other hand and according to Proposition 6, a *quater-grid* G of the following representation:



is a Cayley graph of a cancellative monoid. Precisely and as for the previous graph, we associate to each word $u \in b^* a^*$ the unique vertex $<u>$ accessible from the root by the path labeled by u. By Proposition 6, $(V_G, *_{<\varepsilon>})$ is a cancellative monoid where

$$<b^m a^n> *_{<\varepsilon>} <b^p a^q> \; = \; <b^{m+p} a^{n+q}> \quad \text{for any } m, n, p, q \geq 0$$

and we have $G = \mathcal{C}[\![V_G, \{<a>, <b>\}]\!]$ with $[\![<a>]\!] = a$ and $[\![<b>]\!] = b$.

Recall that a Cayley graph of a semigroup M is a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ such that $M = Q^+$ whose $Q^+ = \{\ q_1 \cdot \ldots \cdot q_n \mid n > 0 \wedge q_1, \ldots, q_n \in Q\ \}$ is the *subsemigroup generated* by Q. Theorem 8 can be easily extended into a characterization of the Cayley graphs of cancellative semigroups. Indeed, a semigroup without an identity is turned into a monoid by just adding an identity. Precisely a *monoid-completion* $\overline{M}$ of a semigroup M is defined by $\overline{M} = M$ if M has an identity element, otherwise $\overline{M} = M \cup \{1\}$ whose 1 is an identity element of $\overline{M}$: $p \cdot 1 = 1 \cdot p = p$ for any $p \in \overline{M}$. This natural completion does not preserve the left-cancellative property but it preserves the cancellative property.

▶ **Lemma 9.** *Any monoid-completion of a cancellative semigroup is a cancellative monoid.*

Proof.
Let $\overline{M} = M \cup \{1\}$ be a monoid-completion of a cancellative semigroup M without an identity element.

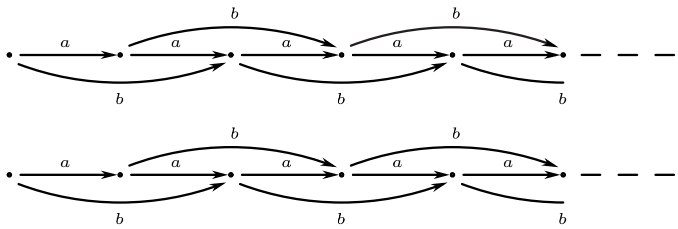
i) Suppose there are $m, e \in M$ such that $m \cdot e = m$.
In this case, let us check that e is an identity element.
We have $m \cdot (e \cdot e) = (m \cdot e) \cdot e = m \cdot e$. As $\cdot$ is left-cancellative, we get $e \cdot e = e$.
Let $n \in M$. So $(n \cdot e) \cdot e = n \cdot (e \cdot e) = n \cdot e$. As $\cdot$ is right-cancellative, we get $n \cdot e = n$.
Finally $e \cdot (e \cdot n) = (e \cdot e) \cdot n = e \cdot n$. As $\cdot$ is left-cancellative, we get $e \cdot n = n$.

ii) By hypothesis M has no identity element. By (i), there are no $m, e \in M$ such that $m \cdot e = m$. Let us show that $\overline{M}$ is left-cancellative.
Let $m \cdot p = m \cdot q$ for some $m, p, q \in \overline{M}$. Let us check that $p = q$.
As M is left-cancellative, we only have to consider the case where $1 \in \{m, p, q\}$.
If $m = 1$ then $p = 1 \cdot p = 1 \cdot q = q$.
Otherwise $m \in M$ and $1 \in \{p, q\}$. By (i), we get $p = q = 1$.

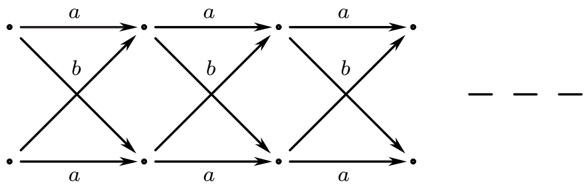
iii) Similarly there are no $m, e \in M$ such that $e \cdot m = m$ hence $\overline{M}$ remains also right-cancellative. ◀

Let us translate the monoid-completion of cancellative semigroups into their Cayley graphs.

A *root-completion* of a graph G is a graph $\overline{G}$ defined by $\overline{G} = G$ if G is rooted, otherwise $G \subset \overline{G} \subseteq G \cup \{r\}{\times}A_G{\times}V_G$ and r is the root of $\overline{G}$; we say that G is *rootable* into $\overline{G}$. For instance the following non connected graph:



is forward vertex-transitive but is not rootable into an forward vertex-transitive graph. On the other hand, a graph consisting of two (isomorphic) deterministic and source-complete trees over $\{a,b\}$ is rootable into a deterministic source-complete tree over $\{a,b\}$. Finally the following graph:



is also rootable into a simple, deterministic, co-deterministic, forward vertex-transitive graph. We can apply Theorem 8.

▶ **Theorem 10.** *A graph is a Cayley graph of a cancellative semigroup if and only if it is rootable into a simple, deterministic, co-deterministic, forward vertex-transitive graph.*

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some cancellative semigroup M and some generating subset Q of M *i.e.* $Q^+ = M$. We have the following two complementary cases.
Case 1: M has an identity element. By Theorem 8, G is rooted, simple, forward vertex-transitive, deterministic and co-deterministic. As G has a root, it is rootable into itself.
Case 2: M is not a monoid. Let $\overline{M} = M \cup \{1\}$ be a monoid-completion of M.
By Lemma 9, $\overline{M}$ remains cancellative. Furthermore $Q^* = \overline{M}$. Let

$$\overline{G} = \mathcal{C}[\![\overline{M},Q]\!] = G \cup \{\, 1 \xrightarrow{[\![q]\!]} q \mid q \in Q \,\}.$$

By Theorem 8, $\overline{G}$ is rooted, simple, forward vertex-transitive, deterministic and co-deterministic. Moreover G is rootable into $\overline{G}$.

$\Longleftarrow$: Let a graph G rootable into a simple, deterministic, co-deterministic, forward vertex-transitive graph $\overline{G}$. We have the following two complementary cases.
Case 1: G is rooted. By Theorem 8 (or Proposition 6), G is a Cayley graph of a cancellative monoid.
Case 2: G has no root. Let r be the root of $\overline{G}$ and $Q = \longrightarrow_{\overline{G}}(r)$.
So $Q \subseteq V_G$ and $V_G = V_{\overline{G}} - \{r\}$.
By Proposition 6, $\overline{G} = \mathcal{C}[\![V_{\overline{G}},Q]\!]$ for the associative and cancellative path-operation $*_r$ on $V_{\overline{G}}$ of identity element r with $V_{\overline{G}}$ generated by Q.
As r is not the target of an edge of $\overline{G}$ and by definition, $*_r$ remains an internal operation on V_G *i.e.* $p *_r q \neq r$ for any $p,q \in V_G$.
Finally $G = \mathcal{C}[\![V_G\,,Q]\!]$ and $(V_G\,,*_r)$ is a cancellative semigroup. ◀

For instance by Theorem 10, the previous graph is a Cayley graph of a cancellative semi-group. It is isomorphic to

$$\begin{array}{rclcl} G & = & \{\, n \xrightarrow{a} n+1 \mid n > 0 \,\} & \cup & \{\, n \xrightarrow{a} n-1 \mid n < 0 \,\} \\ & \cup & \{\, n \xrightarrow{b} -n-1 \mid n > 0 \,\} & \cup & \{\, n \xrightarrow{b} -n+1 \mid n < 0 \,\}. \end{array}$$

We have $G = \mathcal{C}[\![\mathbb{Z} - \{0\}\,,\, \{-1,1\}]\!]$ with $[\![1]\!] = a$ and $[\![-1]\!] = b$ for the following associative and cancellative path-operation $*$ defined by

$$m * n \;=\; sign(m{\times}n)\,(|m| + |n|) \quad \text{for any} \;\; m,n \in \mathbb{Z} - \{0\}.$$

We can now restrict Theorem 8 to the Cayley graphs of groups.

4 Cayley graphs of groups

We present a graph-theoretic characterization for the Cayley graphs of groups: they are the deterministic, co-deterministic, vertex-transitive, simple and connected graphs (Theorem 17). By removing the connectivity condition and under the assumption of the axiom of choice, we get a characterization for the generalized Cayley graphs of groups (Theorem 20).

Recall that a *group* $(M,\cdot)$ is a monoid whose each element $p \in M$ has an inverse p^{-1} : $p{\cdot}p^{-1} = 1 = p^{-1}{\cdot}p$. So $\mathcal{C}(M)$ is strongly connected hence by Proposition 4 is vertex-transitive.

▶ **Fact 11.** Any generalized Cayley graph of a group is vertex-transitive.

Proof.
Let $(M,\cdot)$ be a group and $[\![\;]\!] : M \longrightarrow A$ be an injective mapping.
By Proposition 4, $\mathcal{C}[\![M]\!]$ is forward vertex-transitive.
As $\mathcal{C}[\![M]\!]$ is strongly connected, $\mathcal{C}[\![M]\!]$ is vertex-transitive.
For any $Q \subseteq M$, $\mathcal{C}[\![M,Q]\!] = \mathcal{C}[\![M]\!]^{[\![Q]\!]}$ remains vertex-transitive. ◀

We start by considering the monoid Cayley graphs of a group M which are the generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ with $Q^* = M$.

▶ **Fact 12.** Any monoid Cayley graph of a group is strongly connected.

Proof.
Let $G = \mathcal{C}[\![M,Q]\!]$ for some group M and some $Q \subseteq M$ with $Q^* = M$.
Let $p \in M$. We have to check that $1 \longrightarrow_G^* p \longrightarrow_G^* 1$.
There exists $n \geq 0$ and $q_1,\ldots,q_n \in Q$ such that $p = q_1{\cdot}\ldots{\cdot}q_n$. So $1 \xrightarrow{[\![q_1]\!]\ldots[\![q_n]\!]}_G p$.
For any $1 \leq i \leq n$, we have $q_i^{-1} = q_{i,1}{\cdot}\ldots{\cdot}q_{i,m_i}$ for some $m_i \geq 0$ and $q_{i,1},\ldots,q_{i,m_i} \in Q$.
Thus $p \xrightarrow{u}_G 1$ for $u = [\![q_{n,1}]\!]\ldots[\![q_{n,m_n}]\!] \ldots [\![q_{1,1}]\!]\ldots[\![q_{1,m_1}]\!]$. ◀

Let us complete Proposition 6 in the case where the graph is vertex-transitive. In this case, the path-operation is also invertible.

▶ **Proposition 13.** *For any root r of a deterministic and vertex-transitive graph G, $(V_G, *_r)$ is a group.*

Proof.
It suffices to complete the proof of Proposition 6 when G is in addition vertex-transitive.
Let $s \in V_G$. Let us show that s has an inverse.
There exists $u \in A_G^*$ such that $r \xrightarrow{u} s$.
As $r \simeq_G s$, s is also a root hence there exists v such that $s \xrightarrow{v} r$.

Let $\overline{s}$ be the vertex such that $r \xrightarrow{v} \overline{s}$. So

$$\overline{s} \xrightarrow{u} \overline{s}{\cdot}s \quad \text{and} \quad s \xrightarrow{v} s{\cdot}\overline{s}.$$

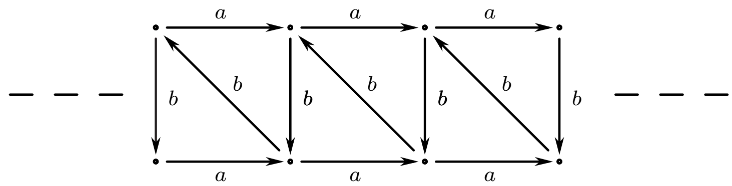
As G is deterministic, we get $s{\cdot}\overline{s} = r$.
As $r \simeq_G s$ and $s \xrightarrow{vu} s$, we get $r \xrightarrow{vu} r$.
As G is deterministic, we get $\overline{s} \xrightarrow{u} r$ hence $\overline{s}{\cdot}s = r$. ◀

We describe the monoid Cayley graphs of groups from the characterization of the Cayley graphs of left-cancellative monoids (Theorem 7) just by replacing the forward vertex-transitivity by the vertex-transitivity.

▶ **Theorem 14.** *A graph is a monoid Cayley graph of a group if and only if it is rooted, simple, deterministic and vertex-transitive.*

Proof.
$\Longrightarrow$: By Theorem 7 and Fact 11.
$\Longleftarrow$: By Propositions 6 and 13. ◀

For instance by Theorem 14, a graph of the following representation:



is a monoid Cayley graph of a group: it is isomorphic to $\mathcal{C}[\![\mathbb{Z}, \{2,-1\}]\!]$ for the group $(\mathbb{Z}, +)$ with $[\![2]\!] = a$ and $[\![-1]\!] = b$.
From Fact 12, we can replace in Theorem 14 the rooted condition by the fact to be strongly connected. By Fact 3, we can also add the co-determinism condition.

▶ **Corollary 15.** *Any rooted, simple, deterministic and vertex-transitive graph is strongly connected and co-deterministic.*

We can now consider a *group Cayley graph* as a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ such that M is a group equal to the *subgroup generated* by Q which is the least subgroup $(Q \cup Q^{-1})^*$ containing Q where $Q^{-1} = \{\ q^{-1} \mid q \in Q\ \}$ is the set of inverses of the elements in Q.
For instance, the *a-line* $\{\ n \xrightarrow{a} n{+}1 \mid n \in \mathbb{Z}\ \}$ is the Cayley graph $\mathcal{C}[\![\mathbb{Z}, \{1\}]\!]$ of the group $(\mathbb{Z}, +)$ with $[\![1]\!] = a$. This unrooted graph is not a monoid Cayley graph.
Let us generalize Theorem 14 to these well-known Cayley graphs. We need to be able to circulate in a graph in the direct and inverse direction of the arrows. Let G be a graph and let $\overline{} : A_G \longrightarrow A - A_G$ be an injective mapping of image $\overline{A_G} = \{\ \overline{a} \mid a \in A_G\ \}$. A *chain* $s \xrightarrow{u}_G t$ is a path labeled by $u \in (A_G \cup \overline{A_G})^*$ where for any $a \in A_G$, we have $s \xrightarrow{\overline{a}}_G t$ for $t \xrightarrow{a}_G s$. Given a vertex r, the path-relation $\mathrm{Path}_G(r)$ is extended into the *chain-relation* $\mathrm{Chain}_G(r)$ as the ternary relation on V_G defined by

$$(s,t,x) \in \mathrm{Chain}_G(r) \text{ if there exists } u \in (A_G \cup \overline{A_G})^* \text{ such that } r \xrightarrow{u}_G t \text{ and } s \xrightarrow{u}_G x.$$

Thus $\mathrm{Path}_G(r) \subseteq \mathrm{Chain}_G(r) = \mathrm{Path}_{\overline{G}}(r)$ for $\overline{G} = G \cup \{\ t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a}_G t\ \}$.
If for any $s, t \in V_G$ there exists a unique x such that $(s,t,x) \in \mathrm{Chain}_G(r)$, we denote by $\overline{*}_r : V_G{\times}V_G \longrightarrow V_G$ the binary *chain-operation* on V_G defined by $(s, t, s \mathbin{\overline{*}_r} t) \in \mathrm{Chain}_G(r)$ for any $s, t \in V_G$; we also write ${}_G\overline{*}_r$ when we need to specify G.
Let us adapt Propositions 6 and 13 to this chain-operation.

▶ **Proposition 16.** *Let r be a vertex of a connected, vertex-transitive, deterministic and co-deterministic graph G.*
Then $(V_G, \overline{*}_r)$ is a group of identity r generated by $\longrightarrow_G(r)$.
If G is simple then $G = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$.

Proof.
i) The graph $\overline{G}$ remains vertex-transitive. As G is deterministic and co-deterministic, $\overline{G}$ is deterministic. As G is connected, $\overline{G}$ is strongly connected. By applying Propositions 6 and 13 to $\overline{G}$, we get that $(V_{\overline{G}}, *_r)$ is a group of identity r and $V_G = V_{\overline{G}} = (\longrightarrow_{\overline{G}}(r))^*$ with $\longrightarrow_{\overline{G}}(r) = \longrightarrow_G(r) \cup \longrightarrow_{G^{-1}}(r)$.
ii) Let us check that $\longrightarrow_{G^{-1}}(r) = (\longrightarrow_G(r))^{-1}$.
$\subseteq$: Let $s \in \longrightarrow_{G^{-1}}(r)$ *i.e.* $r \longrightarrow_{G^{-1}} s$. So $s \xrightarrow{a}_G r$ for some $a \in A$.
As G is source-complete, there exists t such that $r \xrightarrow{a}_G t$. Thus $s \xrightarrow{a}_G s *_r t$.
As G is deterministic, $s *_r t = r$ hence $s = t^{-1} \in (\longrightarrow_G(r))^{-1}$.
$\supseteq$: Let $s \in (\longrightarrow_G(r))^{-1}$ *i.e.* $r \xrightarrow{a}_G s^{-1}$ for some $a \in A$.
So $s \xrightarrow{a}_G s *_r s^{-1} = r$ *i.e.* $s \in \longrightarrow_{G^{-1}}(r)$.
iii) Suppose that in addition G is simple. Note that $\overline{G}$ can be not simple. So we define the graph

$$\widehat{G} = G \cup \{ \, t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a}_G t \not\longrightarrow_G s \, \}.$$

Thus $\widehat{G}$ remains simple, vertex-transitive, deterministic, and is strongly connected.
Let us check that ${}_{\widehat{G}}*_r = {}_G\overline{*}_r$.
As $\widehat{G} \subseteq \overline{G}$, we get ${}_{\widehat{G}}*_r \subseteq {}_{\overline{G}}*_r = {}_G\overline{*}_r$. Let us show the inverse inclusion.
We consider the mapping $\pi : A_G \cup \overline{A_G} \longrightarrow A_{\widehat{G}}$ defined for any $a \in A_{\widehat{G}}$ by $\pi(a) = a$, and for any $\overline{a} \in \overline{A_G} - A_{\widehat{G}}$, $\pi(\overline{a})$ is the unique letter in $A_{\widehat{G}}$ such that $t \xrightarrow{\pi(\overline{a})}_G s$ for any $s \xrightarrow{a}_G t$.
This makes sense because G is deterministic, co-deterministic and vertex-transitive. Thus

$$s \xrightarrow{a}_G t \implies s \xrightarrow{\pi(a)}_{\widehat{G}} t \text{ for any } a \in A_G \cup \overline{A_G}.$$

By extending π by morphism on $(A_G \cup \overline{A_G})^*$, we get

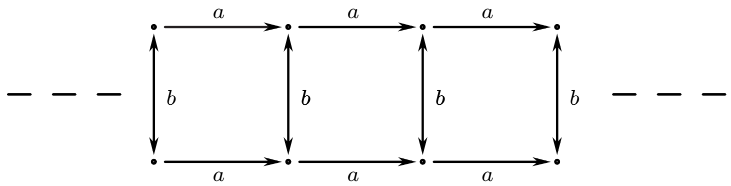
$$s \xrightarrow{u}_G t \implies s \xrightarrow{\pi(u)}_{\widehat{G}} t \text{ for any } u \in (A_G \cup \overline{A_G})^*.$$

This implies that ${}_G\overline{*}_r \subseteq {}_{\widehat{G}}*_r$.
By Proposition 6, $\widehat{G} = \mathcal{C}[\![V_G, \longrightarrow_{\widehat{G}}(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_{\widehat{G}} s$.
Precisely for any $s \in \longrightarrow_{\widehat{G}}(r)$, we have

$$[\![s]\!] = \begin{cases} a & \text{if } r \xrightarrow{a}_G s \\ \overline{a} & \text{if } s \xrightarrow{a}_G r \not\longrightarrow_G s. \end{cases}$$

Finally $G = \widehat{G}^{|A_G} = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!]$. ◀

For instance by Proposition 16, a graph of the following representation:



is a Cayley graph of a group: it is isomorphic to $\mathcal{C}[\![\mathbb{Z}\times\{0,1\}, \{(1,0),(0,1)\}]\!]$ with $[\![(1,0)]\!] = a$, $[\![(0,1)]\!] = b$ and for the chain-operation $(m,i) \; \overline{*}_{(0,0)} \; (n,j) = (m+n, i+j \,(\mathrm{mod} 2))$. It is also isomorphic to the Cayley graph of the group of finite presentation $<a, b \mid ab = ba, b^2 = 1>$.

Let us adapt Theorem 14 to simply describe the Cayley graphs of groups.

▶ **Theorem 17.** *A graph is a Cayley graph of a group if and only if it is connected, simple, deterministic, co-deterministic and vertex-transitive.*

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some group M and $Q \subseteq M$ with $(Q \cup Q^{-1})^* = M$.
By Facts 2, 3, 11, G is simple, vertex-transitive, deterministic and co-deterministic.
By Fact 12, the monoid Cayley graph $\mathcal{C}[\![M,Q \cup Q^{-1}]\!]$ is strongly connected.
Thus $G = \mathcal{C}[\![M,Q \cup Q^{-1}]\!]^{[\![Q]\!]}$ is connected.
$\Longleftarrow$: By Proposition 16. ◀

Theorems 14 and 17 give respectively a characterization of the monoid Cayley graphs of groups, and the group Cayley graphs. We can now deduce a characterization of the generalized Cayley graphs of groups. First, let us apply Corollary 15 and Theorem 17.

▶ **Corollary 18.** *The connected (resp. strongly connected) components of generalized Cayley graphs of groups are the (resp. monoid) Cayley graphs of groups.*

Let us extend Proposition 16 for non connected graphs.
Let a magma $(P,\cdot)$ for P a representative set of $\mathrm{Comp}(G)$.
We define the *extended chain-relation* $\mathrm{Chain}_G(P)$ as the ternary relation on V_G defined by

$$(s,t,x) \in \mathrm{Chain}_G(P) \text{ if there exists } u,v \in (A_G \cup \overline{A_G})^* \text{ such that}$$
$$\pi_P(s) \xrightarrow{u}_G s \text{ and } \pi_P(t) \xrightarrow{v}_G t \text{ and } \pi_P(s)\cdot\pi_P(t) \xrightarrow{uv}_G x.$$

For any connected and deterministic graph G and any vertex r, $\mathrm{Chain}_G(\{r\}) = \mathrm{Chain}_G(r)$.
If for any $s,t \in V_G$ there exists a unique x such that $(s,t,x) \in \mathrm{Chain}_G(P)$, we denote by $\overline{*}_P : V_G{\times}V_G \longrightarrow V_G$ the binary *extended chain-operation* on V_G defined for any $s,t \in V_G$ by $(s,t,s\,\overline{*}_P\,t) \in \mathrm{Chain}_G(P)$; we also write ${}_G\overline{*}_P$ when we need to specify G.
We can extend Proposition 16.

▶ **Proposition 19.** *Let G be a vertex-transitive, deterministic and co-deterministic graph. Let a group on a representative set P of $\mathrm{Comp}(G)$ generated by P_0 and of identity r. Then $(V_G,\overline{*}_P)$ is a group of identity r generated by $P_0 \cup \longrightarrow_G(r)$. If G is simple then $G = \mathcal{C}[\![V_G,\longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$.*

Proof.
i) Let $C \in \mathrm{Comp}(G)$ with $r \in V_C$.
By Proposition 16, $(V_C,\overline{*}_r)$ is a group of identity r and is generated by $\longrightarrow_G(r)$.
We take the group product $(P{\times}V_C,\cdot)$ with $(p,x)\cdot(q,y) = (p\cdot q, x\,\overline{*}_r\,y)$ for any $p,q \in P$ and $x,y \in V_C$. This group is of identity (r,r) and is generated by $P_0{\times}\{r\} \cup \{r\}{\times}{\longrightarrow_G}(r)$.
As G is vertex-transitive, deterministic and co-deterministic, we can define the mapping $f : P{\times}V_C \longrightarrow V_G$ such that $r \xrightarrow{u}_G x \implies p \xrightarrow{u}_G f(p,x)$ for (any) $u \in (A_G \cup \overline{A_G})^*$.
Thus f is a bijection hence $(V_G,\cdot)$ is a group where $f(p,x)\cdot f(q,y) = f(p\cdot q, x\,\overline{*}_r\,y)$ for any $p,q \in P$ and $x,y \in V_C$. This group $(V_G,\cdot)$ is of identity $f(r,r) = r$ and is generated by $f(P_0{\times}\{r\}) \cup f(\{r\}{\times}{\longrightarrow_G}(r)) = P_0 \cup \longrightarrow_G(r)$.
ii) Let us show that the operation $\cdot$ on V_G is equal to $\overline{*}_P$.
Let $p,q \in P$ and $x,y \in V_C$. We have to check that $f(p,x)\,\overline{*}_P\,f(q,y) = f(p,x)\cdot f(q,y)$.
Let $u,v \in (A_G \cup \overline{A_G})^*$ such that $r \xrightarrow{u}_G x$ and $r \xrightarrow{v}_G y$.
By definition of $\overline{*}_P$, we have $x \xrightarrow{v}_G x\,\overline{*}_r\,y$ hence $r \xrightarrow{uv}_G x\,\overline{*}_r\,y$.
By definition of f, we get $p\cdot q \xrightarrow{uv}_G f(p\cdot q, x\,\overline{*}_r\,y)$, $p \xrightarrow{u}_G f(p,x)$ and $q \xrightarrow{v}_G f(q,y)$.

By definition of $\overline{*}_P$, we have $p \cdot q \xrightarrow{uv}_G f(p,x) \,\overline{*}_P\, f(q,y)$.
As G is deterministic, we get $f(p,x) \,\overline{*}_P\, f(q,y) \;=\; f(p \cdot q, x \,\overline{*}_r\, y) \;=\; f(p,x) \cdot f(q,y)$.

iii) Suppose that in addition G is simple. Let $[\![y]\!] = a$ for any $r \xrightarrow{a}_G y$. Thus

$$\begin{aligned} \mathcal{C}[V_G, \longrightarrow_G(r)] &= \{\, s \xrightarrow{a} s \cdot y \mid s \in V_G \wedge r \xrightarrow{a} y \,\} \\ &= \{\, s \xrightarrow{a} s \,\overline{*}_P\, y \mid s \in V_G \wedge r \xrightarrow{a} y \,\} \;=\; G. \end{aligned}$$ ◀

In ZF set theory, the axiom of choice is equivalent to the property that any non-empty set has a group structure [9]. Under the assumption of the axiom of choice, we can characterize the generalized Cayley graphs of groups.

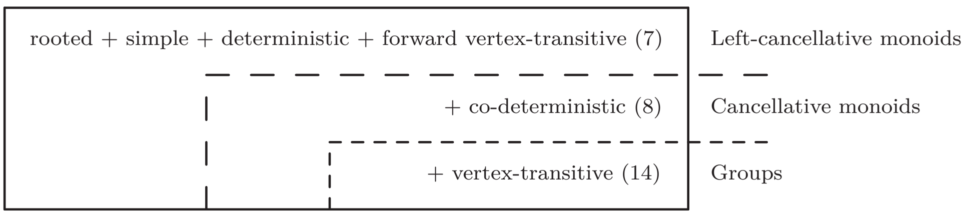
▶ **Theorem 20.** *In ZFC set theory, a graph is a generalized Cayley graph of a group if and only if it is simple, deterministic, co-deterministic, vertex-transitive.*

Proof.
By Facts 2, 3, 11, any generalized Cayley graph of a group is simple, vertex-transitive, deterministic and co-deterministic.
Conversely let G be a simple, deterministic, co-deterministic, vertex-transitive graph.
Using ZFC set theory, there exists a representative set P of $\mathrm{Comp}(G)$ and a binary operation $\cdot$ such that $(P,\cdot)$ is a group. By Proposition 19, $(V_G, \overline{*}_P)$ is a group and $G = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$. ◀

For instance let us consider the following graph:

$$\begin{aligned} G \quad &= \quad \{\, (m,i,p) \xrightarrow{a} (m+1,i,p) \mid m,p \in \mathbb{Z} \wedge i \in \{0,1\} \,\} \\ &\cup \quad \{\, (m,i,p) \xrightarrow{b} (m,1-i,p) \mid m,p \in \mathbb{Z} \wedge i \in \{0,1\} \,\} \end{aligned}$$

of representation the countable repetition of the preceding one.
By Theorem 20, $G = \mathcal{C}[\![\mathbb{Z}{\times}\{0,1\}{\times}\mathbb{Z}, \{(1,0,0),(0,1,0)\}]\!]$ with $[\![(1,0,0)]\!] = a$, $[\![(0,1,0)]\!] = b$ and for the group operation $(m,i,p) \cdot (n,j,q) \;=\; (m+n, i+j \,(\mathrm{mod} 2), p+q)$.
Let us summarize the characterizations obtained for the Cayley graphs.



By relaxing the condition of being rooted by that of connectivity, we have obtained a graph-theoretic characterization for the Cayley graphs of groups (Theorem 17).

5 Generalized Cayley graphs of left-cancellative magmas

Under ZFC set theory, we will give a fully graph-theoretic characterization for generalized Cayley graphs of left-cancellative magmas (Theorem 25), and then when they have an identity (Theorem 27).

Recall that an element e of a magma $(M,\cdot)$ is a *left identity* (resp. *right identity*) if $e{\cdot}p = p$ (resp. $p{\cdot}e = p$) for any $p \in M$. If M has a left identity e and a right identity e' then $e = e'$ which is an *identity* (or neutral element) of M.
In order to characterize the Cayley graphs of left-cancellative magmas with or without an identity, we need to restrict the path-relation. For any graph G and any vertex r, we define

the *edge-relation* $\mathrm{Edge}_G(r)$ as the ternary relation on V_G defined by

$$(s,t,x) \in \mathrm{Edge}_G(r) \ \text{ if there exists } a \in A_G \text{ such that } r \xrightarrow{a}_G t \text{ and } s \xrightarrow{a}_G x.$$

So $\mathrm{Edge}_G(r) \subseteq \mathrm{Path}_G(r)$. If for any $s,t \in V_G$ there exists a unique x such that $(s,t,x) \in \mathrm{Edge}_G(r)$, we denote by $\times_r : V_G{\times}V_G \longrightarrow V_G$ the binary *edge-operation* on V_G defined by $(s,t,s \times_r t) \in \mathrm{Edge}_G(r)$ for any $s,t \in V_G$; we also write ${}_G{\times}_r$ when we need to specify G. Let us give conditions for the existence of this edge-operation. We need to introduce two basic graph notions. We say that a vertex r of a graph G is an 1-*root* if $r \longrightarrow_G s$ for any vertex s of G. Thus a graph is complete if and only if all its vertices are 1-roots.

▶ **Fact 21.** Any left identity of a magma M is an 1-root of $\mathcal{C}(M)$.

Moreover we say that a graph is *loop-complete* if one vertex has an a-loop then all the vertices have an a-loop:

$$\exists\, r \in V_G \ (r \xrightarrow{a}_G r) \implies \forall\, s \in V_G \ (s \xrightarrow{a}_G s).$$

▶ **Fact 22.** Any generalized Cayley graph of a left-cancellative magma with a right identity is loop-complete.

Proof.
Let M be a left-cancellative magma with a right-identity e.
Let $G = \mathcal{C}[\![M,Q]\!]$ be a generalized Cayley graph of M.
Let $p \xrightarrow{[\![q]\!]}_G p$ for some $p \in M$ and $q \in Q$. So $p{\cdot}q = p = p{\cdot}e$.
As M is left-cancellative, $q = e$. Thus $r \xrightarrow{[\![q]\!]}_G r{\cdot}q = r{\cdot}e = r$ for any $r \in M$. ◀

Let us adapt Propositions 6 to the edge-operation.

▶ **Proposition 23.** *Let r be an 1-root of a deterministic source-complete simple graph G. Then $(V_G, \times_r)$ is a left-cancellative magma of left-identity r and*

$$G = \mathcal{C}[\![V_G]\!] \ \text{ with } [\![s]\!] = a \ \text{ for any } r \xrightarrow{a}_G s.$$

If G is loop-complete then r is an identity.

Proof.
i) Let $s,t \in V_G$. Let us check that there is a unique x such that $(s,t,x) \in \mathrm{Edge}_G(r)$.
As r is an 1-root and G is simple, there exists a unique $a \in A_G$ such that $r \xrightarrow{a}_G t$.
As G is source-complete and deterministic, there exists a unique vertex x such that $s \xrightarrow{a}_G x$. Thus $\times_r$ exists and is denoted by $\cdot$ in the rest of this proof.
ii) Let us check that $(V_G, \cdot)$ is left-cancellative. Assume that $s{\cdot}t = s{\cdot}t'$.
As r is an 1-root, there exists $a, a' \in A_G$ such that $r \xrightarrow{a}_G t$ and $r \xrightarrow{a'}_G t'$.
By definition of $\cdot$ we get $s \xrightarrow{a}_G s \cdot t$ and $s \xrightarrow{a'}_G s{\cdot}t' = s{\cdot}t$.
As G is simple, we have $a = a'$. As G is deterministic, it follows that $t = t'$.
iii) Let us check that r is a left identity of $(V_G, \cdot)$.
Let $s \in V_G$. As r is an 1-root, there exists $a \in A_G$ such that $r \xrightarrow{a}_G s$.
By definition of $\cdot$ we have $r \xrightarrow{a}_G r{\cdot}s$. As G is deterministic, we get $r{\cdot}s = s$.
iv) As r is an 1-root and G is simple, we can define the mapping $[\![\ ]\!] : V_G \longrightarrow A_G$ by

$$[\![s]\!] = a \quad \text{for } r \xrightarrow{a}_G s.$$

As G is deterministic, $[\![\ ]\!]$ is an injection. As G is source-complete, $[\![\ ]\!]$ is a bijection.
Let us show that $G = \mathcal{C}[\![V_G]\!]$.
$\subseteq$: Let $s \xrightarrow{a}_G t$. As G is source-complete, there exists r' such that $r \xrightarrow{a}_G r'$.
So $s \xrightarrow{a}_G s{\cdot}r'$. As G is deterministic, $s{\cdot}r' = t$. We have $[\![r']\!] = a$ hence $s \xrightarrow{a}_{\mathcal{C}[\![V_G]\!]} s{\cdot}r' = t$.

$\supseteq$: Let $s \xrightarrow{a}_{\mathcal{C}[\![V_G]\!]} t$. There exists (a unique) $r' \in V_G$ such that $[\![r']\!] = a$.
Thus $t = s{\cdot}r'$ and $r \xrightarrow{a}_G r'$. So $s \xrightarrow{a}_G s{\cdot}r' = t$.

v) Assume that G is loop-complete. Let us check that r is also a right identity.
As r is an 1-root, there is (a unique) $a \in A_G$ such that $r \xrightarrow{a}_G r$.
Let $s \in V_G$. As G is loop-complete, we get $s \xrightarrow{a}_G s$. By definition of $\cdot$ we have $s \xrightarrow{a} s{\cdot}r$.
As G is deterministic, $s = s{\cdot}r$. Thus r is a right identity. ◀

We get a fully graph-theoretic characterization of the Cayley graphs $\mathcal{C}[\![M]\!]$ for any left-cancellative magma M with a left identity.

▶ **Proposition 24.** *A graph is equal to $\mathcal{C}[\![M]\!]$ for some left-cancellative magma M with a left identity if and only if it is simple, deterministic, source-complete and 1-rooted.*

Proof.
$\Longrightarrow$: let $G = \mathcal{C}[\![M]\!]$ for some left-cancellative magma $(M,\cdot)$ with a left identity r, and some injective mapping $[\![\ ]\!]$. By Facts 2 and 3, G is deterministic, source-complete and simple. By Fact 21, r is an 1-root of G.
$\Longleftarrow$: By Proposition 23. ◀

Under the assumption of the axiom of choice, we can characterize the generalized Cayley graphs of left-cancellative magmas.

▶ **Theorem 25.** *In ZFC set theory, the following graphs define the same family:*
a) *the generalized Cayley graphs of left-cancellative magmas,*
b) *the generalized Cayley graphs of left-cancellative magmas with a left identity,*
c) *the simple, deterministic, source-complete graphs.*

Proof.
b) $\Longrightarrow$ **a)** : immediate.
a) $\Longrightarrow$ **c)** : by Facts 2 and 3.
c) $\Longrightarrow$ **b)** : let G be a simple, deterministic and source-complete graph.
Assume the axiom of choice. Let r be a vertex of G with

$$V_G - \longrightarrow_G(r) \;=\; \{\ s \in V_G \mid r \not\longrightarrow_G s\ \} \text{ of minimal cardinality.}$$

For each vertex s, we take an injection f_s from $V_G - \longrightarrow_G(r)$ to $V_G - \longrightarrow_G(s)$ and whose f_r is the identity. We define the graph

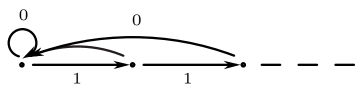
$$<G> \;=\; \{\ s \xrightarrow{p} t \mid \exists\, a\ (s \xrightarrow{a}_G t \wedge r \xrightarrow{a}_G p)\ \} \cup \{\ s \xrightarrow{t} f_s(t) \mid s \in V_G \wedge t \in \mathrm{Dom}(f_s)\ \}.$$

Thus $<G>$ remains simple, deterministic, source-complete with $V_{<G>} = V_G = A_{<G>}$.
Furthermore $r \xrightarrow{s}_{<G>} s$ for any $s \in V_G$ hence r is an 1-root of $<G>$.
By Proposition 23, $<G> = \mathcal{C}(V_G)$ for $(V_G, \times_r)$ left-cancellative, of left identity r with

$$s \xrightarrow{t}_{<G>} s \times_r t \quad \text{for any } s,t \in V_G\,.$$

Finally $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$. ◀

For instance, let $G = \{\ m \xrightarrow{0} 0 \mid m \geq 0\ \} \cup \{\ m \xrightarrow{1} m+1 \mid m \geq 0\ \}$ represented by



It is simple, deterministic, source-complete but without 1-root. By Theorem 25, this graph is a generalized Cayley graph of a left-cancellative magma with a left identity. Precisely we complete G into the graph

$$<G> \;=\; \{\; m \xrightarrow{0} 0 \mid m \geq 0 \;\} \cup \{\; m \xrightarrow{n} m+n \mid m \geq 0 \wedge n > 0 \;\}$$

having 0 as 1-root, and which remains simple, deterministic, source-complete.
By Proposition 23, the magma $(\mathbb{N}, \times_0)$ with the edge-operation $\times_0$ of $<G>$ *i.e.*

$$m \times_0 0 \;=\; 0 \;\text{ and }\; m \times_0 n \;=\; m+n \;\text{ for any }\; m \geq 0 \;\text{ and }\; n > 0$$

is left-cancellative and 0 is a left identity. Furthermore $G \;=\; \mathcal{C}(\mathbb{N}, \{0,1\})$.

We can now characterize the generalized Cayley graphs of the left-cancellative magmas with an identity. We just have to add the loop-complete property to restrict Proposition 24 to left-cancellative magmas with an identity element.

▶ **Proposition 26.** *A graph is equal to $\mathcal{C}[\![M]\!]$ for some left-cancellative magma M with an identity if and only if it is simple, deterministic, source-complete, loop-complete and* 1*-rooted.*

Proof.
$\Longrightarrow$: By Proposition 24 and Fact 22.
$\Longleftarrow$: By Proposition 23. ◀

We restrict Theorem 25 to left-cancellative magmas having a right identity.

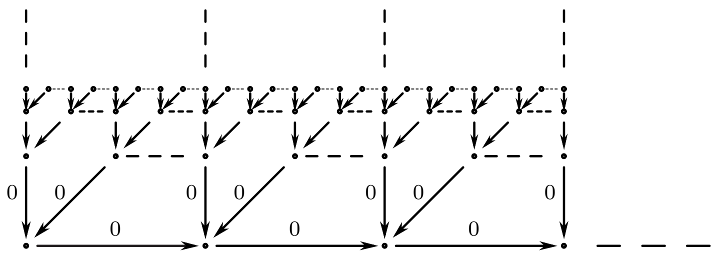
▶ **Theorem 27.** *In ZFC set theory, the following graphs define the same family*:
a) *the generalized Cayley graphs of left-cancellative magmas with a right identity,*
b) *the generalized Cayley graphs of left-cancellative magmas with an identity,*
c) *the simple, deterministic, source-complete and loop-complete graphs.*

Proof.
b) $\Longrightarrow$ **a)**: immediate.
a) $\Longrightarrow$ **c)**: By Facts 2, 3, 22.
c) $\Longrightarrow$ **b)**: Let G be a graph which is simple, deterministic, source and loop-complete.
Let us apply the construction given in the proof of Theorem 25. As G is loop-complete and if $r \not\longrightarrow_G r$, we can add the condition that $f_s(r) = s$ for any $s \in V_G$. Thus $<G>$ remains loop-complete and by Proposition 23, r is an identity for $\times_r$. ◀

For instance, we denote by $\mathbb{N}_+ \;=\; \mathbb{N} - \{0\}$ and we consider the graph

$$G \;=\; \{\; 0^n \xrightarrow{0} 0^{n+1} \mid n \geq 0 \;\} \cup \{\; ui \xrightarrow{0} u \mid u \in 0^*\mathbb{N}_+^* \wedge i \in \mathbb{N}_+ \;\}.$$

of vertex set $V_G \;=\; 0^*\mathbb{N}_+^*$ and represented as follows:



It is simple, deterministic, source-complete and loop-complete (and forward vertex-transitive).
By Theorem 27, this graph is a generalized Cayley graph of a left-cancellative magma having an identity. For instance, we complete G into the graph

$$<G> \;=\; G \cup \{\; 0^m u \xrightarrow{0^n v} 0^{m+n}uv \mid m,n \geq 0 \wedge u,v \in \mathbb{N}_+^* \wedge 0^n v \neq 0 \;\}$$

which remains simple, deterministic, source-complete, loop-complete, with the 1-root ε.

By Proposition 23, $G = \mathcal{C}(V_G, \{0\})$ for the left-cancellative magma $(V_G, \times_\varepsilon)$ of identity ε with the edge-operation $\times_\varepsilon$ of <G> defined for any $m, n \geq 0$, $u, v \in \mathbb{N}_+^*$ and $i \in \mathbb{N}_+$ by

$$0^m ui \times_\varepsilon 0 \ = \ 0^m u \quad \text{otherwise} \quad 0^m u \times_\varepsilon 0^n v \ = \ 0^{m+n} uv.$$

We will see that we can define <G> so that in addition, any vertex is an 1-root *i.e.* <G> is complete (see Theorem 33).

6 Generalized Cayley graphs of left-quasigroups

We can now refine the previous characterization of generalized Cayley graphs from left-cancellative magmas to left-quasigroups (Theorem 32). These algebraic structures define the same family of finitely labeled generalized Cayley graphs (Theorem 33).

A magma $(M, \cdot)$ is a *left-quasigroup* if for each $p, q \in \mathsf{M}$, there is a unique $r \in M$ such that $p{\cdot}r = q$ denoted by $r = p \backslash q$ the *left quotient* of q by p.
This property ensures that each element of M occurs exactly once in each row of the Cayley table for $\cdot$. For instance $\{a, b, c\}$ is a left-quasigroup for $\cdot$ defined by the following Cayley table:

$\cdot$	a	b	c
a	a	b	c
b	b	a	c
c	c	b	a

Note that $\cdot$ is not associative since $c \cdot (b \cdot c) = a$ and $(c \cdot b) \cdot c = c$, and is not right-cancellative since $a{\cdot}b = c{\cdot}b$. The first figure in Section 3 is a representation of $\mathcal{C}(M)$ for the semigroup $M = \{a, b\}$ with $x{\cdot}y = y$ for any $x, y \in M$. This semigroup is also a left-quasigroup which is not right-cancellative.
Any left-quasigroup M is left-cancellative. The converse is true for M finite but is false in general: $(\mathbb{N}, +)$ is cancellative but is not a left-quasigroup. Indeed,

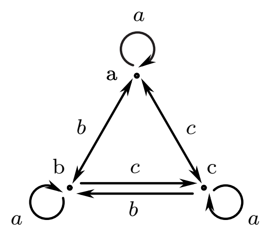
$$M \text{ is a left-quasigroup} \iff M \text{ is left-cancellative and } p{\cdot}M = M \text{ for any } p \in M.$$

Let us refine Proposition 23 in the case where the graph is also complete.

▶ **Proposition 28.** *Let G be a simple, deterministic, complete and source-complete graph. For any vertex r, $(V_G, \times_r)$ is a left-quasigroup.*

Proof.
Let r be a vertex of G. As G is complete, r is an 1-root.
By Proposition 23, $(V_G, \times_r)$ is a left-cancellative magma.
Let $s \in V_G$. It remains to check that $V_G \subseteq s \times_r V_G$. Let $t \in V_G$.
As G is complete, there exists $a \in A_G$ such that $s \xrightarrow{a}_G t$.
As G is source-complete, there exists $t' \in V_G$ such that $r \xrightarrow{a}_G t'$.
By definition of $\times_r$ we have $s \xrightarrow{a}_G s \times_r t'$. As G is deterministic, we get $t = s \times_r t'$. ◀

Let us give a simple characterization of the generalized Cayley graphs for the left-quasigroups.
For the previous left-quasigroup M, its graph $\mathcal{C}(M)$ is the following:



We begin by characterizing these graphs.

▶ **Proposition 29.** *We have the following equivalences:*
a) *a graph is equal to $\mathcal{C}[\![M]\!]$ for some left-quasigroup M (resp. with a right identity),*
b) *a graph is equal to $\mathcal{C}[\![M]\!]$ for some left-quasigroup M with a left identity (resp. identity),*
c) *a graph is simple, deterministic, complete, source-complete (resp. and loop-complete).*

Proof.
b) $\Longrightarrow$ **a)**: immediate.
a) $\Longrightarrow$ **c)**: let $G = \mathcal{C}[\![M]\!]$ for some left-quasigroup $(M,\cdot)$ and injective mapping $[\![\ ]\!]$.
By Facts 2 and 3, G is deterministic, source-complete and simple.
For any $p,q \in M$, we have $p \overset{[\![p\backslash q]\!]}{\longrightarrow}_G q$ hence G is complete.
If in addition M has a right identity then, by Fact 22, G is loop-complete.
c) $\Longrightarrow$ **b)**: by Propositions 23 and 28. ◀

For instance the magma $M = \{0,1\}$ with $i \cdot j = 1 - j$ for any $i,j \in \{0,1\}$, is a left-quasigroup without left and right identity element. By Propositions 23 and 28, the magma $N = \{0,1\}$ with the edge-operation $\times_0$ of $\mathcal{C}(M)$ defined by $i \times_0 j = j$ for any $i,j \in \{0,1\}$ is a left-quasigroup with 0 and 1 are left identities, and $\mathcal{C}(M) = \mathcal{C}[\![N]\!]$ where $[\![0]\!] = 1$ and $[\![1]\!] = 0$.
We now extend Proposition 29 to the generalized Cayley graphs of left-quasigroups. To do this, we must recall and define basic graph notions.
Let G be a graph. For any vertex s, its *out-degree* $\delta^+_G(s) = |G \cap \{s\}\times A \times V_G|$ is the number of edges of source s. The *out-degree* of G is the cardinal $\Delta^+_G = \sup\{\ \delta^+_G(s) \mid s \in V_G\ \}$.
We say that G is of *bounded out-degree* when Δ^+_G is finite.
We have $\Delta^+_G = |A_G|$ for G deterministic and source-complete.

▶ **Fact 30.** For any vertex s of a graph G, we have

$$\delta^+_G(s) \ \leq\ |A_G| \text{ for } G \text{ deterministic, and } |A_G| \ \leq\ \delta^+_G(s) \text{ for } G \text{ source-complete.}$$

In particular by Fact 2 and for any generalized Cayley graph G,

$$G \text{ is of bounded out-degree} \iff G \text{ is finitely labeled.}$$

By removing the labeling of a graph G, we get the binary unlabeled edge relation on V_G:

$$\longrightarrow_G\ =\ \{\ (s,t) \mid \exists\, a \in A_G\ (s,a,t) \in G\ \}.$$

Let $R \subseteq V \times V$ be a binary relation on a set V *i.e.* is an unlabeled graph.
The *image* of $P \subseteq V$ by R is the set $R(P) = \{\ t \mid \exists\, s \in P\ (s,t) \in R\ \}$.
So the out-degree of $s \in V$ is $\delta^+_R(s) = |R(s)|$ and $\Delta^+_R = \sup\{\ \delta^+_R(s) \mid s \in V\ \}$ is the out-degree of R. For any graph G and any vertex s, we have $\delta^+_{\rightarrow_G}(s) \leq \delta^+_G(s)$ hence $\Delta^+_{\rightarrow_G} \leq \Delta^+_G$, and we have equalities for G simple.
A relation R is an *out-regular relation* if all the elements of V have the same out-degree: $|R(s)| = |R(t)|$ for any $s,t \in V$.
Let us give simple conditions on R so that its complement $V \times V - R$ is out-regular.

▶ **Lemma 31.** *Let* $R \subseteq V{\times}V$ *and* $S = V{\times}V - R$ *the complement of* R *w.r.t.* $V{\times}V$.
If R *is out-regular and* $\Delta^+_R < \omega$ *then* S *is out-regular.*
If R *is infinite and* $\Delta^+_R < |V|$ *then* S *is out-regular.*

Proof.
i) When Δ^+_R is finite, we have $S(s) \ = \ |V| - |R(s)|$ for any $s \in V$.
In addition for R out-regular and for any $s, t \in V$, $|R(s)| \ = \ |R(t)|$ hence $|S(s)| \ = \ |S(t)|$.
ii) When R is infinite with $\Delta^+_R < |V|$, we have $|S(s)| \ = \ |V|$ for any $s \in V$.
Hence S is out-regular on V with $\Delta^+_S \ = \ |V|$. ◀

We say that a graph G is *out-regular* if $\longrightarrow_G$ is out-regular *i.e.* all the vertices have the same number of targets. For instance $G \ = \ \{s \overset{a}{\longrightarrow} t\, ,\, t \overset{a}{\longrightarrow} s\, ,\, t \overset{b}{\longrightarrow} s\}$ is out-regular since $\longrightarrow_G(s) = \{t\}$ and $\longrightarrow_G(t) = \{s\}$ while $\delta^+_G(s) = 1$ and $\delta^+_G(t) = 2$.
We also say that G is *co-out-regular* if its unlabeled complement is out-regular *i.e.*

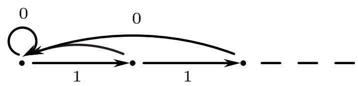
$$(V_G{\times}V_G \ - \longrightarrow_G) \ = \ \not\longrightarrow_G \ \text{ is an out-regular relation on } V_G\,.$$

Under the assumption of the axiom of choice, we can characterize the generalized Cayley graphs of left-quasigroups.

▶ **Theorem 32.** *In ZFC set theory, the following graphs define the same family*:
a) *the generalized Cayley graphs of left-quasigroups (resp. with a right identity),*
b) *the generalized Cayley graphs of left-quasigroups with a left identity (resp. an identity),*
c) *the simple, deterministic, source-complete (resp. and loop-complete) co-out regular graphs.*

Proof.
b) $\Longrightarrow$ **a)** : immediate.
a) $\Longrightarrow$ **c)** : let $G \ = \ \mathcal{C}[\![M, Q]\!]$ for some left-quasigroup M and $Q \subseteq M$.
By Proposition 29, $G \ = \ \mathcal{C}[\![M]\!]^{|[\![Q]\!]}$ remains simple, deterministic and source-complete.
For any $s \in M$, $\delta^+_{\not\to_G}(s) \ = \ |M - Q|$ hence G is co-out-regular.
If in addition M has a right identity then, by Fact 22, G is loop-complete.
c) $\Longrightarrow$ **b)** : let G be a graph which is simple, deterministic, co-out-regular, source-complete (resp. and loop-complete). The co-out-regularity of G means that $|V_G - \longrightarrow_G(s)| \ = \ |V_G - \longrightarrow_G(t)|$ for any $s, t \in V_G$. Let r be a vertex of G. Assume the axiom of choice. Let us apply the construction given in the proof of Theorem 25. As G is co-out-regular, we can now take for each vertex s a bijection f_s from $V_G - \longrightarrow_G(r)$ to $V_G - \longrightarrow_G(s)$ and whose f_r is the identity. As for the proof of Theorem 27, if G is loop-complete and $r \not\longrightarrow_G r$, we add the condition that $f_s(r) = s$ for any $s \in V_G$. The graph obtained $<\!G\!>$ remains simple, deterministic, source-complete (resp. loop-complete) and is in addition a complete graph. By Proposition 28, $(V_G, \times_r)$ is a left-quasigroup. By Proposition 23, r is a left-identity (resp. is an identity) and $G \ = \ \mathcal{C}[\![V_G\, ,\, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \overset{a}{\longrightarrow}_G s$. ◀

For instance, let $G \ = \ \{\ m \overset{0}{\longrightarrow} 0 \mid m \geq 0\ \} \cup \{\ m \overset{1}{\longrightarrow} m+1 \mid m \geq 0\ \}$ be the graph that we had considered after Theorem 25 and represented by



By adding edges, we transform G into the following complete graph:

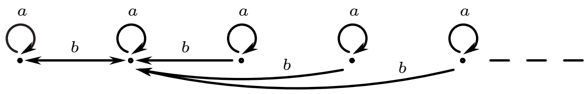
$$<\!G\!> \ = \ G \cup \{\ m \overset{n}{\longrightarrow} n-1 \mid 2 \leq n \leq m+1\ \} \cup \{\ m \overset{n}{\longrightarrow} n \mid m \geq 0 \wedge n > m+1\ \}$$

which remains simple, deterministic and source-complete.

By Propositions 23 and 29, $G = \mathcal{C}(\mathbb{N},\{0,1\})$ for the left-quasigroup $(\mathbb{N},\times_0)$ of left-identity 0 with the edge-operation $\times_0$ of $<G>$ defined for any $m \geq 0$ by

$$m \times_0 0 = 0 \quad ; \quad m \times_0 n = n-1 \quad \forall\, 2 \leq n \leq m+1$$
$$m \times_0 1 = m+1 \quad ; \quad m \times_0 n = n \quad \forall\, n > m+1.$$

Another example is given by a graph G of the following representation:



It is simple, deterministic, co-out-regular, source-complete and loop-complete.
By Theorem 32, this graph is a generalized Cayley graph of a left-quasigroup with an identity.
Indeed, by replacing a by 0 and b by 1, G is isomorphic to the following graph:

$$H = \{\, m \xrightarrow{0} m \mid m \geq 0 \,\} \cup \{\, 1 \xrightarrow{1} 0 \,\} \cup \{\, m \xrightarrow{1} 1 \mid m \geq 0 \wedge m \neq 1 \,\}.$$

We complete H into the graph:

$$<H> = \{\, m \xrightarrow{0} m \mid m \geq 0 \,\} \cup \{\, m \xrightarrow{m} 0 \mid m \geq 0 \,\}$$
$$\cup \quad \{\, m \xrightarrow{n} n \mid m \geq 0 \wedge n > 0 \wedge m \neq n \,\}.$$

So $<H>$ remains simple, deterministic, source-complete and loop-complete.
Furthermore $<H>$ is complete *i.e.* any vertex is an 1-root.
By Proposition 23, the magma $(\mathbb{N},\times_0)$ with the edge-operation $\times_0$ of $<H>$ *i.e.*
$m \times_0 0 = m$; $m \times_0 m = 0$; $m \times_0 n = n$ for any $m,n \geq 0$ with $m \neq n$ and $n > 0$
is a left-quasigroup of identity element 0.
Furthermore G is isomorphic to $\mathcal{C}[\![\mathbb{N},\{0,1\}]\!]$ with $[\![0]\!] = a$ and $[\![1]\!] = b$.

The co-out-regularity in Theorem 32 can not be removed. For instance, consider the monoid $(\mathbb{N},+)$ which is is not a left-quasigroup. Its graph $\mathcal{C}(\mathbb{N}) = \{\, m \xrightarrow{n} m+n \mid m,n \geq 0 \,\}$ is simple, deterministic and source-complete. Furthermore we have $0 \xrightarrow{n}_{\mathcal{C}(\mathbb{N})} n$ for any $n \geq 0$ while there is no edge from 1 to 0. By Proposition 29, this graph is not a generalized Cayley graph of a left-quasigroup.
By Lemma 31, the co-out-regularity in Theorem 32 can be removed for the graphs of bounded out-degree which coincides with the characterization of Theorem 25. In this case, we can also remove the assumption of the axiom of choice.

▶ **Theorem 33.** *For any finitely labeled graph G, the following properties are equivalent:*
a) *G is a generalized Cayley graph of a left-cancellative magma (resp. with a right identity),*
b) *G is a gen. Cayley graph of a left-quasigroups with a left identity (resp. an identity),*
c) *G is simple, deterministic, source-complete (resp. and loop-complete).*

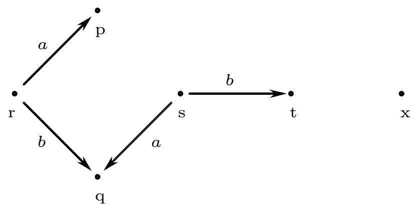
Proof.
b) $\Longrightarrow$ **a)**: immediate.
a) $\Longrightarrow$ **c)**: by Facts 2, 3, 22.
c) $\Longrightarrow$ **b)**: let G be a simple, deterministic and source-complete graph of finite label set. By Fact 30, G is of bounded out-degree. To each injective function $\ell : A_G \longrightarrow A_G$, we associate a permutation $\overline{\ell}$ on A_G extending ℓ *i.e.* $\overline{\ell}(a) = \ell(a)$ for every $a \in \mathrm{Dom}(\ell)$.
Let r be a vertex of G. For each vertex s, we associate the injective function:

$$\ell_s = \{\, (a,b) \mid \exists\, t\ (r \xrightarrow{a}_G t \wedge s \xrightarrow{b}_G t) \,\}.$$

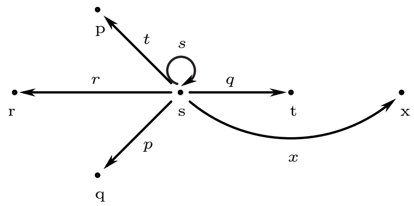
We define the following graph:

$$\begin{aligned} \ll G \gg \;&=\; \{\, s \xrightarrow{p} t \mid \exists\, a\ (r \xrightarrow{a}_G p \wedge\ s \xrightarrow{a}_G t)\,\} \\ &\cup\; \{\, s \xrightarrow{p} t \mid \exists\, a \in A_G - \mathrm{Dom}(\ell_s)\ (r \xrightarrow{a}_G t \wedge\ s \xrightarrow{\overline{\ell_s}(a)}_G p)\,\} \\ &\cup\; \{\, s \xrightarrow{t} t \mid t \in V_G - (\longrightarrow_G(r) \cup \longrightarrow_G(s))\,\}. \end{aligned}$$

For G of vertex set $V_G = \{p,q,r,s,t,x\}$ with the following edges from r and s:



the graph $\ll G \gg$ has the following edges from s:



Thus $\ll G \gg$ remains simple, deterministic, source complete with $V_{\ll G \gg} = V_G = A_{\ll G \gg}$. Furthermore $\ll G \gg$ is complete with $r \xrightarrow{s}_{\ll G \gg} s$ for any $s \in V_G$. By Proposition 23, we have $\ll G \gg = \mathcal{C}(V_G)$ for the left-cancellative magma $(V_G, \times_r)$ of left identity r with

$$s \xrightarrow{t}_{\ll G \gg} s \times_r t \quad \text{for any } s,t \in V_G\,.$$

Finally $G = \mathcal{C}[\![V_G\,,\, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$.

Now suppose that G is loop-complete. We distinguish the two complementary cases below.
Case 1: all the vertices of G have a loop of the same label.

Then $\ll G \gg$ remains loop-complete and by Proposition 23, r is an identity of $\times_r$.

Case 2: G has no loop.

We take a new label $a \in A - A_G$ and we redefine $\ll G \gg$ as being $\ll G' \gg$ for $G' = G \cup \{\, s \xrightarrow{a} s \mid s \in V_G\,\}$. We conclude by Case 1. ◀

For the previous example, we have $\ll H \gg \;=\; <H>$.
For the penultimate example, we have

$$\begin{aligned} \ll G \gg \;&=\; \{\, m \xrightarrow{1} m+1 \mid m \ge 0\,\} \cup \{\, m \xrightarrow{m+1} 1 \mid m \ge 0\,\} \\ &\cup\; \{\, m \xrightarrow{n} n \mid m,n \ge 0 \wedge n \ne 1 \wedge n \ne m+1\,\}. \end{aligned}$$

For the last example of the previous section (after Theorem 27), we have

$$\begin{aligned} \ll G \gg \;&=\; \{\, 0^m \xrightarrow{\varepsilon} 0^m \mid m \ge 0\,\} \cup \{\, 0^m \xrightarrow{0} 0^{m+1} \mid m \ge 0\,\} \\ &\cup\; \{\, 0^m \xrightarrow{0^m} \varepsilon \mid m > 0\,\} \cup \{\, 0^m \xrightarrow{0^{m+1}} 0 \mid m > 1\,\} \cup \{ 0 \xrightarrow{00} \varepsilon \} \\ &\cup\; \{\, 0^m \xrightarrow{u} u \mid u \in 0^*\mathbb{N}_+^* - \{\varepsilon, 0, 0^m, 0^{m+1}\}\,\} \\ &\cup\; \{\, ui \xrightarrow{\varepsilon} ui \mid u \in 0^*\mathbb{N}_+^* \wedge i \in \mathbb{N}_+\,\} \cup \{\, ui \xrightarrow{0} u \mid u \in 0^*\mathbb{N}_+^* \wedge i \in \mathbb{N}_+\,\} \\ &\cup\; \{\, ui \xrightarrow{ui} \varepsilon \mid u \in 0^*\mathbb{N}_+^* - \{\varepsilon\} \wedge i \in \mathbb{N}_+\,\} \cup \{\, ui \xrightarrow{u} 0 \mid u \in 0^*\mathbb{N}_+^* \wedge i \in \mathbb{N}_+\,\} \\ &\cup\; \{\, i \xrightarrow{i} 0 \mid i \in \mathbb{N}_+\,\} \cup \{\, ui \xrightarrow{v} v \mid u,v \in 0^*\mathbb{N}_+^* \wedge i \in \mathbb{N}_+ \wedge v \notin \{\varepsilon, 0, u, ui\}\,\}. \end{aligned}$$

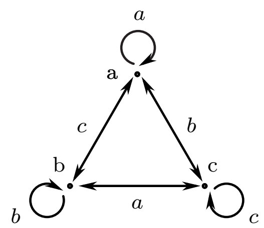
7 Generalized Cayley graphs of quasigroups

We can now refine the previous characterization of generalized Cayley graphs from left-quasigroups (Theorem 32) to quasigroups (Theorem 41).

A magma $(M,\cdot)$ is a *quasigroup* if $\cdot$ obeys the *Latin square* property: for each $p,q \in \mathsf{M}$, there is a unique $r \in \mathsf{M}$ such that $p{\cdot}r = q$ denoted by $r = p\backslash q$ the *left quotient* of q by p, there is a unique $s \in \mathsf{M}$ such that $s{\cdot}p = q$ denoted by $s = q/p$ the *right quotient* of q by p. This property ensures that each element of M occurs exactly once in each row and exactly once in each column of the Cayley table for $\cdot$. The previous finite left-quasigroup is not a quasigroup. On the other hand, $(\{a,b,c\},\cdot)$ is a quasigroup with $\cdot$ defined by the Cayley table:

$\cdot$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Its Cayley graph $\mathcal{C}(\{a,b,c\})$ is represented as follows:



Note that $\cdot$ is not associative: $a\cdot(b\cdot c) = a$ and $(a\cdot b)\cdot c = c$. Furthermore,

$$M \text{ is a quasigroup} \iff M \text{ is cancellative and } p\cdot M = M = M\cdot p \text{ for any } p \in M.$$

Let us refine Proposition 28 in the case where the graph is also co-deterministic and target-complete.

▶ **Proposition 34.** *Let G be a graph which is simple, deterministic and co-deterministic, complete, source-complete and target-complete. For any vertex r, $(V_G, \times_r)$ is a quasigroup.*

Proof.
Let r be a vertex of G and $\cdot$ be the edge-operation $\times_r$.
By Proposition 28, $(V_G,\cdot)$ is a left-quasigroup.
i) Let us check that $(V_G,\cdot)$ is right-cancellative. Assume that $s\cdot t = s'\cdot t$.
As G is complete, there exists a such that $r \xrightarrow{a}_G t$.
By definition of $\cdot$ we have $s \xrightarrow{a}_G s\cdot t$ and $s' \xrightarrow{a}_G s'\cdot t = s\cdot t$.
As G is co-deterministic, we get $s = s'$.
ii) Let $t \in V_G$. Let us check that $V_G \subseteq V_G\cdot t$. Let $s \in V_G$.
As G is complete, there exists $a \in A_G$ such that $r \xrightarrow{a}_G t$.
As G is target-complete, there exists $s' \in V_G$ such that $s' \xrightarrow{a}_G s$.
By definition of $\cdot$ we have $s' \xrightarrow{a}_G s'\cdot t$. As G is deterministic, we get $s = s'\cdot t$. ◀

Let us restrict Proposition 29 to the quasigroups.

▶ **Proposition 35.** *We have the following equivalences:*
a) *a graph is equal to $\mathcal{C}[\![M]\!]$ for some quasigroup M (resp. with a right identity),*
b) *a graph is equal to $\mathcal{C}[\![M]\!]$ for some quasigroup M with a left identity (resp. an identity),*

c) *a graph is simple, deterministic, co-deterministic, complete, source-complete, target-complete (resp. and loop-complete).*

Proof.
b) $\Longrightarrow$ **a)** : immediate.
a) $\Longrightarrow$ **c)** : let $G = \mathcal{C}[\![M]\!]$ for some quasigroup $(M,\cdot)$ and injective mapping $[\![\ ]\!]$.
By Proposition 29, G is simple, deterministic, complete and source-complete.
By Fact 3, G is co-deterministic.
For any $p,q \in M$, we have $p/q \xrightarrow[G]{[\![q]\!]} p$ hence G is target-complete.
If in addition M has a right identity then, by Fact 22, G is loop-complete.
c) $\Longrightarrow$ **b)** : By Propositions 23 and 34. ◀

For instance let us consider the division $\div$ on $\mathbb{R}_+ =]0,+\infty[$. So $(\mathbb{R}_+,\div)$ is a quasigroup of right identity 1. Furthermore $\times_1$ is the multiplication on $\mathbb{R}_+$. Thus $\mathcal{C}(\mathbb{R}_+)$ for the quasigroup $(\mathbb{R}_+,\div)$ is equal to $\mathcal{C}[\![\mathbb{R}_+]\!]$ for the group $(\mathbb{R}_+,\times_1)$ with $[\![x]\!] = \frac{1}{x}$ for any $x > 0$.
We now adapt Theorem 32 to the quasigroups. This will require a more extensive development than what has been done with Theorem 32.
For any vertex s of a graph G, its *in-degree* $\delta^-_G(s) = |G \cap V_G{\times}A{\times}\{s\}| = \delta^+_{G^{-1}}(s)$ is the number of edges of target s, and $\delta_G(s) = \delta^+_G(s) + \delta^-_G(s)$ is the *degree* of s.
The *in-out-degree* of G is the cardinal

$$\Delta_G \;=\; \sup\big(\{\ \delta^+_G(s) \mid s \in V_G\ \} \cup \{\ \delta^-_G(s) \mid s \in V_G\ \}\big).$$

We say that a graph G is of *bounded degree* when Δ_G is finite.
Let $R \subseteq V{\times}V$ be a binary relation on a set V. The in-degree of $s \in V$ is $\delta^-_R(s) = |R^{-1}(s)|$ for $R^{-1} = \{\ (t,s) \mid (s,t) \in R\ \}$ the *inverse* of R. The *in-out-degree* of R is

$$\Delta_R \;=\; \sup\big(\{\ \delta^+_R(s) \mid s \in V\ \} \cup \{\ \delta^-_R(s) \mid s \in V\ \}\big).$$

A relation R is a *regular relation* on V if $|R(s)| = |R^{-1}(s)| = \Delta_R$ for any $s \in V$.
Let us apply Lemma 31 to R and R^{-1}.

▶ **Corollary 36.** *Let $R \subseteq V{\times}V$ and $S = V{\times}V - R$ the complement of R w.r.t. $V{\times}V$.*
If R is regular and $\Delta_R < \omega$ then S is regular.
If R is infinite and $\Delta_R < |V|$ then S is regular.

An *edge-labeling* of R is a mapping $c : R \longrightarrow A$ defining the respective graph and color set

$$R^c \;=\; \{\ (s,c(s,t),t) \mid (s,t) \in R\ \} \quad \text{and} \quad c(R) \;=\; \{\ c(s,t) \mid (s,t) \in R\ \} \;=\; A_{R^c}\,.$$

An *edge-coloring* of R is an edge-labeling c of R such that R^c is a deterministic and co-deterministic graph. In that case, we say that R is $|c(R)|$-*edge-colorable* and we have $|c(R)| \geq \Delta_R$. We will give general conditions for a relation R to be Δ_R-edge-colorable.
An *undirected edge-coloring* of R is an edge-labeling c of R such that two adjacent couples of R have distinct colors: for any $(s,t)\,,\,(s',t') \in R$,

$$\text{if}\ \ (s,t) \neq (s',t') \ \text{ and } \ \{s,t\} \cap \{s',t'\} \neq \emptyset \ \text{ then } \ c(s,t) \neq c(s',t').$$

Any undirected edge-coloring is an edge-coloring.
Let $V' = \{\ s' \mid s \in V\ \}$ be a disjoint copy of V *i.e.* $'$ is a bijection from V to a disjoint set V'. We transform any relation $R \subseteq V{\times}V$ into the relation

$$R' \;=\; \{\ (s,t') \mid (s,t) \in R\ \} \;\subseteq\; V{\times}V'$$

and any edge-labeling c of R into the edge-labeling c' of R' defined by

$$c'(s,t') \;=\; c(s,t) \ \text{ for any } \ (s,t) \in R.$$

So $\Delta_R = \Delta_{R'}$ and for any edge-labeling c of R,

$$c \text{ is an edge-coloring of } R \iff c' \text{ is an edge-coloring of } R'$$
$$\iff c' \text{ is an undirected edge-coloring of } R'.$$

As $R' \subseteq V \times V'$ with $V \cap V' = \emptyset$, R' is a bipartite relation hence for R' finite, and by König's theorem [11], R' has an undirected $\Delta_{R'}$-edge-coloring. This implies that we have an edge-coloring of any finite relation R using Δ_R colors.

▶ **Lemma 37.** *Any finite binary relation R is Δ_R-edge-colorable.*

Proof.
Instead of applying König's theorem to R', we will adapt its standard proof directly to R.
Let $n \geq 0$, $R = \{(s_1, t_1), \ldots, (s_n, t_n)\}$ and $k = \Delta_R$.
By induction on $0 \leq i \leq n$, let us construct an edge-coloring c_i of $R_i = \{(s_1, t_1), \ldots, (s_i, t_i)\}$ in $[k] = \{1, \ldots, k\}$.
For $i = 0$, the empty function c_0 is an edge-coloring of $R_0 = \emptyset$.
Let $0 \leq i < n$ and c_i be an edge-coloring of R_i in $[k]$. We denote by $s = s_{i+1}$, $t = t_{i+1}$,

$$O_s = \{ c_i(s, q) \mid (s, q) \in \mathrm{Dom}(c_i) \} \quad \text{and} \quad I_t = \{ c_i(p, t) \mid (p, t) \in \mathrm{Dom}(c_i) \}.$$

We distinguish the two complementary cases below.
Case 1: $O_s \cup I_t \subset \{1, \ldots, k\}$. We extend c_i to the edge-coloring c_{i+1} of R_{i+1} by defining

$$c_{i+1}(s, t) = \min\{ j \mid j \notin O_s \cup I_t \}.$$

Case 2: $O_s \cup I_t = \{1, \ldots, k\}$.
As $(s, t) \notin \mathrm{Dom}(c_i)$, we have $|O_s| < k$ and $|I_t| < k$. So $\neg(O_s \subseteq I_t)$ and $\neg(I_t \subseteq O_s)$.
Thus there exists $a \in O_s - I_t$ and $b \in I_t - O_s$. In particular $a \neq b$.
As R^{c_i} is deterministic and co-deterministic, there are unique s' and t' such that $c_i(s, t') = a$ and $c_i(s', t) = b$. This is illustrated as follows:

$$t' \overset{a}{\longleftarrow} s \longrightarrow t \overset{b}{\longleftarrow} s'$$

where the labeled relation $\overset{x}{\longrightarrow}$ for any $x \in \{1, \ldots, k\}$ is defined by

$$\overset{x}{\longrightarrow} = \{ (p, q) \in \mathrm{Dom}(c_i) \mid c_i(p, q) = x \}.$$

Let us consider the chain in R^{c_i} of maximal length and of the form

$$s \overset{a}{\longrightarrow} \overset{b}{\longleftarrow} \overset{a}{\longrightarrow} \overset{b}{\longleftarrow} \ldots$$

As $b \notin O_s$ and R^{c_i} is deterministic and co-deterministic, this chain is finite.
As $a \notin I_t$, $s' \overset{b}{\longrightarrow} t$ is not an edge of this chain.
We define another edge-labeling c of R_i by exchanging the labels a and b for the edges of the chain: for any $(p, q) \in \mathrm{Dom}(c_i)$,

$$c(p, q) = \begin{cases} b & \text{if } p \overset{a}{\longrightarrow} q \text{ is an edge of the chain} \\ a & \text{if } p \overset{b}{\longrightarrow} q \text{ is an edge of the chain} \\ c_i(p, q) & \text{otherwise.} \end{cases}$$

Thus R_i^c remains deterministic and co-deterministic *i.e.* c is an edge-coloring of R_i with $c(s, t') = c(s', t) = b$. It remains to add $c(s, t) = a$ to get an edge-coloring of R_{i+1} in $[k]$. ◀

Under the assumption of the axiom of choice (actually under the weaker assumption of the ultrafilter axiom), let us generalize Lemma 37. For this we use a coloring on the vertices instead on the edges. A *vertex-coloring* of $R \subseteq V \times V$ is a mapping $c : V \longrightarrow A$ such that $c(s) \neq c(t)$ for any $(s, t) \in R$, and in that case, we say that R is $|c(V)|$-*vertex-colorable.* Note that a relation with a reflexive pair has no vertex-coloring.
The *dual* of R is the binary relation $D(R)$ on R defined by

$$D(R) = \{ ((r, s), (r, t)) \mid (r, s), (r, t) \in R \wedge s \neq t \}$$
$$\cup \{ ((s, r), (t, r)) \mid (s, r), (t, r) \in R \wedge s \neq t \}.$$

For any edge-labeling c of R,

$$c \text{ is an edge-coloring of } R \iff c \text{ is a vertex-coloring of } D(R).$$

Thus by Lemma 37 and for any finite relation R, $D(R)$ has a Δ_R-vertex-coloring. We can apply the compactness theorem [4] to extend Lemma 37 to any relation of bounded degree.

▶ **Proposition 38.** *In ZFC set theory, any bounded degree relation R has a Δ_R-edge-coloring.*

Proof.
Let k be a positive integer and R be a binary relation on a set V with $\Delta_R = k$.
It is equivalent to show that $D(R)$ is k-vertex-colorable, or that $D(R)$ is vertex-colorable using at most k colors.
By de Bruijn-Erdös theorem [4], it is equivalent to check that any finite subset of $D(R)$ is vertex-colorable with at most k colors. Let $S \subseteq D(R)$ with S finite. Let

$$P \;=\; \{\ s \in V \mid \exists\, t\ (s,t) \in V_S \wedge (t,s) \in V_S\ \}$$

and $R_{|P} \;=\; R \cap P{\times}P$ the induced relation of R by P. So $S \subseteq D(R_{|P})$ which is finite.
By Lemma 37, $R_{|P}$ is edge-colorable using $\Delta_{R_{|P}} \leq \Delta_R = k$ colors.
Finally $D(R_{|P})$ hence S are vertex-colorable using at most k colors. ◀

We now want to color a regular relation in a complete way. First we present a general way to extend an injection into a bijection avoiding given sets.

▶ **Lemma 39.** *Let X, Y be equipotent well orderable infinite sets.*
Let an injection $p : P \longrightarrow Y$ for some subset P of X with $|P| < |X|$.
Let a sequence $(P_x)_{x \in X-P}$ of subsets of Y with $|P_x| < |Y|$ and such that

$$|\{\ x \in X - P \mid y \in P_x\ \}| \;<\; |X| \ \text{ for every } y \in Y - p(P).$$

We can extend p into a bijection $X \longrightarrow Y$ such that $p(x) \notin P_x$ for every $x \in X - P$.

Proof.
Let $<_X$ be an initial well-ordering of X: $\forall\, x \in X,\ |\{\ x' \in X \mid x' < x\ \}| \;<\; |X|$.
Let $<_Y$ be an initial well-ordering of Y.
We define p on $X - P$ by transfinite induction. Let $x \in X - P$.
Let us define $p(x)$ knowing $p(x')$ for any $x' <_X x$.
As $|P_x|\,,\, |P| \;<\; |X| \;=\; |Y|$ with X infinite, the following subset of Y

$$Q_x \;=\; p(P) \,\cup\, P_x \,\cup\, \{\ p(x') \mid x' <_X x\ \}$$

is of cardinal $|Q_x| < |Y|$. So we can define

$$p(x) \;=\; \min\nolimits_{<_Y}(Y - Q_x).$$

Thus p is injective and $p(x) \notin P_x$ for every $x \in X - P$.
Let us check that p is surjective. Assume that $\mathrm{Im}(p) \neq Y$. Let

$$\beta \;=\; \min\nolimits_{<_Y}(Y - \mathrm{Im}(p)).$$

As $|\{\ x \in X - P \mid \beta \in P_x\ \}| \;<\; |X|$, we can define

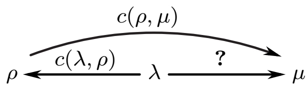
$$\alpha \;=\; \min\{\ x \in X - P \mid\ \beta \notin P_x \wedge \beta <_Y p(x)\ \}.$$

So $\alpha \in X - P$ and $\beta <_Y p(\alpha)$.
As $\beta \notin \mathrm{Im}(p)$ and $\beta \notin P_\alpha$, we have $\beta \notin Q_\alpha$ hence $p(\alpha) \leq_Y \beta$ which is a contradiction. ◀

A *complete* edge-coloring of a regular relation R is an edge-coloring c of R such that R^c is source-complete and target-complete. Under the assumption of the axiom of choice, we can color in a complete way any regular relation.

▶ **Proposition 40.** *In ZFC set theory, any regular relation R has a complete Δ_R-edge-coloring.*

Proof.
Let R be a regular relation on a set V.
We distinguish the two complementary cases below.
Case 1: $\Delta_R < \aleph_0$.
By Proposition 38, R has a Δ_R-edge-coloring c.
So R^c is a deterministic and co-deterministic graph with $|A_{R^c}| = \Delta_R = \Delta_{R^c}$.
Thus R^c is source-complete and target-complete *i.e.* c is a complete edge-coloring.
Case 2: $\Delta_R \geq \aleph_0$.
Under AC, it suffices to show the existence of a complete Δ_R-edge-coloring for R connected.
Under AC and having R connected, we have $|V_R| = |\Delta_R|$.
Thanks to AC, let us consider an initial well-ordering $<$ of V_R.
By transfinite induction, let us define a complete Δ_R-edge-coloring c of R. Let $\lambda \in V_R$.
Let us define c on $\{ (\lambda, \mu) \in R \mid \lambda \leq \mu \} \cup \{ (\mu, \lambda) \in R \mid \lambda \leq \mu \}$
knowing c on $\{ (\mu, \nu) \mid \mu < \lambda \vee \nu < \lambda \}$.
First, we define $c(\lambda, \mu)$ for any $(\lambda, \mu) \in R$ with $\lambda \leq \mu$.
This is illustrated below for $\rho < \lambda \leq \mu$.



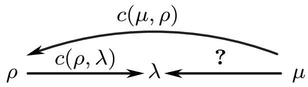
As R is regular, $R(\lambda) = \{ \mu \mid (\lambda, \mu) \in R \}$ has cardinality $|R(\lambda)| = \Delta_R$. Let $P = \{ \rho \in R(\lambda) \mid \rho < \lambda \}$ and $p : P \longrightarrow \Delta_R$ with $p(\rho) = c(\lambda, \rho)$ for any $\rho \in P$.
For any $\mu \in R(\lambda) - P$, we define

$$P_\mu = \{ c(\rho, \mu) \mid (\rho, \mu) \in R \wedge \rho < \lambda \}.$$

By Lemma 39, we can extend p into a bijection $R(\lambda) \longrightarrow \Delta_R$ such that $p(\mu) \notin P_\mu$ for any $\mu \in R(\lambda) - P$. It remains to define

$$c(\lambda, \mu) = p(\mu) \text{ for any } (\lambda, \mu) \in R \text{ and } \lambda \leq \mu.$$

Similarly, we define $c(\mu, \lambda)$ for any $(\mu, \lambda) \in R$ with $\lambda < \mu$.
This is illustrated below for $\rho < \lambda < \mu$.



◀

We say that a graph G is *regular* if $\longrightarrow_G$ is regular. For instance $\{s \xrightarrow{a} t\, ,\, t \xrightarrow{a} s\, ,\, t \xrightarrow{b} s\}$ is a regular graph. We also say that G is *co-regular* if its unlabeled complement is regular: $\not\longrightarrow_G$ is a regular relation on V_G.
Under the assumption of the axiom of choice, we can restrict Theorem 32 to obtain a characterization of the generalized Cayley graphs of quasigroups.

▶ **Theorem 41.** *In ZFC set theory, the following graphs define the same family*:
a) *the generalized Cayley graphs of quasigroups (resp. with a right identity),*
b) *the generalized Cayley graphs of quasigroups with a left identity (resp. an identity),*
c) *the simple, deterministic, co-deterministic, co-regular, source-complete, target-complete (resp. and loop-complete) graphs.*

Proof.
b) $\Longrightarrow$ **a)**: immediate.

a) $\Longrightarrow$ **c)**: let $G = \mathcal{C}[\![M,Q]\!]$ for some quasigroup M and $Q \subseteq M$.
By Proposition 35, $G = \mathcal{C}[\![M]\!]^{|[\![Q]\!]}$ remains simple, deterministic and co-deterministic, source and target-complete.
For any $s \in M$, $\delta^+_{\not\to_G}(s) = \delta^-_{\not\to_G}(s) = |M - Q|$ hence G is co-regular.
If in addition M has a right identity then, by Fact 22, G is loop-complete.

c) $\Longrightarrow$ **b)**: let G be a graph which is simple, deterministic and co-deterministic, source and target-complete. So G is regular with $\Delta_G = |A_G|$.
If G is without loop (hence G is loop-complete) then we take a new label $a \in A - A_G$ and we define $G' = G \cup \{ s \xrightarrow{a} s \mid s \in V_G \}$. If G has at least one loop, we put $G' = G$.
Moreover, suppose also that G is co-regular. By definition, the complement relation of G'

$$S = \{ (s,t) \mid s,t \in V_G \wedge \{s\}{\times}A_G{\times}\{t\} \cap G' = \emptyset \}$$

is a regular relation on V_G.
By Proposition 40, S has a complete Δ_S-edge-coloring c *i.e.* S^c is a deterministic, co-deterministic, source and target-complete graph. By definition, S^c is also simple.
Furthermore we can assume that $A_G \cap A_{S^c} = \emptyset$.
Let $H = G' \cup S^c$. Thus H is source and target-complete. It is also complete, simple, deterministic and co-deterministic. Furthermore for G loop-complete, H is loop-complete.
Let r be a vertex of G. By Proposition 34, $(V_G, \times_r)$ is a quasigroup for the edge-operation $\times_r$ of H. By Proposition 23, r is a left-identity (resp. is an identity) and $H = \mathcal{C}[\![V_G\,, \longrightarrow_H(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_H s$. By label restriction to A_G, $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ is a generalized Cayley graph of $(V_G, \times_r)$. ◀

The co-regularity in Theorem 41 can not be removed. For instance, the following graph:

$$\begin{aligned} \text{PlusMinus} \quad = \quad & \{ i+2mj \xrightarrow{j} i+(2m+1)j \mid m \geq 0 \wedge 0 \leq i < j \} \cup \{ i \xrightarrow{0} i \mid i \geq 0 \} \\ \cup \quad & \{ i+(2m+1)j \xrightarrow{j} i+2mj \mid m \geq 0 \wedge 0 \leq i < j \} \end{aligned}$$

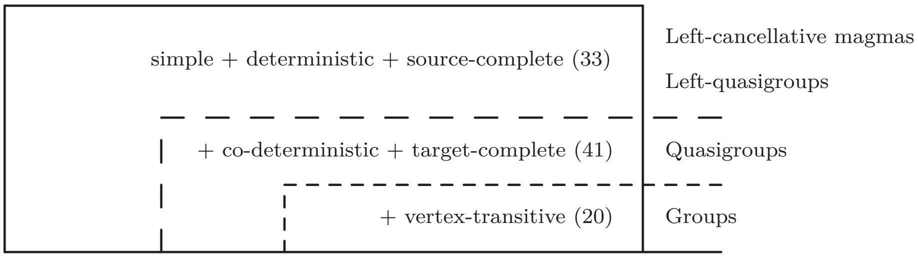
is deterministic, co-deterministic, simple, source-complete and target-complete. Furthermore it is not complete: there is no edge between 1 and 2 and more generally between $i+(2m+1)j$ and $i+(2m+2)j$ for any $m \geq 0$ and $0 \leq i < j$. Finally it is 0-complete: $0 \xrightarrow{j} j$ for any $j \geq 0$. By Proposition 35, PlusMinus is not a generalized Cayley graph of a quasigroup.
By Corollary 36, the co-regularity in Theorem 41 can be removed for the graphs of bounded degree which corresponds by Fact 30 to the finitely labeled graphs G when G and G^{-1} are deterministic and source-complete.

▶ **Corollary 42.** *In ZFC set theory, a finitely labeled graph is a generalized Cayley graph of a quasigroup iff it is deterministic, co-deterministic, simple, source and target-complete.*

For instance the following graph of all the cycles:

$$Cycles = \{ (m,n) \xrightarrow{a} (m, n+1\,(\mathrm{mod}\, m)) \mid m > n \geq 0 \}$$

is by Corollary 42 a generalized Cayley graph of a quasigroup.
Let us summarize the characterizations obtained in ZFC theory for the finitely labeled generalized Cayley graphs.



For all the generalized Cayley graphs (not necessarily finitely labeled), we need the co-out-regularity for the left-quasigroups, and the co-regularity for the quasigroups.

8 Decidability results

We have given graph-theoretic characterizations of generalized Cayley graphs of various basic algebraic structures. These characterizations are adapted to decide whether a graph G is a generalized Cayley graph, and if so, we got

$$G \;=\; \mathcal{C}[\![V_G\,,\,\longrightarrow_G(r)]\!] \quad \text{with} \quad [\![s]\!] = a \;\text{ for any }\; r \overset{a}{\longrightarrow}_G s$$

for the operation on V_G which is either the path-operation $*_r$ with r a root, or the chain-operation $\overline{*}_r$, or the extended chain-operation $\overline{*}_P$ with P a representative set of $\mathrm{Comp}(G)$, or the edge-operation ${}_{<G>}\!\times_r$ for some completion $<G>$ of G and for any vertex r.
We will show the effectiveness of these characterizations and their associated operations for a general family of infinite graphs. We restrict to the family of end-regular graphs of finite degree [13] which admits an external characterization by finite decomposition by distance which allows to decide the isomorphism problem, and an internal characterization as suffix graphs of word rewriting systems which have a decidable monadic second-order theory.

8.1 End-regular graphs

A *marked graph* is a couple (G,P) of a graph G with a vertex subset $P \subseteq V_G$. We extend the graph isomorphism to the marked graphs: $(G,P) \equiv (G',P')$ if $G \equiv_g G'$ for some isomorphism g such that $g(P) = P'$, and we also write $(G,P) \equiv_g (G',P')$.
Let G be a graph. The *frontier* $\mathrm{Fr}_G(H)$ of $H \subseteq G$ is the set of vertices common to H and $G - H$:

$$\mathrm{Fr}_G(H) \;=\; V_H \cap V_{G-H}$$

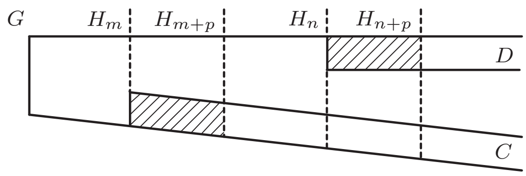
and we denote by $\mathrm{End}_G(H)$ the set of *ends* obtained by removing H in G:

$$\mathrm{End}_G(H) \;=\; \{\; (C, \mathrm{Fr}_G(C)) \mid C \in \mathrm{Comp}(G-H) \;\}.$$

We say that G is *end-regular* if there exists an increasing sequence $H_0 \subseteq \ldots \subseteq H_n \subseteq \ldots$ of finite subgraphs H_n of G such that

$$G \;=\; \bigcup_{n\geq 0} H_n \quad \text{and} \quad \bigcup_{n\geq 0} \mathrm{End}_G(H_n) \text{ is of finite index for } \equiv$$

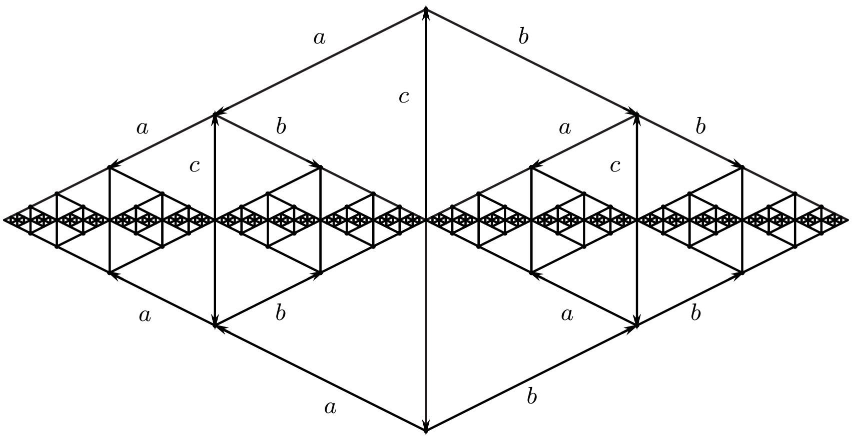
and two isomorphic ends with nonempty frontiers have the same decomposition:
for any $m,n \geq 0$ and any $(C,P) \in \mathrm{End}_G(H_m)$ and $(D,Q) \in \mathrm{End}_G(H_n)$ with $P,Q \neq \emptyset$,
if $(C,P) \equiv_g (D,Q)$ then $C \cap H_{m+p} \equiv_{g'} D \cap H_{n+p}$ for every $p \geq 0$
with g' the restriction of g to the vertices of $C \cap H_{m+p}$. This is illustrated as follows:



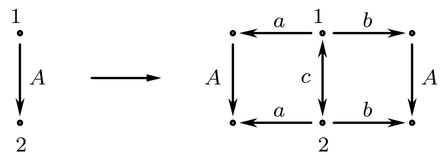
Note that any end-regular graph is finitely labeled and of finite or countable vertex set. Furthermore any end-regular graph of finite degree is of bounded degree. Moreover, every end-regular graph has only a finite number of non-isomorphic connected components.
Any end-regular graph can be finitely presented by a deterministic graph grammar [6].
Any finite graph is end-regular. Any regular tree (having a finite number of non isomorphic subtrees) is end-regular. Except for the quater-grid and the graph *Cycles*, all other graphs in this article are end-regular. The following infinite graph:

$$\begin{aligned}\Xi \quad &= \quad \{\ ui \xrightarrow{x} uxi \mid u \in \{a,b\}^* \wedge x \in \{a,b\} \wedge i \in \{0,1\}\ \} \\ &\cup \quad \{\ ui \xrightarrow{c} u(1-i) \mid u \in \{a,b\}^* \wedge i \in \{0,1\}\ \}\end{aligned}$$

of vertex set $\{a,b\}^*.\{0,1\}$ is represented below.



This graph is formed by two disjoint source-complete $\{a,b\}$-trees whose every node of a tree is connected by a c-edge to the corresponding node of the other tree. The graph Ξ is end-regular since it is generated by the graph grammar [6] reduced to this unique rule:



Note that Ξ is rooted and of bounded degree, simple, deterministic and co-deterministic, source-complete but not target-complete, forward vertex-transitive but not vertex-transitive. By Theorem 8, Ξ is a Cayley graph of a cancellative monoid. Precisely $\Xi = \mathcal{C}[\![V_\Xi, \{a0, b0, 1\}]\!]$ with $[\![a0]\!] = a$, $[\![b0]\!] = b$, $[\![1]\!] = c$ for the cancellative monoid $(V_\Xi, *_0)$ with the path-operation $*_0$ defined for any $u, v \in \{a,b\}^*$ and $i, j \in \{0,1\}$ by

$$ui *_0 vj = uvk \quad \text{with} \quad k = i + j \pmod 2.$$

By Theorem 33, Ξ is also a generalized Cayley graph of a left-quasigroup with an identity, namely $(\{a,b\}^*.\{0,1\}, \cdot)$ for $\cdot$ the edge-operation $\times_0$ of a completion $\ll\!\Xi\!\gg$ that we can define for any $u \in \{a,b\}^*$, $x \in \{a,b\}$, $i \in \{0,1\}$ by

$$ui \cdot x0 = uxi \quad ; \quad ui \cdot uxi = x0 \ \text{ for } ui \neq x0 \quad ; \quad x0 \cdot xx0 = 0$$

$$ui \cdot 0 = ui \;\; ; \;\; ui \cdot 1 = u(1-i) \;\; ; \;\; ui \cdot u(1-i) = 1$$
$$ui \cdot v = v \;\text{ otherwise.}$$

The end-regularity of a graph can also be expressed on the vertices. A graph G is *vertex-end-regular* if $V_G = \bigcup_{n\geq 0} V_n$ with

$$V_0 \subseteq \ldots \subseteq V_n \subseteq \ldots \text{ finite and } \bigcup_{n\geq 0} \mathrm{End}_G(G_{|V_n}) \text{ is of finite index for } \equiv .$$

This notion of vertex-end-regularity corresponds to the (edge-)end-regularity.

▶ **Lemma 43.** *A graph is end-regular if and only if it is vertex-end-regular.*

Proof.
$\Longleftarrow$: $G_n = G_{|V_n}$ suits.
$\Longrightarrow$: $V_n = V_{G_n}$ suits since $G - G_{|V_n} = G - \big(G_n \cup \bigcup_{K\in \mathrm{Comp}(G-G_n)} G_{|\mathrm{Fr}_G(K)}\big)$. ◀

We say that a graph G is *end-regular by distance* from a vertex r if it is vertex-end-regular for the sequence defined by $V_n = \{\ s \mid \mathrm{d}_G(r,s) \leq n\ \}$ for any $n \geq 0$. In that case, G is connected and of finite degree. Furthermore the sequence $(H_n)_{n\geq 0}$ defined by

$$H_n = G_{|V_n} = \{\ (s,a,t) \in G \mid \mathrm{d}_G(r,s) \leq n \wedge \mathrm{d}_G(r,t) \leq n\ \}$$

is a finite decomposition of G. The regularity by distance is a normal form for the connected end-regular graphs of finite degree [6].

▶ **Proposition 44.** *For any connected graph G of finite degree,*

G is end-regular $\iff$ G is end-regular by distance from some vertex
$\iff$ G is end-regular by distance from any vertex.

This normalization of the regularity by distance implies that for any end-regular graph G of finite degree, the isomorphism problem is decidable: from any finite decomposition of G, we can decide whether $s \simeq_G t$ by comparing by distance G from s with G from t [6].

▶ **Corollary 45.** *For any end-regular graph G of finite degree, $\simeq_G$ is decidable.*

The representation of an end-regular graph G by a graph grammar is an *external representation* of G, namely which is up to isomorphism: the vertices of G are not taken into account. To recall decidable logical properties on end-regular graphs, we present an *internal representation* of these graphs by naming their vertices by words.

8.2 Suffix recognizable graphs

Another way to describe the end-regular graphs of finite degree is through rewriting systems. A *labeled word rewriting system* R over an alphabet N is a finite A-graph of vertex set $V_R \subset N^*$ *i.e.* $R \subset N^* \times A \times N^*$ and R is finite. Each edge $u \xrightarrow{a}_R v$ is a *rule* labeled by a, of left hand side u and right hand side v. The *suffix graph* of R is the graph

$$N^*.R = \{\ wu \xrightarrow{a} wv \mid (u,a,v) \in R \wedge w \in N^*\ \}.$$

For instance let us consider the rewriting system R over $N = \{a,b,0,1\}$ defined by

$$0 \xrightarrow{a} a0 \qquad 1 \xrightarrow{a} a1 \qquad 0 \xrightarrow{b} b0 \qquad 1 \xrightarrow{b} b1 \qquad 0 \xrightarrow{c} 1 \qquad 1 \xrightarrow{c} 0$$

The connected component of the suffix graph $N^*.R$ of vertex 0 is equal to the graph Ξ. These suffix graphs give an internal representation of the end-regular graphs of finite degree [6].

▶ **Proposition 46.** *A connected graph of finite degree is end-regular if and only if it is isomorphic to a connected component of a suffix graph.*

Any suffix graph $N^*.R$ can be obtained by a first order interpretation in the source-complete N-tree $T_N = \{ u \xrightarrow{a} ua \mid u \in N^* \wedge a \in N \}$ *i.e.*

$$N^*.R = \{ u \xrightarrow{a} v \mid T_N \models \phi_a(u,v) \}$$

where for any $a \in A_R$, ϕ_a is the following first order formula

$$\phi_a(x,y) : \bigvee_{(u,a,v)\in R} \exists\, z\ (z \xrightarrow{u} x \wedge z \xrightarrow{v} y)$$

and the path relation $x \xrightarrow{a_1 \ldots a_n} y$ for $n \geq 0$ and $a_1, \ldots, a_n \in A_R$ can be expressed by the first order formula

$$\exists\, z_0, \ldots, z_n\ (z_0 \xrightarrow{a_1} z_1 \wedge \ldots \wedge z_{n-1} \xrightarrow{a_n} z_n \wedge z_0 = x \wedge z_n = y).$$

As the existence of a chain between two vertices can be expressed by a monadic (second order) formula, any connected component of $N^*.R$ can be obtained by a monadic interpretation in T_N. As T_N has a decidable monadic theory [14] and by Proposition 46, any connected end-regular graph has a decidable monadic theory. This is generalized [1, 12] to the *suffix recognizable graphs* over N which are the graphs of the form

$$\bigcup_{i=1}^{n} W_i(U_i \xrightarrow{a_i} V_i) \text{ where } n \geq 0 \text{ and } U_1, V_1, W_1, \ldots, U_n, V_n, W_n \in \mathrm{Rec}(N^*)$$

for $\mathrm{Rec}(N^*)$ the family of recognizable (regular) languages over N. These graphs are exactly the ones we obtain by monadic interpretations in the N-tree [1, 12].

▶ **Proposition 47.** *The suffix recognizable graphs over N are the graphs obtained by monadic interpretations in T_N hence have a decidable monadic second order theory.*

In particular, we can decide whether a suffix recognizable graph G is rooted or is connected. For any vertex u of G, the subset of vertices connected to u is an effective regular language [6]. It follows that we can extract a regular set of representents of the connected components. Precisely let $<_{ll}$ be the length-lexicographic order extending a linear order on N.

▶ **Lemma 48.** *For any suffix recognizable graph G over N,*
the set $\{ \min_{<_{ll}}(V_C) \mid C \in \mathrm{Comp}(G) \}$ is an effective regular language.

We can also decide first order properties like the simplicity which can be expressed by the following first order formula:

$$\forall\, x, y \bigwedge_a (x \xrightarrow{a} y \implies \neg \bigvee_{b\neq a} x \xrightarrow{b} y)$$

and this the same for the properties of being deterministic, co-deterministic, source-complete, target-complete, and loop-complete.
We still have to consider the decidability of the forward vertex-transitivity and the vertex-transitivity of end-regular graphs of finite degree. The suffix recognizable graphs form a strict extension of end-regular graphs that coincide for graphs of finite degree [6].

▶ **Proposition 49.** *Any end-regular graph is isomorphic to a suffix recognizable graph. Any suffix recognizable graph of finite degree is end-regular.*

Note that a suffix recognizable graph over N of finite degree is of the form

$$\bigcup_{i=1}^{n} W_i(u_i \xrightarrow{a_i} v_i) \text{ where } n \geq 0,\ u_1, v_1, \ldots, u_n, v_n \in N^* \text{ and } W_1, \ldots, W_n \in \mathrm{Rec}(N^*).$$

By Proposition 49, these graphs constitute an internal representation of end-regular graphs of finite degree. Note also that $\{ A^m \xrightarrow{a} A^{m+n} \mid m, n \geq 0 \} = A^*(\varepsilon \xrightarrow{a} A^*)$ is a suffix recognizable graph which is not an end-regular graph.
By monadic interpretation (or by a simple saturation method on graph grammars), the family of end-regular graphs is closed under accessibility.

▶ **Corollary 50.** *For any end-regular graph G and any vertex r, $G_{\downarrow r}$ is end-regular.*

By Corollaries 45 and 50, we can decide whether $s \downarrow_G t$ for G end-regular of finite degree.

▶ **Corollary 51.** *For any end-regular graph G of finite degree, $\downarrow_G$ is decidable.*

The forward vertex-transitivity of rooted graphs can be reduced to the accessible-isomorphism of a root with its successors.

▶ **Lemma 52.** *A graph G of root r is forward vertex-transitive iff $r \downarrow_G s$ for any $r \longrightarrow_G s$.*

Proof.
Let G be a graph with a root r such that $r \downarrow_G s$ for any $r \longrightarrow_G s$.
Let us check that G is forward vertex-transitive *i.e.* $r \downarrow_G s$ for any $r \longrightarrow_G^* s$.
The proof is done by induction on $n \geq 0$ for $r \longrightarrow_G^n s$.
For $n = 0$, we have $r = s$. For $n > 0$, let t be a vertex such that $r \longrightarrow_G^{n-1} t \longrightarrow_G s$.
By induction hypothesis, we have $r \downarrow_G t$ *i.e.* $f(r) = t$ for some isomorphism f from $G_{\downarrow r}$ to $G_{\downarrow t}$. As $t \longrightarrow_G s$, there exists r' such that $r \longrightarrow_G r'$ and $f(r') = s$. So $r' \downarrow_G s$.
By hypothesis $r \downarrow_G r'$. By transitivity of $\downarrow_G$, we get $r \downarrow_G s$. ◀

Let us transpose Lemma 52 to the vertex-transitive graphs. The forward vertex-transitivity of connected graphs can be reduced to the isomorphism of a vertex with its adjacent vertices.

▶ **Lemma 53.** *A connected graph with a vertex r is vertex-transitive if and only if*

$$r \simeq_G s \text{ for any } r \longrightarrow_{G \cup G^{-1}} s.$$

Let us apply Lemma 52 and 53 with Corollaries 45 and 51.

▶ **Corollary 54.** *We can decide whether a rooted (resp. any) end-regular graph of finite degree is forward vertex-transitive (resp. vertex-transitive).*

In this corollary, we do not need the connected condition for the vertex-transitivity since any end-regular graph has only a finite number of non-isomorphic connected components. We can establish the effectiveness of previous Cayley graph characterizations for the regular graphs of finite degree. This decidability result does not require the assumption of the axiom of choice.

▶ **Theorem 55.** *We can decide whether a suffix recognizable graph G of finite degree is a Cayley graph of a left-cancellative monoid, of a cancellative monoid, of a group, and whether G is a generalized Cayley graph of a left-quasigroup, of a quasigroup, of a group.*
In the affirmative, $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ where $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$ and with a computable suitable binary operation on V_G and vertex r.

Proof.
i) *Cayley graph of a left-cancellative or cancellative monoid.*
By Proposition 47 and Corollary 54, we can decide whether G is rooted, simple, forward vertex-transitive, deterministic (resp. and co-deterministic) *i.e.* by Theorem 7 (resp. Theorem 8) whether G is a Cayley graph of a left-cancellative monoid (resp. cancellative monoid).
In the affirmative and by Proposition 6, $(V_G, *_r)$ is a left-cancellative (resp. cancellative) monoid where r is a root of G, and $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ where $[\![s]\!] = a$ for any $r \xrightarrow{a}_G s$.
It remains to check that the path-operation $*_r$ is computable.
We just need that G is deterministic and forward vertex-transitive.
Let $s, t \in V_G$. By Proposition 6, $s *_r t$ is a vertex that we can determine. The label set $\mathrm{L}_G(r,t) = \{\ u \in A_G^* \mid r \xrightarrow{u}_G t\ \}$ of the paths from r to t is an effective non empty context-free language [6]: we can construct a pushdown automaton recognizing $\mathrm{L}_G(r,t)$

hence we can compute a word $u \in L_G(r,t)$.
Thus $s *_r t$ is the target of the path from s labeled by u *i.e.* $s \xrightarrow{u}_G s *_r t$.

ii) *Cayley graph of a group.*
By Proposition 47 and Corollary 54, we can decide whether G is connected, vertex-transitive, deterministic and co-deterministic *i.e.* by Theorem 17, whether G is a Cayley graph of a group. In the affirmative and by Proposition 16, it remains to check that the chain-operation $\overline{*}_r$ is computable where r is any vertex of G. We have seen that ${}_G\overline{*}_r = {}_{\overline{G}}*_r$. As r is a root of $\overline{G}$ which is deterministic and forward vertex-transitive and by (i), $\overline{*}_r$ is computable.

iii) *Generalized Cayley graph of a left-quasigroup.*
As G has a decidable first order theory, we can decide whether G is simple, deterministic, source-complete *i.e.* by Theorem 33, whether G is a generalized Cayley graph of a left-quasigroup. In the affirmative and by Propositions 23 and 28, it remains to check that the edge-operation ${}_{\ll G \gg}\times_r$ is computable where r is any vertex and $\ll G \gg$ is the completion of G defined in the proof of Theorem 33. This edge-operation that we denote by $\cdot$ has been defined for any $s,t \in V_G$ by

$$\begin{aligned} s &\xrightarrow{a}_G s\cdot t && \text{for } r \xrightarrow{a}_G t \\ r &\xrightarrow{a}_G s\cdot t && \text{for } s \xrightarrow{b}_G t \text{ and } (a,b) \in \overline{\ell_s} - \ell_s \\ s\cdot t &= t && \text{for } t \in V_G - (\longrightarrow_G(r) \cup \longrightarrow_G(s)) \end{aligned}$$

where $\ell_s = \{ (a,b) \mid \exists\, t\, (r \xrightarrow{a}_G t \wedge s \xrightarrow{b}_G t) \}$ and to each injective function $\ell : A_G \longrightarrow A_G$, we have associated a permutation $\overline{\ell}$ on A_G extending ℓ. Thus $\cdot$ is computable.
Moreover we can check that $\cdot$ is an effective ternary suffix-recognizable relation.

iv) *Generalized Cayley graph of a quasigroup.*
As G has a decidable first order theory, we can decide whether G is simple, deterministic, co-deterministic, source and target-complete *i.e.* by Corollary 42 and under the assumption of the axiom of choice, whether G is a generalized Cayley graph of a quasigroup.
Assume that G is simple, deterministic, co-deterministic, source and target-complete.
We do not need the assumption of the axiom of choice: we will define a computable quasigroup operation on V_G.
Let r be any vertex of G. By Proposition 34 (and as for the proof of Theorem 41), it is sufficient to define a complete graph $H \supseteq G$ of vertex set V_G with the same properties as G such that ${}_H\times_r$ is computable.
When V_G is finite and by Lemma 37, such a completion H is effective hence ${}_H\times_r$ is computable. We have to deal with the case where V_G is infinite.
The set V_G is a regular language over some finite alphabet that we order totally.
For any integer $i \geq 0$, we can compute the i-th vertex v_i by length-lexicographic order.
We can assume that the finite label set A_G is disjoint of $\mathbb{N}$. Let us define a mapping T from $\mathbb{N}\times\mathbb{N}$ into $A_G \cup \mathbb{N}$ such that the completion of G is $H = \{ v_i \xrightarrow{T(i,j)} v_j \mid i,j \geq 0 \}$.
By Lemma 39 and Proposition 40, T is defined for any $i,j \geq 0$ by

$$T(i,j) = \begin{cases} a & \text{if } (v_i,a,v_j) \in G \\ \min\big(\mathbb{N} - \{T(i,0),\dots,T(i,j-1),T(0,j),\dots,T(i-1,j)\}\big) & \text{otherwise.} \end{cases}$$

Thus T is a Latin square: for each $i \in \mathbb{N}$ and $a \in A_G \cup \mathbb{N}$, there are unique $j,k \in \mathbb{N}$ such that $T(i,j) = a = T(k,i)$. Therefore T is computable hence also ${}_H\times_r$.

v) *Generalized Cayley graph of a group.*
As G has a decidable first order theory and by Corollary 54, we can decide whether G is simple, vertex-transitive, deterministic and co-deterministic *i.e.* by Theorem 20 and under the assumption of the axiom of choice, whether G is a generalized Cayley graph of a group.

Assume that G is simple, vertex-transitive, deterministic and co-deterministic.
We do not need the hypothesis of the axiom of choice: we will define a computable extended chain-operation.
Let N be the alphabet of the words of G. We take a linear order on N. By Lemma 48,

$$P \;=\; \{\; \min_{<_{ll}}(V_C) \mid C \in \mathrm{Comp}(G) \;\}$$

is an effective regular language.
Let $\mathrm{Rk}(u) \;=\; |\{\; v \in P \mid v <_{ll} u \;\}|$ be the rank of $u \in P$ according to $<_{ll}$ *i.e.* u is the $\mathrm{Rk}(u)$-word in P by $<_{ll}$. We have a group $(P,+)$ for $u+v$ defined for any $u,v \in P$ by

$$\mathrm{Rk}(u+v) \;=\; \mathrm{Rk}(u) + \mathrm{Rk}(v) \pmod{|P|} \quad \text{for } P \text{ finite,}$$

and for P countable, we consider the bijection $\|\;\;\| : P \longrightarrow \mathbb{Z}$ defined for any $u \in P$ by

$$\|\, u \,\| \;=\; \begin{cases} \frac{\mathrm{Rk}(u)}{2} & \text{if } \mathrm{Rk}(u) \text{ is even,} \\ -\frac{\mathrm{Rk}(u)+1}{2} & \text{if } \mathrm{Rk}(u) \text{ is odd} \end{cases}$$

and we define $u+v \in P$ by $\|\, u+v \,\| \;=\; \|\, u \,\| + \|\, v \,\|$.
Let $x \in V_G$. It is connected to $v_x \;=\; \min_{<_{ll}}(V_C)$ for $C \in \mathrm{Comp}(G)$ and $x \in V_C$.
The label set $\mathrm{L}_x \;=\; \{\; u \in (A_G \cup \overline{A_G})^* \mid v_x \overset{u}{\longrightarrow}_G x \;\}$ of the chains between v_x and x is an effective context-free language.
As G is vertex-transitive, the extended chain-operation $x \cdot y$ has been defined by

$$v_{x+y} \overset{\ell_x \ell_y}{\longrightarrow}_G x \cdot y \text{ with } \ell_x \in \mathrm{L}_x \text{ and } \ell_y \in \mathrm{L}_y\,.$$

Thus $\cdot$ is an effective group operation. ◀

We can consider the generalization of Theorem 55 to all the suffix-recognizable graphs (allowing vertices of infinite degree) which form the first level of a stack hierarchy for which any graph has a decidable monadic theory [5]. To extend Theorem 55 to any graph of this hierarchy, we have to decide on the forward vertex-transitivity (resp. vertex-transitivity) when these graphs are deterministic (resp. and co-deterministic).
The decidability result given by Theorem 55 is a first application of the Cayley graph characterizations presented in this paper. Another application is to describe differently a generalized Cayley graph by defining another operation on its vertex set. A trivial example is given by the quasigroup $(\mathbb{Z},-)$ of right identity 0. Its Cayley graph $\mathcal{C}(\mathbb{Z})$ is strongly connected, vertex-transitive, deterministic and co-deterministic. Its path-operation from 0 is $*_0 \;=\; +$ hence by Theorem 14, it is equal to $\mathcal{C}[\![\mathbb{Z}]\!]$ for the group $(\mathbb{Z},+)$ with $[\![n]\!] = -n$ for any $n \in \mathbb{Z}$. In particular for any $P \subseteq \mathbb{Z}$, the generalized Cayley graph $\mathcal{C}[\![\mathbb{Z},P]\!]$ of the quasigroup $(\mathbb{Z},-)$ is equal to the generalized Cayley graph $\mathcal{C}[\![\mathbb{Z},-P]\!]'$ of the group $(\mathbb{Z},+)$ with $[\![-n]\!]' = [\![n]\!]$ for any $n \in P$. Similarly by Theorem 33, any finitely labeled generalized Cayley graph G of a left-cancellative magma is a generalized Cayley graph of a left-quasigroup and its operation is computable for G end-regular.

9 Conclusion

We obtained simple graph-theoretic characterizations for Cayley graphs of elementary algebraic structures. We have shown the effectiveness of these characterizations for infinite graphs having a structural regularity. This is only a first approach in the structural description and its effectiveness of Cayley graphs of algebraic structures.

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