On Cayley graphs of basic algebraic structures

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Abstract

We present simple graph-theoretic characterizations of Cayley graphs for monoids, semigroups and groups. We extend these characterizations to commutative monoids, semilattices, and abelian groups.

Arthur Cayley was the first to define in 1854 [2] the notion of a group as well as the table of its operation known as the Cayley table. To describe the structure of a group $(G,\cdot)$, Cayley also introduced in 1878 [3] the concept of graph for G according to a generating subset S, namely the set of labeled oriented edges $g \xrightarrow{s} g{\cdot}s$ for every g of G and s of S. Such a graph, called Cayley graph, is directed and labeled in S (or an encoding of S by symbols called letters or colors). A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [5] : an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. Following a question asked by Hamkins in 2010 [4]: 'Which graphs are Cayley graphs?', we gave simple graph-theoretic characterizations of Cayley graphs for groups, as well as for left-cancellative and cancellative monoids [1]. Here we present a generalization of this last characterization to the Cayley graphs of monoids, and of semigroups. We also strengthen all these characterizations to commutative monoids, semilattices and abelian groups.

Generalized Cayley graphs of magmas

Let A be an arbitrary (finite or infinite) set. A directed A-*graph* (V,G) is defined by a set V of *vertices* and a subset $G \subseteq V{\times}A{\times}V$ of *edges*. Any edge $(s,a,t) \in G$ is from *source* s to *target* t with *label* a, and is also written by the *transition* $s \xrightarrow{a}_G t$ or directly $s \xrightarrow{a} t$ if G is clear from the context. The sources and targets of edges form the set $V_G = \{\ s \mid \exists\, a,t\ \ (s \xrightarrow{a} t \vee t \xrightarrow{a} s)\ \}$ of *non-isolated vertices* of G, and $A_G = \{\ a \mid \exists\, s,t\ \ (s \xrightarrow{a} t)\ \}$ is the set of its edge labels. We assume that any graph (V,G) is without isolated vertex: $V = V_G$ hence the graph can be identified with its edge set G. A *path* $s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_n} s_n$ of source s_0 of target s_n and of label $u = a_1 \ldots a_n$ is also denoted by $s_0 \xrightarrow{u} s_n$. We write $s \xrightarrow{u,v}$ for $u,v \in A_G^*$ if $s \xrightarrow{u} t$ and $s \xrightarrow{v} t$ for some t.
Recall that a *magma* is a set M equipped with a binary operation $\cdot : M{\times}M \longrightarrow M$ that sends any two elements $p,q \in M$ to the element $p \cdot q$. Given a subset $Q \subseteq M$ and an injective mapping $[\![\ ]\!] : Q \longrightarrow A$, we define the *generalized Cayley graph* $\mathcal{C}[\![M,Q]\!] = \{\ p \xrightarrow{[\![q]\!]} p{\cdot}q \mid p \in M \wedge q \in Q\ \}$ of vertex set M and of label set $\{\ [\![q]\!] \mid q \in Q\ \}$. Such a graph G is *deterministic*: there are no two edges of the same source and label *i.e.* $(r \xrightarrow{a}_G s \wedge r \xrightarrow{a}_G t) \implies s = t$. It is also *source-complete*: for all vertex s and label a, there is an a-edge from s *i.e.* $\forall\, s \in V_G\ \ \forall\, a \in A_G\ \ \exists\, t\ (s \xrightarrow{a}_G t)$.
Recall that an element e of M is a *left identity* (resp. *right identity*) if $e{\cdot}p = p$ (resp. $p{\cdot}e = p$) for any $p \in M$. A left identity e of a magma is an *out-simple vertex* of any generalized Cayley graph: it is not source of two edges with the same target *i.e.* $(e \xrightarrow{a} s \wedge e \xrightarrow{b} s) \implies a = b$. If M has a left identity e and a right identity e' then $e = e \cdot e' = e'$ is the *identity* of M. The identity e of a magma is a *loop-propagating vertex* for its generalized Cayley graphs: if e has a loop labeled by a then any vertex has a loop labeled by a *i.e.* $e \xrightarrow{a} e \implies \forall\, s \in M\ (s \xrightarrow{a} s)$. These properties characterize the generalized Cayley graphs of magmas with a left identity, or with an identity.

Theorem 1. *A graph is a generalized Cayley graph of a magma with a left identity* (*resp. an identity*) if and only if *it is deterministic, source-complete with an out-simple* (*resp. and loop-propagating*) *vertex.*

Recall that a magma M is *commutative* if $p \cdot q = q \cdot p$ for any $p,q \in M$. Any left identity e of a commutative magma is a *locally commutative vertex* of its generalized Cayley graphs: if $e \xrightarrow{ab} s$ then $e \xrightarrow{ba} s$ for any $s \in M$ and labels a,b. We strengthen Theorem 1 to commutative magmas.

Theorem 2. *A graph is a generalized Cayley graph of a commutative magma with an identity* iff *it is deterministic, source-complete with an out-simple, loop-propagating, locally commutative vertex.*

Cayley graphs of semigroups

Recall that a magma $(M, \cdot)$ is a *semigroup* if $\cdot$ is associative: $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ for any $p, q, r \in M$. A *Cayley graph of a semigroup* M is a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ such that M is *generated* by Q: $M = \{ \, q_1 \cdot \ldots \cdot q_n \mid n > 0 \wedge q_1, \ldots, q_n \in Q \, \}$. Let us describe these graphs.

Theorem 3. *A graph G is a Cayley graph of a* (*resp. commutative*) *semigroup* if and only if *it is deterministic and there is an injection i from A_G into V_G such that*

$$G \text{ is accessible from } i(A_G) : \forall\, s \in V_G \;\; \exists\, a \in A_G \;\; \exists\, u \in A_G^* \;\; i(a) \xrightarrow{u} s,$$

$$i(a) \xrightarrow{u} \xleftarrow{v} i(b) \implies s \xrightarrow{au,bv} \text{ for any } s \in V_G\,,\; a, b \in A_G \text{ and } u, v \in A_G^*\,,$$

$$(\text{resp. and } i(a) \xrightarrow{b} \xleftarrow{a} i(b) \text{ for any } a, b \in A_G).$$

A *semilattice* M is a commutative semigroup which is also *idempotent*: $p \cdot p = p$ for any $p \in M$. We strengthen Theorem 3 to the semilattices by adding the following condition:

$$i(a) \xrightarrow{u} s \implies s \xrightarrow{au} s \quad \text{for any } s \in V_G\,,\; a \in A_G \text{ and } u \in A_G^*\,.$$

Cayley graphs of monoids

A *monoid* M is a semigroup with an identity 1. So 1 is a *propagating vertex* of any generalized Cayley graph G of M: if $1 \xrightarrow{u,v}_G$ for $u, v \in A_G^*$ then $s \xrightarrow{u,v}_G$ for any $s \in V_G$. A graph $\mathcal{C}[\![M, Q]\!]$ is a *Cayley graph of a monoid* M if M is *generated* by Q: $M = \{ \, q_1 \cdot \ldots \cdot q_n \mid n \geq 0 \wedge q_1, \ldots, q_n \in Q \, \}$. For such a graph G, 1 is a *root*: $\forall\, s \in V_G \;\; \exists\, u \in A_G^* \;\; 1 \xrightarrow{u} s$.
Let us give a simple graph-theoretic characterization for the Cayley graphs of monoids.

Theorem 4. *A graph is a Cayley graph of a* (*resp. commutative*) *monoid* if and only if *it is deterministic, source-complete with a propagating* (*resp. and locally commutative*) *out-simple root.*

Cayley graphs of groups

A *group* M is a monoid whose each element p has an *inverse* p^{-1}: $p \cdot p^{-1} = p^{-1} \cdot p = 1$. Any generalized Cayley graph G of a group is also *co-deterministic* (resp. *target-complete*) *i.e.* $\overline{G} = \{ \, t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a} t \, \}$, for $\{ \, \overline{a} \mid a \in A_G \, \}$ a disjoint copy of A_G, is deterministic (resp. source-complete). Furthermore 1 is an *in-simple vertex* of G: it is an out-simple vertex of $\overline{G}$. Finally 1 is a *chain-propagating vertex* of G *i.e.* 1 is propagating for $G \cup \overline{G}$. A graph $\mathcal{C}[\![M, Q]\!]$ is a *Cayley graph of a group* M if M is *generated* by Q: $M = \{ \, q_1 \cdot \ldots \cdot q_n \mid n \geq 0 \wedge q_1, \ldots, q_n \in Q \cup Q^{-1} \, \}$. Such a graph is connected. These properties give another description for the Cayley graphs [1].

Theorem 5. *A graph is a Cayley graph of a* (*resp. commutative*) *group* if and only if *it is connected, deterministic and co-deterministic, source-complete and target-complete, with a chain-propagating* (*resp. and locally commutative*) *in-simple and out-simple vertex.*

A full version with proofs, examples and other characterizations is available in arxiv 1903.06521

References

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[5] G. Sabidussi, *On a class of fixed-point-free graphs*, Proceedings of the American Mathematical Society 9-5, 800–804 (1958).