# On Cayley graphs of basic algebraic structures

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**Abstract**

We present simple graph-theoretic characterizations of Cayley graphs for monoids, semigroups and groups. We strengthen these characterizations to commutative monoids, semilattices, and abelian groups.

## 1 Introduction

Arthur Cayley was the first to define in 1854 [2] the notion of a group as well as the table of its operation known as the Cayley table. To describe the structure of a group $(G,\cdot)$, Cayley also introduced in 1878 [3] the concept of graph for $G$ according to a generating subset $S$, namely the set of labeled oriented edges $g \xrightarrow{s} g{\cdot}s$ for every $g$ of $G$ and $s$ of $S$. Such a graph, called Cayley graph, is directed and labeled in $S$ (or an encoding of $S$ by symbols called letters or colors). A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [5] : an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. Following a question asked by Hamkins in 2010 [4]: 'Which graphs are Cayley graphs?', we gave simple graph-theoretic characterizations of Cayley graphs for groups, as well as for left-cancellative and cancellative monoids [1]. In this paper, we generalize this last characterization to Cayley graphs of monoids, then to semigroups. We also strengthen all these characterizations to commutative monoids, semilattices and abelian groups.

To structurally characterize the Cayley graphs (of groups), we selected four basic properties of these graphs. First and by definition, any Cayley graph is deterministic: there are no two edges of the same source and label. The right-cancellative property of groups induces the co-determinism of their graphs: there are no two edges of the same target and label. The left-cancellative property of groups implies that their graphs are simple: there are no two edges of the same source and goal. Finally, any Cayley graph is according to a generating subset hence is connected: there is a chain from the identity element to any vertex. To these four basic conditions is added the well known symmetry property of vertex-transitivity: all the vertices are isomorphic. These five properties satisfied by the Cayley graphs are sufficient to characterize them [1]. Similarly, we obtained a graph-theoretic characterization for the Cayley graphs of cancellative monoids: first, they are now rooted since there is a path from the identity element to any vertex, and then by relaxing the vertex transitivity to the forward vertex-transitivity: all the vertices are accessible-isomorphic *i.e.* the induced subgraphs by vertex accessibility are isomorphic [1].

To characterize the Cayley graphs of all monoids (not necessarily cancellative), we must weaken the forward vertex-transitivity. We say that a vertex is propagating if there is a homomorphism from its accessible subgraph to the accessible subgraph from any vertex. Thus, the identity of a monoid is a propagating vertex for each of its Cayley graphs. The identity is also an out-simple vertex: it is not source of two edges with the same target. Moreover, any Cayley graph is source-complete: for any label of the graph and from any vertex, there is at least one edge. These properties are sufficient to characterize the Cayley graphs of monoids: they are the deterministic and source-complete graphs with a propagating out-simple root. It follows a graph-theoretic characterization for the Cayley graphs of semigroups (see Theorem 37) and of cancellative semigroups (see Theorem 39).

For the Cayley graphs of commutative monoids, we just have to add the condition that any vertex $s$ is locally commutative: for any path from $s$ labeled by (two letters) $ab$, there is a path from $s$ labeled by $ba$ of the same target. The locally-commutativity can be restricted to a single vertex: the Cayley graphs of commutative monoids are the deterministic and source-complete graphs with a locally commutative propagating out-simple root. It follows a graph-theoretic characterization for the Cayley graphs of semilattices (see Theorem 40). By extending to chains the vertex propagation, we can restrict the vertex-transitivity of a Cayley graph to the existence of a single propagating vertex: the Cayley graphs of (resp. abelian) groups are the deterministic and co-deterministic, simple and connected graphs with a chain-propagating (resp. and locally commutative) source and target-complete vertex.

## 2 Directed labeled graphs

We recall some basic concepts on directed labeled graphs, especially the vertex-transitivity and the forward vertex-transitivity.

Let $A$ be an arbitrary (finite or infinite) set. We denote by $A^*$ the set of tuples (words) over $A$ (the free monoid generated by $A$) and by $\varepsilon$ the 0-tuple (the identity element called the empty word). A directed $A$-*graph* $(V,G)$ is defined by a set $V$ of *vertices* and a subset $G \subseteq V{\times}A{\times}V$ of *edges*. Any edge $(s,a,t) \in G$ is from the *source* $s$ to the *target* $t$ with *label* $a$, and is also written by the *transition* $s \xrightarrow{a}_G t$ or directly $s \xrightarrow{a} t$ if $G$ is clear from the context. The sources and targets of edges form the set $V_G$ of *non-isolated vertices* of $G$ and we denote by $A_G$ the set of edge labels:

$$V_G \;=\; \{\; s \mid \exists\, a,t \;\; (s \xrightarrow{a} t \vee t \xrightarrow{a} s) \;\} \quad \text{and} \quad A_G \;=\; \{\; a \mid \exists\, s,t \;\; (s \xrightarrow{a} t) \;\}.$$

We say that $G$ is *finitely labeled* if $A_G$ is finite. The set $V - V_G$ is the set of *isolated vertices*. From now on, we assume that any graph $(V,G)$ is without isolated vertex (*i.e.* $V = V_G$), hence the graph can be identified with its edge set $G$. We also exclude the empty graph $\emptyset$: every graph is a non-empty set of labeled edges. A vertex $s$ is an *out-simple vertex* if there are no two edges of source $s$ with the same target: $(s \xrightarrow{a} t \wedge s \xrightarrow{b} t) \implies a = b$. A graph is *simple* if all its vertices are out-simple. We also say that $s$ is an *in-simple vertex* if there are no two edges with the same source and target $s$: $(t \xrightarrow{a} s \wedge t \xrightarrow{b} s) \implies a = b$. Thus an in-simple vertex for $G$ is an out-simple vertex for $G^{-1} \;=\; \{\; (t,a,s) \mid (s,a,t) \in G \;\}$ the *inverse* of $G$. The *vertex-restriction* $G_{|P}$ of $G$ to a set $P$ is the induced subgraph of $G$ by $P \cap V_G$:

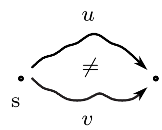
$$G_{|P} \;=\; \{\; (s,a,t) \in G \mid s,t \in P \;\}.$$

The *label-restriction* $G^{|P}$ of $G$ to a set $P$ is the subset of all its edges labeled in $P$:

$$G^{|P} \;=\; \{\; (s,a,t) \in G \mid a \in P \;\}.$$

Let $\longrightarrow_G$ be the unlabeled edge relation *i.e.* $s \longrightarrow_G t$ if $s \xrightarrow{a}_G t$ for some $a \in A$. We write $\longleftrightarrow_G$ for the unlabeled *adjacency relation* $\longrightarrow_{G \cup G^{-1}}$ *i.e.* $s \longleftrightarrow_G t$ for $s \longrightarrow_G t$ or $t \longrightarrow_G s$. We denote by $\longrightarrow_G(s) \;=\; \{\; t \mid s \longrightarrow_G t \;\}$ the set of *successors* of $s \in V_G$. We write $s \not\longrightarrow_G t$ if there is no edge in $G$ from $s$ to $t$ *i.e.* $G \cap \{s\}{\times}A{\times}\{t\} \;=\; \emptyset$. The *accessibility* relation $\longrightarrow_G^* \;=\; \bigcup_{n \geq 0} \longrightarrow_G^n$ is the reflexive and transitive closure under composition of $\longrightarrow_G$. A graph $G$ is *accessible* from $P \subseteq V_G$ if for any $s \in V_G$, there is $r \in P$ such that $r \longrightarrow_G^* s$. We denote by $G_{\downarrow P}$ the induced subgraph of $G$ to the vertices accessible

from $P$ which is the greatest subgraph of $G$ accessible from $P$. A *root* $r$ is a vertex from which $G$ is accessible *i.e.* $G_{\downarrow\{r\}}$ also denoted by $G_{\downarrow r}$ is equal to $G$. A *co-root* of $G$ is a root of $G^{-1}$. A graph $G$ is *strongly connected* if every vertex is a root: $s \longrightarrow_G^* t$ for all $s, t \in V_G$. A vertex $r$ of a graph $G$ is an 1-*root* if $r \longrightarrow_G s$ for any vertex $s$ of $G$. A graph is *complete* if all its vertices are 1-roots *i.e.* there is an edge between any couple of vertices: $\forall\, s, t \in V_G\;\; \exists\, a \in A_G\; (s \xrightarrow{a}_G t)$. An 1-*coroot* of $G$ is an 1-root of $G^{-1}$.
A graph $G$ is *co-accessible* from $P \subseteq V_G$ if $G^{-1}$ is accessible from $P$. A graph $G$ is *connected* if $G \cup G^{-1}$ is strongly connected.
A *path* $(s_0, a_1, s_1, \ldots, a_n, s_n)$ of *length* $n \geq 0$ in a graph $G$ is a sequence $s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_n} s_n$ of $n$ consecutive edges, and we write $s_0 \overset{a_1 \ldots a_n}{\longrightarrow} s_n$ for indicating the source $s_0$, the target $s_n$ and the label word $a_1 \ldots a_n \in A_G^*$ of the path; such a path is *elementary* if it goes through distinct vertices: $s_0 \neq \ldots \neq s_n$ and we write $s_0 \underset{\neq}{\overset{a_1 \ldots a_n}{\longrightarrow}} s_n$. We write $s \overset{u,v}{\longrightarrow}_G t$ if $s \xrightarrow{u}_G t$ and $s \xrightarrow{v}_G t$; we also denote by $s \overset{u,v}{\longrightarrow}_G$ if $s \overset{u,v}{\longrightarrow}_G t$ for some $t$.
Let $u = a_1 \ldots a_m$ and $v = b_1 \ldots b_n$ where $m, n \geq 0$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$. We write $s \underset{\neq}{\overset{u,v}{\longrightarrow}}_G$ if $uv \neq \varepsilon$ and there exists paths $s \xrightarrow{a_1}_G s_1 \ldots \xrightarrow{a_m} s_m$ and $s \xrightarrow{b_1}_G t_1 \ldots \xrightarrow{b_n} t_n = s_m$ forming an elementary cycle: $s \neq s_1 \neq \ldots \neq s_m \neq t_1 \neq \ldots \neq t_{n-1}$ which is illustrated as follows:

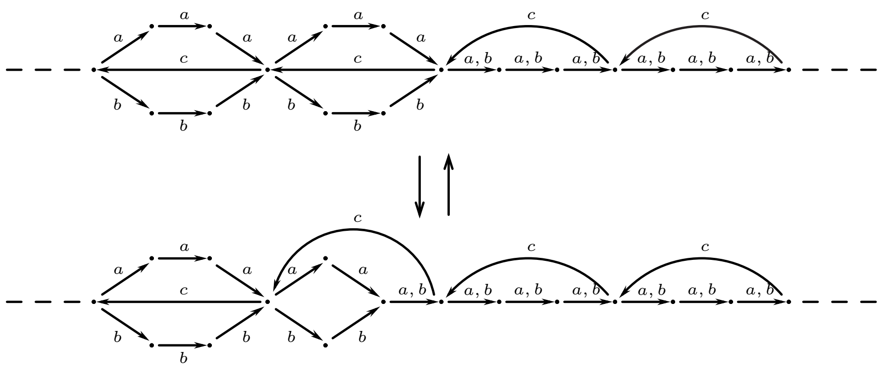



In particular for a loop $s \xrightarrow{a} s$, we have $s \underset{\neq}{\overset{\varepsilon,a}{\longrightarrow}}$ and for two edges $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$ of the same source and goal, we have $s \underset{\neq}{\overset{a,b}{\longrightarrow}}$.
Recall that a *morphism* from a graph $G$ into a graph $H$ is a mapping $h$ from $V_G$ into $V_H$ such that $s \xrightarrow{a}_G t \implies h(s) \xrightarrow{a}_H h(t)$; we write $G \longrightarrow_h H$. If, in addition, $h$ is bijective and $h^{-1}$ is a morphism, $h$ is called an *isomorphism* from $G$ to $H$; we write $G \equiv_h H$ or directly $G \equiv H$ if we do not specify an isomorphism, and we say that $G$ and $H$ are *isomorphic*. An *automorphism* of $G$ is an isomorphism from $G$ to $G$.

▶ **Lemma 1.** *For any deterministic graphs $G$ and $H$ rooted respectively by $s$ and $t$, if $G \longrightarrow_g H$ and $H \longrightarrow_h G$ with $g(s) = t$ and $h(t) = s$ then $G \equiv_g H$.*

**Proof.**
**i)** Let us check that $h(g(x)) = x$ for any $x \in V_G$.
The proof is done by induction on $\ell(x) = \min\{\, n \mid s \longrightarrow_G^n x \,\}$.
$\ell(x) = 0$: $x = s$ and $h(g(s)) = h(t) = s$.
$\ell(x) > 0$: There is $x' \xrightarrow{a} x$ such that $\ell(x') = \ell(x) - 1$. As $g$ is a morphism, $g(x') \xrightarrow{a} g(x)$.
By induction hypothesis and as $h$ is a morphism, $x' = h(g(x')) \xrightarrow{a} h(g(x))$.
As $G$ is deterministic, we get $h(g(x)) = x$.
**ii)** By (i), $g$ is injective. By exchanging $g$ with $h$ and by (i), $g(h(y)) = y$ for any $y \in V_H$.
In particular $g$ is surjective. Thus $g$ is bijective and $h = g^{-1}$. So $G \equiv_g H$. ◀

In Lemma 1, even if $g$ and $h$ are surjective, the condition $g(s) = t$ and $h(t) = s$ is necessary. For instance, the two non-isomorphic graphs below are rooted, deterministic, co-deterministic, and there is a surjective morphism from one into the other.




Two vertices $s, t$ of a graph $G$ are *isomorphic* and we write $s \simeq_G t$ if $t = h(s)$ for some automorphism $h$ of $G$. A graph $G$ is *vertex-transitive* if all the vertices are isomorphic *i.e.* $s \simeq_G t$ for every $s, t \in V_G$. Two vertices $s, t$ of a graph $G$ are *accessible-isomorphic* and we write $s \downarrow_G t$ if $t = h(s)$ for some isomorphism $h$ from $G_{\downarrow s}$ to $G_{\downarrow t}$. A graph $G$ is *forward vertex-transitive* if all its vertices are accessible-isomorphic: $s \downarrow_G t$ for every $s, t \in V_G$. Any vertex-transitive graph is forward vertex-transitive. The semiline $\{ n \xrightarrow{a} n+1 \mid n \in \mathbb{N} \}$ is forward vertex-transitive but is not vertex-transitive. Any strongly connected forward vertex-transitive graph is vertex-transitive.
We need to circulate in a graph $G$ in the direct and inverse direction of the arrows.
Let $\overline{\phantom{a}} : A_G \longrightarrow A - A_G$ be an injective mapping of image $\overline{A_G} = \{ \overline{a} \mid a \in A_G \}$ a disjoint copy of $A_G$. This allows to define the graph

$$\overline{G} \;=\; G \cup \{ t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a}_G t \}$$

in order that $V_{\overline{G}} = V_G$ and $A_{\overline{G}} = A_G \cup \overline{A_G}$ with

$$\begin{array}{rcl} G \text{ deterministic and co-deterministic} & \Longrightarrow & \overline{G} \text{ deterministic and co-deterministic} \\ G \text{ source and target-complete} & \Longrightarrow & \overline{G} \text{ source and target-complete} \\ G \text{ connected} & \Longrightarrow & \overline{G} \text{ strongly connected} \\ s \simeq_G t & \Longleftrightarrow & s \downarrow_{\overline{G}} t \text{ for any } s, t \in V_G \end{array}$$

hence $G$ is vertex-transitive if and only if $\overline{G}$ is forward vertex-transitive.
A path of $\overline{G}$ *i.e.* $s \xrightarrow{u}_{\overline{G}} t$ with $u \in (A_G \cup \overline{A_G})^*$ is a *chain* of $G$ also denoted by $s \xrightarrow{u}_G t$ where $s \xrightarrow{\overline{a}}_G t$ means that $t \xrightarrow{a}_G s$ for any $a \in A_G$. Thus

$$s \xrightarrow{u}_G t \quad\Longleftrightarrow\quad t \xrightarrow{\overline{\widetilde{u}}}_G s \quad \text{for any } u \in (A_G \cup \overline{A_G})^*$$

such that for $u = a_1 \ldots a_n$ with $n \geq 0$ and $a_1, \ldots, a_n \in A_G \cup \overline{A_G}$, $\overline{u} = \overline{a_1} \ldots \overline{a_n}$ where $\overline{\overline{a}} = a$ for any $a \in A_G \cup \overline{A_G}$, and $\widetilde{u} = a_n \ldots a_1$ is the *mirror* of $u$.
Let us give basic properties on (forward) vertex-transitive graphs.

▶ **Fact 2.** Any forward vertex-transitive graph is source-complete.
Any vertex-transitive graph is source and target-complete.

The forward vertex-transitivity of a rooted graph is reduced to the accessible-isomorphism of a root with its successors. The vertex-transitivity of a connected graph is reduced to the isomorphism of a vertex with its adjacent vertices.

▶ **Lemma 3.** *A graph $G$ of root $r$ is forward vertex-transitive iff $r \downarrow_G s$ for any $r \longrightarrow_G s$.*
*A connected graph with a vertex $r$ is vertex-transitive* if and only if
$r \simeq_G s$ *for any* $r \longleftrightarrow_G s$.

**Proof.**
Let $G$ be a graph with a root $r$ such that $r \downarrow_G s$ for any $r \longrightarrow_G s$.

Let us check that $G$ is forward vertex-transitive *i.e.* $r \downarrow_G s$ for any $r \longrightarrow_G^* s$.
The proof is done by induction on $n \geq 0$ for $r \longrightarrow_G^n s$.
For $n = 0$, we have $r = s$. For $n > 0$, let $t$ be a vertex such that $r \longrightarrow_G^{n-1} t \longrightarrow_G s$.
By induction hypothesis, we have $r \downarrow_G t$ *i.e.* $f(r) = t$ for some isomorphism $f$ from $G_{\downarrow r}$ to $G_{\downarrow t}$. As $t \longrightarrow_G s$, there exists $r'$ such that $r \longrightarrow_G r'$ and $f(r') = s$. So $r' \downarrow_G s$.
By hypothesis $r \downarrow_G r'$. By transitivity of $\downarrow_G$, we get $r \downarrow_G s$.
We get the second equivalence using the first one for $\overline{G}$. ◀

## 3 Commutative and propagating graphs

We recall when a graph is deterministic, co-deterministic, source-complete, target-complete, commutative. All these notions are equivalent when they are defined globally by paths or locally by edges. We introduce the propagation of joined paths which allows to express differently accessible-isomorphic vertices for deterministic graphs. The propagation can be restricted to elementary paths for deterministic and source-complete graphs. Finally we extend to chains the commutation and the propagation.

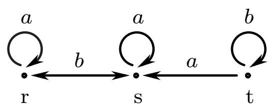
A graph is *deterministic* if there are no two paths with the same source and label word:

$$(r \xrightarrow{u} s \wedge r \xrightarrow{u} t) \implies s = t \quad \text{for any } r, s, t \in V_G \text{ and } u \in A_G^* .$$

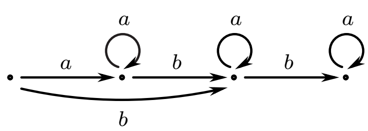
This definition coincides with the local property that there are no two edges with the same source and label: $(r \xrightarrow{a} s \wedge r \xrightarrow{a} t) \implies s = t$ for any $r, s, t \in V_G$ and $a \in A_G$ .
A graph is *co-deterministic* if its inverse is deterministic: there are no two paths (resp. edges) with the same target and label word (resp. label).
A graph $G$ is a *source-complete graph* if for all vertex $s$ and label word $u$, there exists a path from $s$ labeled by $u$: $\forall\, s \in V_G \;\; \forall\, u \in A_G^* \;\; \exists\, t \in V_G \; (s \xrightarrow{u}_G t)$. Locally a vertex $r$ is a *source-complete vertex* if for any label $a \in A_G$ there exists $s \in V_G$ such that $r \xrightarrow{a}_G s$. Thus

$$G \text{ is source-complete} \iff \text{all its vertices are source-complete.}$$

Similarly a vertex $r$ of a graph $G$ is a *target-complete vertex* if $r$ is source-complete for $G^{-1}$ *i.e.* $\forall\, a \in A_G \;\; \exists\, s \in V_G \; (s \xrightarrow{a}_G r)$. A graph $G$ is a *target-complete graph* if its inverse is source-complete *i.e.* all the vertices of $G$ are target-complete.
Let us recall the path commutation in a graph. Let $\approx_A$ be the binary *commutative relation* on $A^*$ defined by $uabv \approx_A ubav$ for any $a, b \in A$ and $u, v \in A^*$. By reflexivity and transitivity, we extend $\approx_A$ to the *commutation congruence* $\cong_A$ . A vertex $r$ of a graph $G$ is a *commutative vertex* if $r \xrightarrow{u}_G s \implies r \xrightarrow{v}_G s$ for any $s \in V_G$ and any $u, v \in A^*$ such that $u \cong_A v$ . For the following deterministic and source-complete graph:




the vertices $r$ and $s$ are commutative but $t$ is not commutative since $t \xrightarrow{ab} r$ and $t \xrightarrow{ba} s$.
We say that $G$ is a *commutative graph* if all its vertices are commutative. For instance, the following deterministic graph is commutative:



Let us restrict this commutation from paths to edges. We say that a vertex $r$ of a graph $G$ is a *locally commutative vertex* if $r \xrightarrow{ab}_G s \implies r \xrightarrow{ba}_G s$ for any $s \in V_G$ and any $a, b \in A_G$. The commutation of all vertices may be restricted to the local commutation.

▶ **Lemma 4.** *A graph is commutative* if and only if *all its vertices are locally commutative.*

**Proof.**
$\Longrightarrow$: Any commutative vertex is locally commutative.
$\Longleftarrow$: Let a graph $G$ whose any vertex is locally commutative. Let a path $s \xrightarrow{u}_G t$.
We have to check that $s \xrightarrow{v}_G t$ for any $v \cong_A u$.
By induction on the minimum number of commutations between $u$ and $v$, we can restrict to $u \approx_A v$ *i.e.* $u = xaby$ and $v = xbay$ for some $a, b \in A_G$ and $x, y \in A_G^*$.
Thus $s \xrightarrow{x} s' \xrightarrow{ab} t' \xrightarrow{y} t$ for some vertices $s', t'$.
As $s'$ is locally commutative, we have $s' \xrightarrow{ba} t'$ hence $s \xrightarrow{v} t$. ◀

We will now express the accessible-isomorphism of vertices by propagation of confluent paths. We start with the propagation of loops. We say that a vertex $r$ of a graph $G$ is a *loop-propagating vertex* when we have the following property: if $r$ has a loop labeled by $a \in A$ then any vertex has a loop labeled by $a$: $r \xrightarrow{a}_G r \implies \forall\, s \in V_G\ (s \xrightarrow{a}_G s)$.

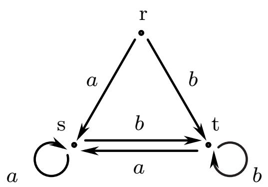
▶ **Lemma 5.** *Any locally commutative* 1*-root of a deterministic graph is loop-propagating.*

**Proof.**
Let $G$ be a deterministic graph and $r$ be a locally commutative 1-root.
Let a loop $r \xrightarrow{a} r$ and a vertex $s$. As $r$ is an 1-root, there exists $b \in A$ such that $r \xrightarrow{b} s$.
So $r \xrightarrow{ab} s$. As $r$ is locally commutative, $r \xrightarrow{ba} s$. As $G$ is deterministic, we get $s \xrightarrow{a} s$. ◀

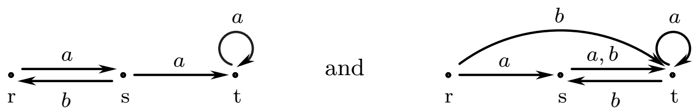
We extend the propagation of loops to paths.
A vertex $r$ is *propagating* (resp. 1-*propagating*) if for any $u, v \in A_G^*$ (resp. $A_G$),

$$r \xrightarrow{u,v}_G \quad \Longrightarrow \quad \forall\, s \in V_G\ (s \xrightarrow{u,v}_G).$$

The restriction of this implication to any $u \in A_G$ with $v = \varepsilon$ is the loop-propagating notion.
The restriction of the implication to $u = v$ means that $r \xrightarrow{u}_G \implies \forall\, s \in V_G\ (s \xrightarrow{u}_G)$.
In particular for the graph $\{r \xrightarrow{a} s\}$, the vertex $r$ is not 1-propagating and $s$ is propagating.
For the following deterministic and source-complete graph:




the vertex $r$ is propagating but the vertices $s, t$ are not propagating: we have $s \xrightarrow{\varepsilon,a}$ (resp. $t \xrightarrow{\varepsilon,b}$) which is not the case for the other vertices. All the vertices are 1-propagating.
For the following two deterministic connected graphs:




the vertices $r, t$ are 1-propagating (but not propagating) and $s$ is not 1-propagating.
A propagating vertex of a deterministic graph is a vertex from which there is a morphism linking it to any vertex.

▶ **Lemma 6.** *For any deterministic graph* $G$ *and vertices* $r,s$, *we have*

$$r \xrightarrow{u,v}_G \implies s \xrightarrow{u,v}_G \quad \textit{for any } u,v \in A_G^*$$

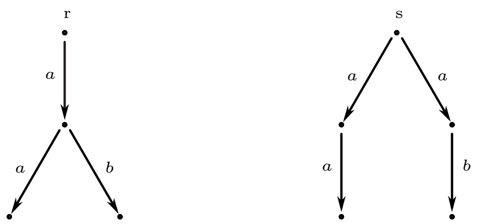
if and only if *there is a morphism* $h$ *from* $G_{\downarrow r}$ *to* $G_{\downarrow s}$ *such that* $h(r) = s$.

**Proof.**
$\Longleftarrow$: Immediate for any graph.
$\Longrightarrow$: As $G$ is deterministic, it allows to define the mapping $h : V_{G_{\downarrow r}} \longrightarrow V_{G_{\downarrow s}}$ by

$$h(p) = q \text{ if } r \xrightarrow{u}_G p \text{ and } s \xrightarrow{u}_G q \text{ for some } u \in A^*.$$

Thus $h(r) = s$. It remains to check that $h$ is a morphism.
Let $p \xrightarrow{a}_{G_{\downarrow r}} q$. There exists $u \in A_G^*$ such that $r \xrightarrow{u}_G p$.
As $r \xrightarrow{ua}_G$, we have $s \xrightarrow{ua}_G$ *i.e.* $s \xrightarrow{u}_G p' \xrightarrow{a}_G q'$ for some vertices $p', q'$.
As $G$ is deterministic, $h(p) = p'$ and $h(q) = q'$ hence $h(p) \xrightarrow{a}_{G_{\downarrow s}} h(q)$. ◀

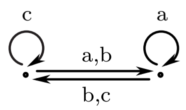
The determinism condition in Lemma 6 is necessary: for the following non deterministic (and non connected) graph $G$:




we have $(r \xrightarrow{u,v}_G \implies s \xrightarrow{u,v}_G)$ for any $u,v \in A_G^*$ and there is no morphism from $G_{\downarrow r}$ into $G$ linking $r$ to $s$. Let us give other basic properties on 1-propagating vertices.

▶ **Fact 7.** Any graph with a source-complete 1-propagating vertex is source-complete.
For any source-complete graph, any out-simple vertex is 1-propagating.

Here is a source-complete and deterministic graph without 1-propagating vertex.




The existence of a source-complete propagating vertex in a deterministic graph allows to reduce the commutativity of the graph to the locally commutativity of the vertex.

▶ **Lemma 8.** *Let a deterministic graph* $G$ *with a source-complete and propagating vertex* $r$.
*If* $r$ *is locally commutative then* $G$ *is commutative.*

**Proof.**
We have to show that $G$ is commutative.
Let $s \xrightarrow{ab} t$ with $a,b \in A$. By Lemma 4, it remains to check that $s \xrightarrow{ba} t$.
By Fact 7, $G$ is source-complete. So $r \xrightarrow{ab} r'$ for some vertex $r'$.
As $r$ is locally commutative, $r \xrightarrow{ba} r'$.
As $r \xrightarrow{ab,ba}$ and $r$ is propagating, we get $s \xrightarrow{ab,ba} t'$ for some vertex $t'$.
As $G$ is deterministic, $t = t'$ thus $s \xrightarrow{ba} t$. ◀

When a graph is deterministic, two vertices are accessible-isomorphic means that each vertex is propagating for the other.

▶ **Lemma 9.** *For any deterministic graph $G$ and any vertices $s,t$, we have*

$$s \downarrow_G t \iff \forall\, u,v \in A_G^* \; (s \xrightarrow{u,v}_G \iff t \xrightarrow{u,v}_G ).$$

**Proof.**

$\Longrightarrow$: Obvious for any graph $G$.

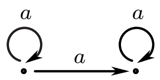
$\Longleftarrow$: Let $s,t \in V_G$ such that $s \xrightarrow{u,v}_G$ if and only if $t \xrightarrow{u,v}_G$ for any $u,v \in A^*$.

As $G$ is deterministic, we define the mappings $g : V_{G_{\downarrow s}} \longrightarrow V_{G_{\downarrow t}}$ and $h : V_{G_{\downarrow t}} \longrightarrow V_{G_{\downarrow s}}$ by

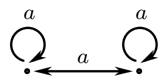
$g(p) = q$ and $h(q) = p$ if $s \xrightarrow{u}_G p$ and $t \xrightarrow{u}_G q$ for some $u \in A^*$.

As yet done in the proof of Lemma 6, $g$ and $h$ are morphisms with $g(s) = t$ and $h(t) = s$. By Lemma 1, $g$ is an isomorphism from $G_{\downarrow s}$ to $G_{\downarrow t}$ *i.e.* $s \downarrow_G t$. ◀

A graph $G$ is a *propagating graph* (resp. 1-*propagating graph*) if all its vertices are propagating (resp. 1-propagating) *i.e.* for any $u,v \in A_G^*$ (resp. $A_G$) and any $s,t \in V_G$, $s \xrightarrow{u,v}_G \implies t \xrightarrow{u,v}_G$. For the following two connected graphs:




and




the first graph is propagating but not forward vertex-transitive, and the second graph is forward vertex-transitive hence propagating. The 1-propagating property and the source-complete property coincide for the simple graphs.

▶ **Fact 10.** Any 1-propagating graph is source-complete.
Any 1-propagating graph with an out-simple vertex is a simple graph.
Any source-complete and simple graph is 1-propagating.

Let us restrict the path propagation to elementary paths.
A vertex $r$ is an *elementary-propagating vertex* if for any $u,v \in A_G^*$,

$$r \;{}_{\neq}\!\!\xrightarrow{u,v}_G \implies \forall\, s \in V_G \; (s \;{}_{\neq}\!\!\xrightarrow{u,v}_G ).$$

An *elementary-propagating graph* is a graph whose any vertex is elementary-propagating. For instance, the deterministic and simple graph $\{r \xrightarrow{a} s\}$ is elementary-propagating but is not source-complete, hence is not 1-propagating. The elementary-propagating property coincides with the propagating property for the deterministic and source-complete graphs.

▶ **Proposition 11.** **a)** *A source-complete graph $G$ is propagating* if and only if

$$s \;{}_{\neq}\!\!\xrightarrow{u,v}_G \implies t \xrightarrow{u,v}_G \;\; \textit{for any } s,t \in V_G \textit{ and } u,v \in A_G^* .$$

**b)** *A deterministic source-complete graph is propagating* iff *it is elementary-propagating.*
**c)** *A deterministic graph is propagating* if and only if *it is forward-vertex transitive.*
**d)** *A deterministic graph $G$ with a root $r$ is a propagating graph* if and only if

$$r \xrightarrow{u,v}_G \iff s \xrightarrow{u,v}_G \;\; \textit{for any } r \longrightarrow_G s \textit{ and } u,v \in A_G^* .$$

**e)** *A deterministic target-complete graph with a root $r$ is propagating* iff *$r$ is propagating.*

**Proof.**

**i)** Let us check (a).

$\Longrightarrow$: Immediate for any propagating graph.

$\Longleftarrow$: Assume that $s \;{}_{\neq}\!\!\xrightarrow{u,v}_G \implies t \xrightarrow{u,v}_G$ for any $s,t \in V_G$ and $u,v \in A_G^*$.

Let us show that $s \xrightarrow{u,v}_G \implies t \xrightarrow{u,v}_G$ for any $s,t \in V_G$ and $u,v \in A_G^*$.

The proof is done by induction on $|uv| \geq 0$.

$|uv| = 0$: $u = v = \varepsilon$ hence $t \xrightarrow{u,v}_G t$ for any vertex $t$.

$|uv| > 0$: Let $s \xrightarrow{u,v}_G$ with $u,v \in A_G^*$ and let $t \in V_G$. We can assume that $|u| \leq |v|$.

We distinguish the two complemantary cases below.

*Case 1*: $u = \varepsilon$. So $s \;{}_{\neq}\!\!\xrightarrow{\varepsilon,v'}_G s \xrightarrow{\varepsilon,v''}_G$ for some $v'v'' = v$.

By hypothesis $t \xrightarrow{\varepsilon,v'}_G t$. By induction hypothesis $t \xrightarrow{\varepsilon,v''}_G$ hence $t \xrightarrow{\varepsilon,v}_G$.

*Case 2*: $u \neq \varepsilon$. We distinguish the two subcases below.

*Case 2.1*: $s \xrightarrow{a}_G s' \xrightarrow{u',v'}_G$ with $a \in A_G$, $u = au'$ and $v = av'$.

As $G$ is source-complete, $t \xrightarrow{a}_G t'$ for some vertex $t'$.

By induction hypothesis, $t' \xrightarrow{u',v'}_G$ hence $t \xrightarrow{u,v}_G$.

*Case 2.2*: $s \;{}_{\neq}\!\!\xrightarrow{u',v'}_G s' \xrightarrow{u'',v''}_G$ for some $s'$ with $u = u'u''$ and $v = v'v''$.

By hypothesis $t \xrightarrow{u',v'}_G t'$ for some vertex $t'$.

By induction hypothesis $t' \xrightarrow{u'',v''}_G$ hence $t \xrightarrow{u,v}_G$.

**ii)** Let us check (b).

$\Longleftarrow$: by (a) for $G$ source-complete and elementary-propagating.

$\Longrightarrow$: Let $G$ be a deterministic and propagating graph.

Let us check that $G$ is elementary-propagating.

Let $s \;{}_{\neq}\!\!\xrightarrow{u,v}_G$ with $u,v \in A_G^*$ and let $t \in V_G$. So $t \xrightarrow{u,v}_G$ and $uv \neq \varepsilon$.

As $G$ is deterministic and for $u,v \neq \varepsilon$, the words $u$ and $v$ have not the same first letter.

As yet done for (a), there exists $u'u'' = u$ and $v'v'' = v$ such that $t \;{}_{\neq}\!\!\xrightarrow{u',v'}_G \xrightarrow{u'',v''}_G$.

So $s \xrightarrow{u',v'}_G$. For $G$ deterministic, $u'' = v'' = \varepsilon$ hence $t \;{}_{\neq}\!\!\xrightarrow{u,v}_G$.

**iii)** (c) follows from Lemma 9.

**iv)** (d) follows from (c) and Lemmas 3 and 9.

**v)** Let us check (e). Let $r$ be a propagating root of a deterministic and target-complete graph $G$. We have to show that $G$ is propagating.

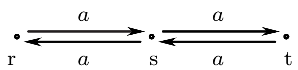
Let $s \xrightarrow{u,v}_G$ with $u,v \in A_G^*$. We have to check that $r \xrightarrow{u,v}_G$.

As $r$ is a root, $r \xrightarrow{w}_G s$ for some $w \in A_G^*$.

As $G$ is target complete, there exists a vertex $r_0$ such that $r_0 \xrightarrow{w}_G r$.

As $r \xrightarrow{wu,wv}_G$ and $r$ is propagating, $r_0 \xrightarrow{wu,wv}_G$. As $G$ is deterministic, $r \xrightarrow{u,v}_G$. ◀

Note that the determinism condition in Proposition 11 (b) and (c) is necessary. For instance the following propagating and non deterministic graph:




is not elementary-propagating (and not forward vertex-transitive) since $s \;{}_{\neq}\!\!\xrightarrow{aa,aa}$ but $\neg(r \;{}_{\neq}\!\!\xrightarrow{aa,aa})$.
Note also that the rooted condition in Proposition 11 (e) is necessary. For instance, the inverse semiline $\{\; n+1 \xrightarrow{a} n \mid n \in \mathbb{N} \;\}$ is deterministic, target-complete, and is not a propagating graph while 0 is a propagating co-root.

Let us generalize to the chains the path commutation in a graph.
A vertex $r$ is a *chain-commutative vertex* for a graph $G$ if $r$ is commutative for $\overline{G}$ *i.e.*
$r \xrightarrow{u}_G s \implies r \xrightarrow{v}_G s$ for any $s \in V_G$ and any $u,v \in (A_G \cup \overline{A_G})^*$ such that $u \cong_A v$.
We say that $G$ is a *chain-commutative graph* if $\overline{G}$ is a commutative graph *i.e.* all the vertices are chain-commutative for $G$.
As in Lemma 4, we can express locally the chain-commutation. A vertex $r$ is a *locally chain-commutative vertex* for a graph $G$ if $r$ is locally commutative for $\overline{G}$ *i.e.* $r \xrightarrow{ab}_G s \implies r \xrightarrow{ba}_G s$ for any $s \in V_G$ and $a,b \in A_G \cup \overline{A_G}$. Let us apply Lemma 4.

▶ **Corollary 12.** *A graph is chain-commutative* iff *all its vertices are locally chain-commutative.*

Any chain-commutative graph is commutative and the converse is true when the graph is deterministic, co-deterministic, source and target-complete.

▶ **Lemma 13.** *Let $G$ be a deterministic, co-deterministic, source and target-complete graph. If $G$ is commutative then $G$ is chain-commutative.*

**Proof.**
Let $s \xrightarrow{ab}_G t$ where $a,b \in A_G \cup \overline{A_G}$. By Corollary 12, it remains to check that $s \xrightarrow{ba}_G t$.
We distinguish the four complementary cases below.
*Case 1*: $a,b \in A_G$. As $s$ is locally commutative, $s \xrightarrow{ba}_G t$.
*Case 2*: $\overline{a},\overline{b} \in A_G$. So $t \xrightarrow{\overline{b}\,\overline{a}}_G s$. As $t$ is locally commutative, $t \xrightarrow{\overline{a}\,\overline{b}}_G s$ *i.e.* $s \xrightarrow{ba}_G t$.
*Case 3*: $\overline{a},b \in A_G$. So $s \xleftarrow{\overline{a}}_G r \xrightarrow{b}_G t$ for some vertex $r$.
As $G$ is source-complete, $s \xrightarrow{b}_G r'$ for some vertex $r'$. So $r \xrightarrow{\overline{a}b}_G r'$.
As $r$ is locally commutative, $r \xrightarrow{b\overline{a}}_G r'$.
As $G$ is deterministic, $t \xrightarrow{\overline{a}}_G r'$ hence $s \xrightarrow{ba}_G t$.
*Case 4*: $a,\overline{b} \in A_G$. So $s \xrightarrow{a}_G r \xleftarrow{\overline{b}}_G t$ for some vertex $r$.
As $G$ is target-complete, $r' \xrightarrow{a}_G t$ for some vertex $r'$. So $r' \xrightarrow{a\overline{b}}_G r$.
As $r'$ is locally commutative, $r' \xrightarrow{\overline{b}a}_G r$.
As $G$ is co-deterministic, $r' \xrightarrow{\overline{b}}_G s$ hence $s \xrightarrow{ba}_G t$. ◀

Let us apply Lemmas 8 and 13.

▶ **Lemma 14.** *Let $G$ be a deterministic, co-deterministic, source and target-complete graph. If $G$ has a propagating locally commutative vertex then $G$ is chain-commutative.*

Let us generalize to the chains the path propagation in a graph. Note that

$$s \xrightarrow{u,v}_G \iff s \xrightarrow{u\widetilde{\overline{v}}}_G s \quad \text{for any } s \in V_G \text{ and } u,v \in (A_G \cup \overline{A_G})^*.$$

A vertex $r$ is a *chain-propagating vertex* for a graph $G$ if $r$ is propagating for $\overline{G}$ *i.e.*

$$\text{for any } u \in (A_G \cup \overline{A_G})^*, \quad r \xrightarrow{\varepsilon,u}_G \implies \forall\, s \in V_G\ (s \xrightarrow{\varepsilon,u}_G).$$

Let us apply Lemma 9.

▶ **Corollary 15.** *For any deterministic and co-deterministic graph $G$ and any vertices $s,t$,*

$$s \simeq_G t \iff \forall\, u \in (A_G \cup \overline{A_G})^*,\ (s \xrightarrow{\varepsilon,u}_G \iff t \xrightarrow{\varepsilon,u}_G).$$

A *chain-propagating graph* is a graph whose all its vertices are chain-propagating. The line $\{\, n \xrightarrow{a} n+1 \mid n \in \mathbb{Z} \,\}$ is chain-propagating but the semiline $\{\, n \xrightarrow{a} n+1 \mid n \in \mathbb{N} \,\}$ is not chain-propagating since $\neg(0 \xrightarrow{\overline{a}})$. We denote by $C_G(s) = \{\, u \in (A_G \cup \overline{A_G})^* \mid s \underset{\neq}{\xrightarrow{\varepsilon,u}}_G \,\}$ the language of non empty words labeling the elementary cycles from $s$.
Let us apply Proposition 11.

▶ **Corollary 16.** **a)** *A source and target-complete graph $G$ is chain-propagating* if and only if

$$s \underset{\neq}{\xrightarrow{\varepsilon,u}}_G \implies t \xrightarrow{\varepsilon,u}_G \quad \textit{for any } s,t \in V_G \textit{ and } u \in (A_G \cup \overline{A_G})^*.$$

**b)** *Let $G$ be a deterministic and co-deterministic, source and target-complete graph,*

$$G \textit{ is chain-propagating} \iff C_G(s) = C_G(t) \textit{ for any } s,t \in V_G.$$

**c)** *A deterministic and co-deterministic graph is chain-propagating* iff *it is vertex transitive.*
**d)** *Let $G$ be a connected deterministic and co-deterministic graph and $r$ be a vertex,*
*$G$ is chain-propagating* iff $r \xrightarrow{\varepsilon,u}_G \iff s \xrightarrow{\varepsilon,u}_G$ *for any $r \longleftrightarrow_G s$ and $u \in (A_G \cup \overline{A_G})^*$.*
**e)** *For any connected, deterministic and co-deterministic graph $G$,*
*$G$ is vertex-transitive* iff *$G$ is source and target-complete with a chain-propagating vertex.*

Note that the connected condition in Corollary 16 (e) is necessary. For instance, the non-connected graph $\{\ n \xrightarrow{a} n+1 \mid n \in \mathbb{Z}\ \} \cup \{\ \omega \xrightarrow{a} \omega\ \}$ is not vertex-transitive but it is deterministic, co-deterministic, source and target-complete, and any $n \in \mathbb{Z}$ is a chain-propagating vertex.

## 4 Generalized Cayley graphs of magmas

We present graph-theoretic characterizations for the generalized Cayley graphs of magmas with a left identity, and with an identity (see Theorem 25). These characterizations are then refined to the commutative magmas with an identity (see Theorem 26).

A *magma* is a set $M$ equipped with a binary operation $\cdot : M \times M \longrightarrow M$ that sends any two elements $p,q \in M$ to the element $p \cdot q$.
Given a subset $Q \subseteq M$ and an injective mapping $[\![\ ]\!] : Q \longrightarrow A$, we define the graph

$$\mathcal{C}[\![M,Q]\!] \;=\; \{\ p \xrightarrow{[\![q]\!]} p \cdot q \mid p \in M \wedge q \in Q\ \}$$

which is called a *generalized Cayley graph* of $M$. It is of vertex set $M$ and of label set $[\![Q]\!] = \{\ [\![q]\!] \mid q \in Q\ \}$. We denote $\mathcal{C}[\![M,Q]\!]$ by $\mathcal{C}(M,Q)$ when $[\![\ ]\!]$ is the identity. We also write $\mathcal{C}[\![M]\!]$ instead of $\mathcal{C}[\![M,M]\!]$ and $\mathcal{C}(M) = \mathcal{C}(M,M) = \{\ p \xrightarrow{q} p \cdot q \mid p,q \in M\ \}$.
Let us give basic properties of these generalized Cayley graphs.

▶ **Fact 17.** Any generalized Cayley graph of a magma is deterministic and source-complete.

A magma $(M,\cdot)$ is *left-cancellative* if $r \cdot p = r \cdot q \implies p = q$ for any $p,q,r \in M$.
Similarly $(M,\cdot)$ is *right-cancellative* if $p \cdot r = q \cdot r \implies p = q$ for any $p,q,r \in M$.
A magma is *cancellative* if it is both left-cancellative and right-cancellative.

▶ **Fact 18.** Any generalized Cayley graph of a left-cancellative magma is simple.
Any generalized Cayley graph of a right-cancellative magma is co-deterministic.

Recall that an element $e$ of $M$ is a *left identity* (resp. *right identity*) if $e{\cdot}p = p$ (resp. $p{\cdot}e = p$) for any $p \in M$.

▶ **Fact 19.** Any left identity of a magma $M$ is an out-simple 1-root of $\mathcal{C}(M)$.

If $M$ has a left identity $e$ and a right identity $e'$ then $e = e \cdot e' = e'$ is the *identity* of $M$ which is called a *unital magma*.

▶ **Fact 20.** The identity of any unital magma $M$ is loop-propagating for $\mathcal{C}(M)$.

**Proof.**
Assume that $M$ has an identity $1$ with an $a$-loop on $\mathcal{C}(M)$: $1 \xrightarrow{a}_{\mathcal{C}(M)} 1$ with $a \in A$.
By definition of $\mathcal{C}(M)$, we have $1{\cdot}a = 1$ *i.e.* $a = 1$.
For any vertex $s$, we have $s \xrightarrow{a}_{\mathcal{C}(M)} s{\cdot}a = s{\cdot}1 = s$. ◀

Recall that a magma $M$ is *commutative* if $p \cdot q = q \cdot p$ for any $p,q \in M$.

▶ **Fact 21.** Any left identity of a commutative magma is locally commutative for $\mathcal{C}(M)$.

We say that (the operation $\cdot$ of) a magma $M$ is *left-invertible* (resp. *right-invertible*) with respect to $e \in M$ if for any $p \in M$ there exists $\overline{p} \in M$ such that $\overline{p} \cdot p = e$ (resp. $p \cdot \overline{p} = e$).
A unital magma is left (resp. right)-invertible if it is w.r.t. its unit element.

▶ **Fact 22.** For any left-invertible (resp. right-invertible) magma $M$ w.r.t. $e$,
$e$ is a target-complete vertex (resp. is an 1-coroot) for $\mathcal{C}(M)$.

**Proof.**
For any $p \in M$, $\overline{p} \xrightarrow{p}_{\mathcal{C}(M)} e$ for $M$ left-invertible, and $p \xrightarrow{\overline{p}}_{\mathcal{C}(M)} e$ for $M$ right-invertible. ◀

The previous facts are basic properties of the generalized Cayley graphs of magmas with a left-identity that characterize them.

▶ **Lemma 23.** *Let $r$ be an 1-propagating 1-root of a deterministic graph $G$.*
*We can define the* edge-operation $\times_r$ *for any* $s, t \in V_G$ *by*

$$s \xrightarrow{a}_G s \times_r t \text{ if } r \xrightarrow{a}_G t \text{ for some } a \in A.$$

*Then* $(V_G, \times_r)$ *is a magma of left identity* $r$.
*If* $r$ *is loop-propagating then* $r$ *is an identity.*
*If* $r$ *is locally commutative then* $\times_r$ *is commutative.*
*If* $G$ *is* 1*-propagating then* $\times_r$ *is left-cancellative.*
*If* $G$ *is co-deterministic then* $\times_r$ *is right-cancellative.*
*If* $r$ *is target-complete then* $\times_r$ *is left-invertible w.r.t.* $r$.
*If* $r$ *is a source-complete* 1*-coroot then* $\times_r$ *is right-invertible w.r.t.* $r$.
*For any* $B \subseteq A_G$ *such that* $r$ *is source-complete and out-simple for* $G^{|B}$,
$G^{|B} = \mathcal{C}[\![V_G, Q]\!]$ *for* $Q = \{\, q \mid \exists\, b \in B\ (r \xrightarrow{b}_G q)\,\}$ *with* $[\![q]\!] = b$ *for* $b \in B$ *and* $r \xrightarrow{b}_G q$.

**Proof.**
**i)** Let $s, t \in V_G$. Let us check that $s \times_r t$ is well-defined.
As $r$ is an 1-root, there exists $a \in A$ such that $r \xrightarrow{a}_G t$.
As $r$ is 1-propagating, there exists $x$ such that $s \xrightarrow{a}_G x$.
Let $r \xrightarrow{b}_G t$. So $r \xrightarrow{a,b}_G$. As $r$ is 1-propagating, we have $s \xrightarrow{a,b}_G$.
As $G$ is deterministic, we get $s \xrightarrow{b}_G x$. Thus $s \times_r t = x$.
The operation $\times_r$ is illustrated as follows:


We denote $\times_r$ by $\cdot$ in the rest of this proof.
**ii)** Let us check that $r$ is a left identity of $(V_G, \cdot)$.
Let $s \in V_G$. As $r$ is an 1-root, there exists $a \in A$ such that $r \xrightarrow{a}_G s$.
By definition of $\cdot$ we have $r \xrightarrow{a}_G r{\cdot}s$. As $G$ is deterministic, we get $r{\cdot}s = s$.
**iii)** Assume that $r$ is loop-propagating. Let us check that $r$ is also a right identity.
As $r$ is an 1-root, there is $a \in A_G$ such that $r \xrightarrow{a}_G r$.
Let $s \in V_G$. As $r$ is loop-propagating, we get $s \xrightarrow{a}_G s$.
By definition of $\cdot$ we have $s \xrightarrow{a} s{\cdot}r$.
As $G$ is deterministic, $s = s{\cdot}r$. Thus $r$ is a right identity.
**iv)** Assume that $r$ is locally commutative. Let us check that $\cdot$ is commutative.
Let $s, t \in V_G$. There exists $a, b \in A_G$ such that $r \xrightarrow{a} s$ and $r \xrightarrow{b} t$.
So $s \xrightarrow{b} s{\cdot}t$ and $t \xrightarrow{a} t{\cdot}s$. Thus $r \xrightarrow{ab} s{\cdot}t$ and $r \xrightarrow{ba} t{\cdot}s$.
As $r$ is locally commutative, $r \xrightarrow{ba} s{\cdot}t$. As $G$ is deterministic, we get $s{\cdot}t = t{\cdot}s$.
**v)** Assume that $G$ is 1-propagating. Let us check that $\cdot$ is left-cancellative.
Assume that $s{\cdot}t = s{\cdot}t'$.
As $r$ is an 1-root, there exists $a, a' \in A_G$ such that $r \xrightarrow{a}_G t$ and $r \xrightarrow{a'}_G t'$.
By definition of $\cdot$ we get $s \xrightarrow{a}_G s \cdot t$ and $s \xrightarrow{a'}_G s{\cdot}t' = s{\cdot}t$. Thus $s \xrightarrow{a,a'}_G$.
As $G$ is 1-propagating, $r \xrightarrow{a,a'}_G$. As $G$ is deterministic, it follows that $t = t'$.

**vi)** Assume that $G$ is co-deterministic. Let us check that $\cdot$ is right-cancellative.
Let $s, s', t \in V_G$ such that $s{\cdot}t = s'{\cdot}t$.
There exists $a \in A_G$ such that $r \overset{a}{\longrightarrow} t$. So $s \overset{a}{\longrightarrow} s{\cdot}t$ and $s' \overset{a}{\longrightarrow} s'{\cdot}t = s{\cdot}t$.
As $G$ is co-deterministic, we get $s = s'$.

**vii)** Assume that $r$ is target-complete. Let us check that $\cdot$ is left-invertible w.r.t. $r$.
Let $s \in V_G$. As $r$ is an 1-root, there exists $a \in A_G$ such that $r \overset{a}{\longrightarrow}_G s$.
As $r$ is target-complete, there exists $\overline{s} \in V_G$ such that $\overline{s} \overset{a}{\longrightarrow}_G r$.
So $\overline{s} \overset{a}{\longrightarrow}_G \overline{s} \cdot s$. As $G$ is deterministic, $\overline{s} \cdot s = r$.

**viii)** Assume that $r$ is source-complete and is an 1-coroot.
Let us check that $\cdot$ is right-invertible w.r.t. $r$.
Let $s \in V_G$. As $r$ is an 1-coroot, there exists $a \in A_G$ such that $s \overset{a}{\longrightarrow}_G r$.
As $r$ is source-complete, there exists $\overline{s} \in V_G$ such that $r \overset{a}{\longrightarrow}_G \overline{s}$.
So $s \overset{a}{\longrightarrow}_G s \cdot \overline{s}$. As $G$ is deterministic, $s \cdot \overline{s} = r$.

**ix)** Let $B \subseteq A_G$ such that $r$ is out-simple and source-complete for $H = G^{|B}$.
Let $Q = \{\, q \mid \exists\, b \in B\ (r \overset{b}{\longrightarrow}_G q)\,\} = \{\, q \mid \exists\, b \in A\ (r \overset{b}{\longrightarrow}_H q)\,\}$.
As $r$ is out-simple for $H$, we can define the mapping $[\![\ ]\!] : Q \longrightarrow B$ by $[\![q]\!] = b$ for $r \overset{b}{\longrightarrow}_H q$.
As $H$ is deterministic, $[\![\ ]\!]$ is an injection. As $r$ is source-complete for $H$, $[\![\ ]\!]$ is a bijection.
Let us show that $H = \mathcal{C}[\![V_G, Q]\!]$.
$\subseteq$: Let $s \overset{b}{\longrightarrow}_H t$. As $r$ is source-complete for $H$, there exists $q$ such that $r \overset{b}{\longrightarrow}_H q$.
So $s \overset{b}{\longrightarrow}_G s{\cdot}q$. As $G$ is deterministic, $s{\cdot}q = t$. As $[\![q]\!] = b$, we get $s \overset{b}{\longrightarrow}_{\mathcal{C}[\![V_G,\,Q]\!]} s{\cdot}q = t$.
$\supseteq$: Let $s \overset{b}{\longrightarrow}_{\mathcal{C}[\![V_G,\,Q]\!]} t$. There exists $q \in Q$ such that $[\![q]\!] = b$ and $s{\cdot}q = t$.
Thus $b \in B$ and $r \overset{b}{\longrightarrow}_G q$. So $s \overset{b}{\longrightarrow}_H s{\cdot}q = t$. ◀

We get a fully graph-theoretic characterization for the Cayley graphs $\mathcal{C}[\![M]\!]$ of any magma $M$ with a left identity.

▶ **Proposition 24.** *A graph $G$ is equal to $\mathcal{C}[\![M]\!]$ for some magma $M$ with a left identity $r$ (resp. commutative magma, with an identity, left-cancellative, right-cancellative, left-invertible w.r.t. $r$, right-invertible w.r.t. $r$)* if and only if
*$G$ is a deterministic and source-complete graph and $r$ is an out-simple 1-root (and resp. $r$ is locally commutative, $r$ is loop-propagating, $G$ is simple, $G$ is co-deterministic, $r$ is target-complete, $r$ is an 1-coroot).*

**Proof.**
$\Longrightarrow$: Let $G = \mathcal{C}[\![M]\!]$ for some magma $(M, \cdot)$ with a left identity $r$, and some injective mapping $[\![\ ]\!]$. By Fact 17, $G$ is deterministic and source-complete.
By Fact 19, $r$ is out-simple and is an 1-root of $G$.
If $M$ is commutative then by Fact 21, $r$ is locally commutative.
If $r$ is an identity then by Fact 20, $r$ is loop-propagating.
If $\cdot$ is left-cancellative (resp. right-cancellative) then by Fact 18, $G$ is simple (resp. co-deterministic).
If $M$ is left-invertible (resp. right-invertible) w.r.t. $r$ then by Fact 22, $r$ is target-complete (resp. is an 1-coroot).
$\Longleftarrow$: By Fact 7 and by Lemma 23 (resp. by Fact 10). ◀

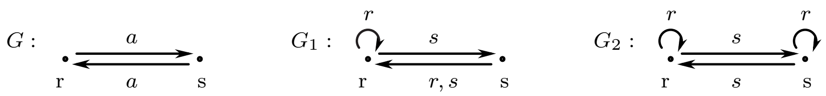
We can now present a graph-theoretic characterization for the generalized Cayley graphs of magmas with a left identity, or with an identity.

▶ **Theorem 25.** *A graph is a generalized Cayley graph of a magma with a left identity* (*resp. an identity*) if and only if *it is deterministic, source-complete with an out-simple* (*resp. and loop-propagating*) *vertex.*

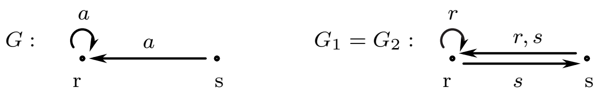
**Proof.**
$\Longrightarrow$: By Facts 17, 19, (resp. 20), or by Proposition 24 having $\mathcal{C}[\![M,Q]\!] = \mathcal{C}[\![M]\!]^{[\![Q]\!]}$.
$\Longleftarrow$: Let $G$ be a deterministic and source-complete graph with an out-simple vertex $r$.
Let $Q = \{\ q \mid r \longrightarrow_G q\ \}$ be the set of successors of $r$. We define the graph

$$\begin{aligned} G_1 \quad = \quad & \{\ s \xrightarrow{q} t \mid \exists\, a\ (s \xrightarrow{a}_G t \ \wedge\ r \xrightarrow{a}_G q)\ \} \cup \{\ r \xrightarrow{s} s \mid s \in V_G - Q\ \} \\ \cup \quad & \{\ s \xrightarrow{t} r \mid s \in V_G - \{r\} \ \wedge\ t \in V_G - Q\ \}. \end{aligned}$$

Here is an example with $r \notin Q$ loop-propagating:




and another example with $r \in Q$ not loop-propagating:




Thus $G_1$ remains deterministic and source-complete with $V_{G_1} = V_G = A_{G_1}$.
As $r \xrightarrow{s}_{G_1} s$ for any $s \in V_G$, $r$ is an out-simple 1-root of $G_1$.
By Fact 7, $r$ is 1-propagating.
By Lemma 23 applied to $G_1$ and $r$, $(V_G, \times_r)$ is a magma of left identity $r$ and $G_1 = \mathcal{C}(V_G)$ *i.e.*

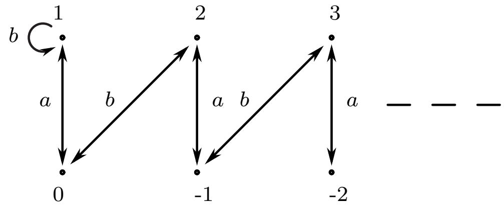
$$s \xrightarrow{t}_{G_1} s \times_r t \quad \text{for any } s, t \in V_G\,.$$

Thus $G = \mathcal{C}[\![V_G]\!]^{[\![Q]\!]} = \mathcal{C}[\![V_G\,,\,Q]\!]$ with $[\![q]\!] = a$ for any $r \xrightarrow{a}_G q$.
Note that for $r \notin Q$ and $s \neq r$ (as for the first example), $r$ is not loop-propagating for $G_1$ since $r \xrightarrow{r}_{G_1} r$ and $s \xrightarrow{r}_{G_1} r$. Thus we refine $G_1$ in the following graph:

$$\begin{aligned} G_2 \quad = \quad & \{\ s \xrightarrow{q} t \mid \exists\, a\ (s \xrightarrow{a}_G t \ \wedge\ r \xrightarrow{a}_G q)\ \} \cup \{\ r \xrightarrow{s} s \mid s \in V_G - Q\ \} \\ \cup \quad & \{\ s \xrightarrow{r} s \mid s \in V_G - \{r\} \ \wedge\ r \notin Q\ \} \ \cup\ \{\ s \xrightarrow{t} r \mid s \in V_G - \{r\} \ \wedge\ t \in V_G - (Q \cup \{r\})\ \}. \end{aligned}$$

Note that $G_2$ remains deterministic and source-complete with $V_{G_2} = V_G = A_{G_2}$.
As $r \xrightarrow{s}_{G_2} s$ for any $s \in V_G$, $r$ is an out-simple 1-root of $G_2$.
Assume that $r$ is loop-propagating for $G$. In particular $G_1 = G_2$ for $r \in Q$.
Let us check that $r$ remains loop-propagating for $G_2$. Let $r \xrightarrow{s}_{G_2} r$ and $t \in V_G$.
  Either $s \notin Q$. Thus $s = r$. So $r \notin Q$ hence $t \xrightarrow{r}_{G_2} t$.
  Or $s \in Q$ *i.e.* $r \xrightarrow{a}_G s$ for some $a \in A$. So $r \xrightarrow{s}_{G_2} s$.
  As $G_2$ is deterministic, $s = r$. Thus $r \xrightarrow{a}_G r$.
  As $r$ is loop-propagating for $G$, we get $t \xrightarrow{a}_G t$ hence $t \xrightarrow{s}_{G_2} t$.
By Lemma 23, $(V_G, \times_r)$ is a magma of identity $r$ with $G_2 = \mathcal{C}(V_G)$.
Thus $G = \mathcal{C}[\![V_G]\!]^{[\![Q]\!]} = \mathcal{C}[\![V_G\,,\,Q]\!]$ with $[\![q]\!] = a$ for any $r \xrightarrow{a}_G q$. ◀

For instance consider the magma $(\mathbb{Z}, \cdot)$ where $m \cdot n = -m + n$ for any $m, n \in \mathbb{Z}$. There is no right identity and $0$ is the unique left identity. The generalized Cayley graph $G = \mathcal{C}[\![\mathbb{Z}, \{1,2\}]\!]$ with $[\![1]\!] = a$ and $[\![2]\!] = b$ is the following simple graph:




Any vertex other than 1 is loop-propagating. By Theorem 25, $G$ is a generalized Cayley graph of a unital magma. Precisely by Lemma 23, $G = \mathcal{C}[\![\mathbb{Z}, \{1, 2\}]\!]$ with $[\![1]\!] = a$ and $[\![2]\!] = b$ for the unital magma $(\mathbb{Z}, \times_0)$ defined for any $n$ by $0 \times_0 n = n$ and for any $m \neq 0$ by

$$m \times_0 n \;=\; \begin{cases} m & \text{if } \; n = 0 \\ -m + n & \text{if } \; n = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

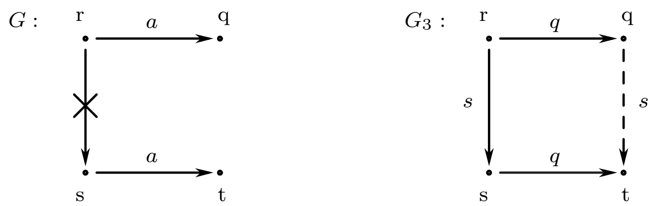
We strengthen Theorem 25 for any commutative magma with a unit element.

▶ **Theorem 26.** *A graph is a generalized Cayley graph of a commutative unital magma* iff *it is deterministic, source-complete with an out-simple, loop-propagating and locally commutative vertex.*

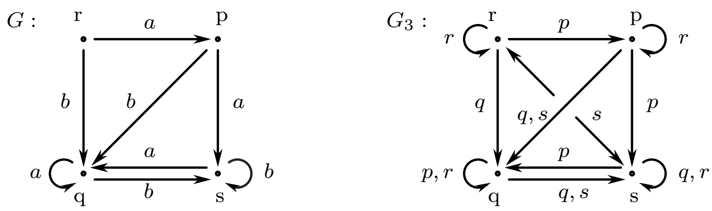
**Proof.**
$\Longrightarrow$: By Facts 17, 19, 20, 21.
$\Longleftarrow$: Let $G$ be a deterministic and source-complete graph.
Let $r$ be an out-simple, loop-propagating and locally commutative vertex.
Let $Q = \{\, q \mid r \longrightarrow_G q \,\}$ and $Q_r = Q \cup \{r\}$.
The graph $G_2$ defined in the proof of Theorem 25 is refined in the following graph:

$$\begin{array}{rll} G_3 \;=& \{\, s \xrightarrow{q} t \mid \exists\, a\; (s \xrightarrow{a}_G t \;\wedge\; r \xrightarrow{a}_G q) \,\} & \cup\;\; \{\, r \xrightarrow{s} s \mid s \in V_G - Q \,\} \\ \cup & \{\, s \xrightarrow{r} s \mid s \in V_G - \{r\} \;\wedge\; r \notin Q \,\} & \cup\;\; \{\, s \xrightarrow{t} r \mid s, t \in V_G - Q_r \,\} \\ \cup & \{\, q \xrightarrow{s} t \mid \exists\, a\; (s \xrightarrow{a}_G t \;\wedge\; r \xrightarrow{a}_G q \neq r) \;\wedge\; s \notin Q_r \,\} & \end{array}$$

The last subset of $G_3$ is illustrated as follows:



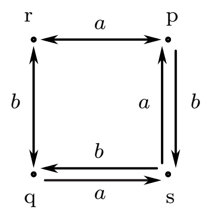

Here is an example with $r \notin Q$:




The five subsets defining $G_3$ are disjoint. Thus $G_3$ remains deterministic.
Furthermore $G_3$ is source-complete and $V_{G_3} = V_G = A_{G_3}$.
Let $s \in V_G$. Let us check that $r \xrightarrow{s}_{G_3} s$.
Due to the second subset defining $G_3$, this is true when $s \notin Q$.
Assume that $s \in Q$. So $r \xrightarrow{a}_G s$ for some $a \in A$.
The first subset defining $G_3$ gives $r \xrightarrow{s}_{G_3} s$.

So $r$ is an out-simple 1-root of $G_3$.
Let $s \in V_G$. Let us check that $s \xrightarrow{r}_{G_3} s$.
Due to the third subset defining $G_3$, this is true when $r \notin Q$.
Assume that $r \in Q$. So $r \xrightarrow{a}_G r$ for some $a \in A$.
As $r$ is loop-propagating, $s \xrightarrow{a}_G s$. The first subset defining $G_3$ gives $s \xrightarrow{r}_{G_3} s$.
By Lemma 5, we just have to check that $r$ remains locally commutative for $G_3$ since by Lemma 23, $(V_G, \times_r)$ is a commutative magma of identity $r$ with $G_3 = \mathcal{C}(V_G)$ which implies that $G = \mathcal{C}[\![V_G]\!]^{[\![Q]\!]} = \mathcal{C}[\![V_G\,, Q]\!]$ with $[\![q]\!] = a$ for any $r \xrightarrow{a}_G q$.
Let $r \xrightarrow{xy}_{G_3} t$ for some $x, y \in V_G$. We have to show that $r \xrightarrow{yx}_{G_3} t$.
As $r \xrightarrow{x}_{G_3} x$ and $G_3$ is deterministic, we have $x \xrightarrow{y}_{G_3} t$.
As $r \xrightarrow{y}_{G_3} y$, we have to check that $y \xrightarrow{x}_{G_3} t$.
Let us start by checking it for $x = r$ or $y = r$.
Case $x = r$: So $r \xrightarrow{y}_{G_3} t$ hence $y = t \xrightarrow{r}_{G_3} t$.
Case $y = r$: So $x \xrightarrow{r}_{G_3} t$ hence $y = r \xrightarrow{x}_{G_3} x = t$.
It remains the four complementary cases below.
*Case 1*: $x, y \notin Q_r$.
So $x \xrightarrow{y}_{G_3} t$ is only defined by the fourth subset hence $t = r$.
By this fourth subset, we have $y \xrightarrow{x}_{G_3} r = t$.
*Case 2*: $x \notin Q_r$ and $y \in Q - \{r\}$.
So $x \xrightarrow{y}_{G_3} t$ is only defined by the first subset: we have $x \xrightarrow{a}_G t$ for $r \xrightarrow{a}_G y$.
By the fifth subset, we get $y \xrightarrow{x}_{G_3} t$.
*Case 3*: $x \in Q - \{r\}$ and $y \notin Q_r$.
So $x \xrightarrow{y}_{G_3} t$ is only defined by the fifth subset: we have $y \xrightarrow{a}_G t$ for $r \xrightarrow{a}_G x$.
By the first subset, we get $y \xrightarrow{x}_{G_3} t$.
*Case 4*: $x, y \in Q - \{r\}$. There exists $a, b \in A$ such that $r \xrightarrow{a}_G x$ and $r \xrightarrow{b}_G y$.
As $G$ is source-complete, there exists a vertex $z$ such that $x \xrightarrow{b}_G z$.
By the first subset, we have $x \xrightarrow{y}_{G_3} z$. As $G_3$ is deterministic, $t = z$.
So $r \xrightarrow{ab}_G t$. As $r$ is locally commutative for $G$, we get $r \xrightarrow{ba}_G t$.
As $G$ is deterministic, $y \xrightarrow{a}_G t$. By the first subset, we have $y \xrightarrow{x}_{G_3} t$. ◀

For instance, let us consider the following deterministic and source-complete graph $G$:




which is also simple and loop-propagating. The vertex $r$ is locally commutative whereas $G$ is not a commutative graph since $p \xrightarrow{ab} q$ and $p \xrightarrow{ba} p$. By Theorem 26, $G$ is a generalized Cayley graph of a commutative unital magma.
Precisely by Lemma 23, $G = \mathcal{C}[\![\{p, q, r, s\}, \{p, q\}]\!]$ where $[\![p]\!] = a$ and $[\![q]\!] = b$ for the commutative unital magma $(\{p, q, r, s\}, \times_r)$ defined by the following Cayley table:

| $\times_r$ | $r$ | $p$ | $q$ | $s$ |
|---|---|---|---|---|
| $r$ | $r$ | $p$ | $q$ | $s$ |
| $p$ | $p$ | $r$ | $s$ | $p$ |
| $q$ | $q$ | $s$ | $r$ | $q$ |
| $s$ | $s$ | $p$ | $q$ | $r$ |

## 5 Cayley graphs of monoids

We present graph-theoretic characterizations for the Cayley graphs of monoids, and of commutative monoids (see Theorem 33), and when they are left-cancellative and cancellative (see Theorems 35 and 36).

Recall that a magma $(M,\cdot)$ is a *semigroup* if $\cdot$ is associative: $(p\cdot q)\cdot r = p\cdot(q\cdot r)$ for any $p,q,r\in M$. A semigroup with an identity element is a monoid. The identity is a propagating vertex for its generalized Cayley graphs.

▶ **Fact 27.** For any generalized Cayley graph of a monoid, 1 is propagating and out-simple.

**Proof.**
Let $G = \mathcal{C}[\![M,Q]\!]$ for some monoid $(M,\cdot)$ and $Q\subseteq M$.
By Fact 19, the identity 1 is an out-simple vertex. Let us check that 1 is propagating.
Let $1 \xrightarrow{u,v}_G$ with $u,v\in A_G^*$.
So $u = [\![p_1]\!]\ldots[\![p_m]\!]$ and $v = [\![q_1]\!]\ldots[\![q_n]\!]$ for some $m,n\geq 0$ and $p_1,\ldots,p_m,q_1,\ldots,q_n\in Q$.
So $(\ldots(1\cdot p_1)\ldots)\cdot p_m = (\ldots(1\cdot q_1)\ldots)\cdot q_n$. As $\cdot$ is associative, $p_1\cdot\ldots\cdot p_m = q_1\cdot\ldots\cdot q_n$.
Let $s$ be any vertex of $G$. So $s \xrightarrow{u}_G (\ldots(s\cdot p_1)\ldots)\cdot p_m = s\cdot(p_1\cdot\ldots\cdot p_m)$ and
$s \xrightarrow{v}_G s\cdot(q_1\cdot\ldots\cdot q_n) = s\cdot(p_1\cdot\ldots\cdot p_m)$. Thus $s \xrightarrow{u,v}_G$. ◀

The commutativity of a semigroup is transposed on its generalized Cayley graphs.

▶ **Fact 28.** Any generalized Cayley graph of a commutative semigroup is commutative.

**Proof.**
Let $G = \mathcal{C}[\![M,Q]\!]$ for some commutative semigroup $M$ and some $Q\subseteq M$.
Let us show that $G$ is commutative.
Let $s \xrightarrow{[\![p]\!][\![q]\!]}_G t$ for some $p,q\in Q$. By Lemma 4, it remains to check that $s \xrightarrow{[\![q]\!][\![p]\!]}_G t$.
We have $t = (s\cdot p)\cdot q = s\cdot(p\cdot q) = s\cdot(q\cdot p) = (s\cdot q)\cdot p$ hence $s \xrightarrow{[\![q]\!][\![p]\!]}_G t$. ◀

When a monoid is left-cancellative, its generalized Cayley graphs are forward vertex-transitive.

▶ **Fact 29.** Any generalized Cayley graph of a left-cancellative monoid is propagating.

**Proof.**
Let $G = \mathcal{C}[\![M,Q]\!]$ for some left-cancellative monoid $(M,\cdot)$ and some $Q\subseteq M$.
Let $s \xrightarrow{u,v}_G$ with $u,v\in A_G^*$. Let $t\in M$. We have to check that $t \xrightarrow{u,v}_G$.
So $u = [\![p_1]\!]\ldots[\![p_m]\!]$ and $v = [\![q_1]\!]\ldots[\![q_n]\!]$ for some $m,n\geq 0$ and $p_1,\ldots,p_m,q_1,\ldots,q_n\in Q$.
Thus $s\cdot(p_1\cdot\ldots\cdot p_m) = s\cdot(q_1\cdot\ldots\cdot q_n)$. As $\cdot$ is left-cancellative, $p_1\cdot\ldots\cdot p_m = q_1\cdot\ldots\cdot q_n$.
Hence $t \xrightarrow{u,v}_G t\cdot(p_1\cdot\ldots\cdot p_m) = t\cdot(q_1\cdot\ldots\cdot q_n)$. ◀

The *submonoid generated* by $Q\subseteq M$ is the least submonoid containing $Q$ *i.e.*

$$Q^* = \{\ q_1\cdot\ldots\cdot q_n \mid n\geq 0 \wedge q_1,\ldots,q_n\in Q\ \}.$$

▶ **Fact 30.** A monoid $M$ is generated by $Q$ $\iff$ 1 is a root of $\mathcal{C}[\![M,Q]\!]$.

A *Cayley graph of a monoid* $M$ is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ for $M$ generated by $Q$. The commutativity of such a graph coincides with the commutativity of $M$.

▶ **Fact 31.** A Cayley graph of a monoid $M$ is commutative iff $M$ is commutative.

**Proof.**
$\Longleftarrow$: By Fact 28.
$\Longrightarrow$: Let a commutative graph $G = \mathcal{C}[\![M, Q]\!]$ for $M$ generated by $Q$. Let $s, t \in M$.
So $s = p_1 \cdot \ldots \cdot p_m$ and $t = q_1 \cdot \ldots \cdot q_n$ for some $m, n \geq 0$ and $p_1, \ldots, p_m, q_1, \ldots, q_n \in Q$.
So $1 \xrightarrow{u}_G s \xrightarrow{v}_G s \cdot t$ for $u = [\![p_1]\!] \cdot \ldots \cdot [\![p_m]\!]$ and $v = [\![q_1]\!] \cdot \ldots \cdot [\![q_n]\!]$.
As $G$ is commutative, $1 \xrightarrow{vu}_G s \cdot t$.
As $G$ is deterministic and $1 \xrightarrow{v}_G t \xrightarrow{u}_G t \cdot s$, we get $s \cdot t = t \cdot s$. ◀

In order to show that Facts 17, 27, 30 characterize the Cayley graphs of monoids, we just have to apply Lemma 23 in the case where the root is propagating.

▶ **Lemma 32.** *Let $r$ be a propagating root of a deterministic graph $G$.*
*We can define the* path-operation $*_r$ *for any $s, t \in V_G$ by*

$$s \xrightarrow{u}_G s *_r t \quad \text{if} \quad r \xrightarrow{u}_G t \text{ for some } u \in A^*.$$

*Then $(V_G, *_r)$ is a monoid of identity $r$.*
*If $r$ is commutative then $*_r$ is commutative.*
*If $G$ is propagating then $*_r$ is left-cancellative.*
*If $G$ is co-deterministic then $*_r$ is right-cancellative.*
*If $G$ is target-complete then $*_r$ is left-invertible.*
*If $r$ is a source-complete co-root then $*_r$ is right-invertible.*
*If $r$ is source-complete then $\longrightarrow_G(r)$ generates $V_G$ by $*_r$; moreover for $r$ out-simple,*

$$G = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!] \text{ where } [\![q]\!] = a \text{ for any } r \xrightarrow{a}_G q.$$

**Proof.**
We have to apply Lemma 23 to the graph $\{\ s \xrightarrow{<u>} t \mid u \in A_G^* \wedge s \xrightarrow{u}_G t\ \}$ where $< > : A_G^* \longrightarrow A$ is an injective mapping. The operation $*_r$ is illustrated as follows:

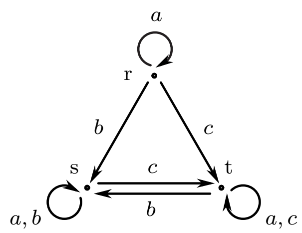

Note that by Fact 7, if $r$ is a source-complete vertex then $G$ is a source-complete graph.
Let us check the associativity of $*_r$ denoted by $\cdot$ in the sequel.
Let $x, y, z \in V_G$. We have to check that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
As $r$ is a root, there exists $v, w \in A_G^*$ such that $r \xrightarrow{v} y$ and $r \xrightarrow{w} z$.
Thus $x \xrightarrow{v} x \cdot y \xrightarrow{w} (x \cdot y) \cdot z$ and $y \xrightarrow{w} y \cdot z$. So $r \xrightarrow{vw} y \cdot z$ hence $x \xrightarrow{vw} x \cdot (y \cdot z)$.
As $G$ is deterministic, we get $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
It remains to check that $\longrightarrow_G(r)$ is a generating subset of $V_G$ by $\cdot$. Let $s \in V_G$.
There exists a path $r = s_0 \xrightarrow{a_1} s_1 \ldots s_{n-1} \xrightarrow{a_n} s_n = s$.
As $r$ is source-complete, there exists $r_1, \ldots, r_n$ such that $r \xrightarrow{a_1} r_1, \ldots, r \xrightarrow{a_n} r_n$.
For every $1 \leq i \leq n$, $s_i = s_{i-1} \cdot r_i$ hence $s = (\ldots(r \cdot r_1) \cdot \ldots) \cdot r_n = r_1 \cdot \ldots \cdot r_n \in Q^*$. ◀

We get a graph-theoretic characterization for the Cayley graphs of monoids.

▶ **Theorem 33.** *A graph is a Cayley graph of a (resp. commutative) monoid* if and only if *it is deterministic, source-complete with a propagating (resp. and locally commutative) out-simple root.*

**Proof.**
$\Longrightarrow$: By Facts 17, 27, 30 (resp. and Fact 28).
$\Longleftarrow$: By Lemma 32 (resp. and Lemma 8). ◀

For instance, the following finite graph $G$:



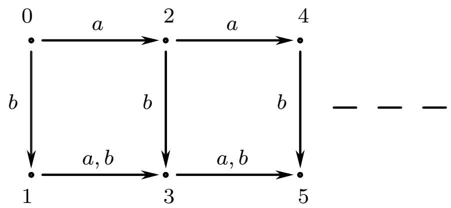

is deterministic, source-complete, and $r$ is a propagating out-simple root. By Theorem 33, $G$ is a Cayley graph of a monoid. Precisely by Lemma 32, $G = \mathcal{C}[\![\{r,s,t\}]\!]$ where $[\![r]\!] = a$, $[\![s]\!] = b$, $[\![t]\!] = c$ for the monoid $(\{r,s,t\},*_r)$ defined by the following Cayley table:

| $*_r$ | $r$ | $s$ | $t$ |
|---|---|---|---|
| $r$ | $r$ | $s$ | $t$ |
| $s$ | $s$ | $s$ | $t$ |
| $t$ | $t$ | $s$ | $t$ |

Another example is given by the following infinite graph:

$$G \ = \ \{\ n \xrightarrow{a} n{+}2 \mid n \in \mathbb{N}\ \} \cup \{\ 2n \xrightarrow{b} 2n{+}1 \mid n \in \mathbb{N}\ \} \cup \{\ 2n{+}1 \xrightarrow{a,b} 2n{+}3 \mid n \in \mathbb{N}\ \}$$

represented as follows:




Such a graph is deterministic, source-complete, and the root $0$ is out-simple, propagating and locally commutative. By Theorem 33, $G$ is a Cayley graph of a commutative monoid. Precisely by Lemma 32, $G = \mathcal{C}[\![\mathbb{N},\{1,2\}]\!]$ where $[\![1]\!] = b$ and $[\![2]\!] = a$ for the path-operation $*$ from $0$ defined for any $p,q \in \mathbb{N}$ by

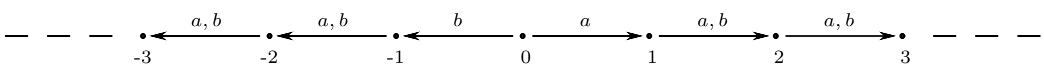
$$p * q \ = \ \begin{cases} p+q & \text{if } p \text{ or } q \text{ is even,} \\ p+q+1 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

which is indeed a commutative monoid.
Let us adapt Theorem 33 to right-cancellative monoids.

▶ **Theorem 34.** *A graph is a Cayley graph of a right-cancellative* (*resp. commutative*) *monoid* if and only if *it is deterministic and co-deterministic, source-complete with a propagating* (*resp. and locally commutative*) *out-simple root.*

**Proof.**
$\Longrightarrow$: By Facts 17, 18, 27, 30 (resp. and Fact 28).
$\Longleftarrow$: By Lemma 32 (resp. and Lemma 8). ◀

By Theorem 34, the following graph:




is a Cayley graph of a right-cancellative monoid. By Lemma 32, this graph is equal to $\mathcal{C}[\![\mathbb{Z},\{1,-1\}]\!]$ where $[\![1]\!] = a$ and $[\![-1]\!] = b$ for the right-cancellative monoid $(\mathbb{Z},*)$ where the path-operation $*$ from $0$ is defined for any $m,n \in \mathbb{Z}$ by

$$m * n \;=\; \begin{cases} m+|n| & \text{if } \; m \geq 0, \\ m-|n| & \text{if } \; m < 0. \end{cases}$$

We also deduce a known graph-theoretic characterization for the Cayley graphs of left-cancellative and cancellative monoids [1].

▶ **Theorem 35.** *A graph is a Cayley graph of a left-cancellative (resp. cancellative) monoid* iff *it is rooted, deterministic (resp. and co-deterministic), simple, propagating or forward vertex-transitive.*

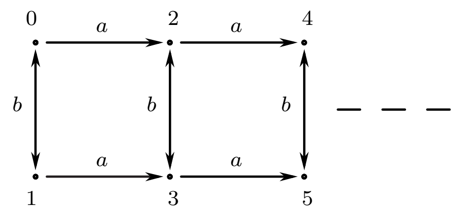
**Proof.**
By Proposition 11 (c) and for any deterministic graph, it is equivalent for a graph to be propagating and forward vertex transitive.
$\Longrightarrow$: By Facts 17, 18, 29, 30.
$\Longleftarrow$: By Fact 10 and Lemma 32. ◀

Let us strengthen Theorem 35 to the commutative case.

▶ **Theorem 36.** *For any graph $G$, the following properties are equivalent:*
**a)** *$G$ is a Cayley graph of a left-cancellative (resp. cancellative) commutative monoid,*
**b)** *$G$ is deterministic (resp. and co-deterministic), propagating with a locally commutative and out-simple root,*
**c)** *$G$ is rooted, deterministic (resp. and co-deterministic), simple, forward vertex-transitive and commutative.*

**Proof.**
It suffices to apply Facts 10 and 28 with Lemma 8 to Theorem 35. ◀

For instance the following graph




is rooted, deterministic and co-deterministic, simple, forward vertex-transitive and commutative. By Theorem 36, it is a Cayley graph of a cancellative and commutative monoid. By Lemma 32, this graph is equal to $\mathcal{C}[\![\mathbb{N},\{1,2\}]\!]$ where $[\![1]\!] = b$ and $[\![2]\!] = a$ for the path-operation $*$ from $0$ defined for any $p,q \in \mathbb{N}$ by

$$p * q \;=\; \begin{cases} p+q & \text{if } \; p \text{ or } q \text{ is even,} \\ p+q-2 & \text{if } \; p \text{ and } q \text{ are odd} \end{cases}$$

which is indeed a cancellative and commutative monoid.

## 6 Cayley graphs of semigroups and semilattices

We apply the previous characterizations for the Cayley graphs of monoids to the Cayley graphs of semigroups (see Theorem 37) and of semilattices (see Theorem 40).

Recall that a *Cayley graph of a semigroup* $M$ is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ such that $M = Q^+$ whose $Q^+ = \{\, q_1\cdot\ldots\cdot q_n \mid n > 0 \wedge q_1,\ldots,q_n \in Q \,\}$ is the *subsemigroup generated* by $Q$. Let us extend Theorem 33 into a characterization of these graphs.

▶ **Theorem 37.** *A graph $G$ is a Cayley graph of a (resp. commutative) semigroup* iff *it is deterministic and there is an injection $i$ from $A_G$ into $V_G$ such that*

*$G$ is accessible from $i(A_G)$,*

$i(a) \overset{u}{\longrightarrow} \overset{v}{\longleftarrow} i(b) \implies s \overset{au,bv}{\longrightarrow}$ *for any $s \in V_G$, $a,b \in A_G$ and $u,v \in A_G^*$*

*(resp. and $i(a) \overset{b}{\longrightarrow} \overset{a}{\longleftarrow} i(b)$ for any $a,b \in A_G$).*

**Proof.**

$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some semigroup $(M,\cdot)$ generated by $Q$.
By Fact 17, $G$ is deterministic.
If $M$ is not a monoid *i.e.* it has no identity, we turn $M$ into a monoid $M' = M \cup \{1\}$ by just adding an identity 1 *i.e.* $p{\cdot}1 = 1{\cdot}p = p$ for any $p \in M'$.
Let $M' = M$ when $M$ is a monoid.
In both cases, $M'$ is a monoid of identity 1. Let $G' = \mathcal{C}[\![M',Q]\!] = G \cup \{\, 1 \overset{[\![q]\!]}{\longrightarrow} q \mid q \in Q \,\}$.
Let $i : A_G \longrightarrow Q$ defined by $i([\![q]\!]) = q$ for any $q \in Q$.
As $M$ is generated by $Q$, the graph $G$ is accessible from $Q = i(A_G)$.
Let $i([\![p]\!]) \overset{u}{\longrightarrow}_G {}_G\overset{v}{\longleftarrow} i([\![q]\!])$ for some $p,q \in Q$ and $u,v \in A_G^*$.
So $p \overset{u}{\longrightarrow}_G {}_G\overset{v}{\longleftarrow} q$ hence $1 \overset{[\![p]\!]u\,,[\![q]\!]v}{\longrightarrow}_{G'}$. By Fact 27, 1 is a propagating vertex of $G'$.
Let $s \in V_G$. Therefore $s \overset{[\![p]\!]u\,,[\![q]\!]v}{\longrightarrow}_{G'}$ hence $s \overset{[\![p]\!]u\,,[\![q]\!]v}{\longrightarrow}_{G}$.
If $M$ is commutative, we have

$$i([\![p]\!]) = p \overset{[\![q]\!]}{\longrightarrow}_G p\cdot q = q\cdot p \;{}_G\overset{[\![p]\!]}{\longleftarrow} q = i([\![q]\!]) \quad \text{for any } p,q \in Q.$$

$\Longleftarrow$: Let $G$ be a deterministic graph.
Let an injection $i : A_G \longrightarrow V_G$ such that $G$ is accessible from $Q = i(A_G)$ and such that

$$s \overset{au,bv}{\longrightarrow}_G \text{ when } i(a) \overset{u}{\longrightarrow}_G {}_G\overset{v}{\longleftarrow} i(b) \text{ for any } s \in V_G,\ a,b \in A_G \text{ and } u,v \in A_G^*. \qquad (1)$$

Note that by (1), we get $s \overset{a}{\longrightarrow}_G$ for any $s \in V_G$ and $a \in A_G$, hence $G$ is source-complete.
We take a new vertex $r$ and we define the graph

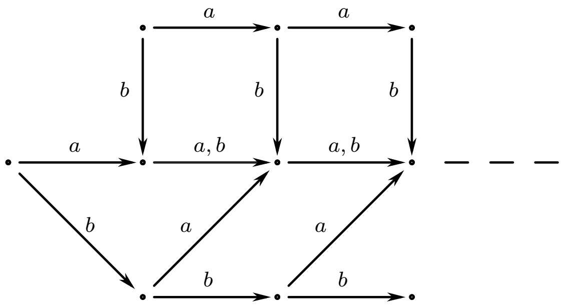
$$\widehat{G} = G \cup \{\, r \overset{a}{\longrightarrow} i(a) \mid a \in A_G \,\}.$$

So $\widehat{G}$ remains deterministic and source-complete, and $r$ is out-simple.
As $G$ is accessible from $Q$, $r$ is a root of $\widehat{G}$. By condition (1), $r$ is propagating.
By Lemma 32, $V_{\widehat{G}} = V_G \cup \{r\}$ is a monoid for the path operation $*_r$ of identity $r$ and generated by $Q$ with $\widehat{G} = \mathcal{C}[\![V_{\widehat{G}},Q]\!] = \mathcal{C}[\![V_G,Q]\!] \cup \{\, r \overset{[\![q]\!]}{\longrightarrow} q \mid q \in Q \,\}$.
As $r$ is not the target of an edge of $\widehat{G}$ and by definition, $*_r$ remains an internal operation on $V_G$ *i.e.* $p *_r q \neq r$ for any $p,q \in V_G$. Thus $*_r$ remains associative on $V_G$.
Finally $G = \widehat{G}_{|V_G} = \mathcal{C}[\![V_G\,,Q]\!]$ and $(V_G\,,*_r)$ is a semigroup.
Let us assume that $i(a) \overset{b}{\longrightarrow}_G {}_G\overset{a}{\longleftarrow} i(b)$ for any $a,b \in A_G$.
So $r$ is locally commutative for $\widehat{G}$ and by Lemmas 8 and 32, $*_r$ is commutative. ◀

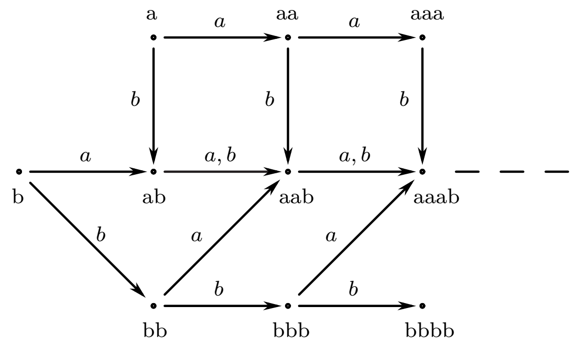
For instance, the two semilines $G_a = \{\, n \overset{a}{\longrightarrow} n{+}1 \mid n \in \mathbb{N}{-}\{0\} \,\} \cup \{\, n \overset{a}{\longrightarrow} n{-}1 \mid n \in \mathbb{Z}{-}\mathbb{N} \,\}$ is a generalized Cayley graph of a monoid (see the previous example) but by Theorem 37, $G_a$ is not a Cayley graph of a semigroup since it is not connected and has a unique label. On the other hand, $G_a \cup G_b$ is a Cayley graph of a (non commutative) semigroup. By Lemma 32, $G_a \cup G_b = \mathcal{C}[\![\mathbb{Z}-\{0\},\{1,-1\}]\!]$ where $[\![1]\!] = a$ and $[\![-1]\!] = b$ for the associative path-operation $*$ defined by

$$m * n = m + sgn(m)\,|n| \quad \text{for any } m,n \in \mathbb{Z}-\{0\}$$

where $sgn(m) = 1$ for any $m > 0$ and $sgn(m) = -1$ for any $m < 0$.
Another example is given by the following graph:




By Theorem 37, this graph is a Cayley graph of a commutative semigroup. Precisely and by coding the vertices as follows:




and by Lemma 32, this graph is the Cayley graph $\mathcal{C}(V, \{a, b\})$ for the regular vertex set $V = a^*a \cup a^*b \cup b^*bb$ and for the associative and commutative binary operation $*$ on $V$ defined for any $m, n > 0$, $i, j \geq 0$, $p, q > 1$ by the following Cayley table:

| $*$ | $a^n$ | $a^jb$ | $b^q$ |
|---|---|---|---|
| $a^m$ | $a^{m+n}$ | $a^{j+m}b$ | $a^{m+q-1}b$ |
| $a^ib$ | $a^{i+n}b$ | $a^{i+j+1}b$ | $a^{i+q}b$ |
| $b^p$ | $a^{n+p-1}b$ | $a^{j+p}b$ | $b^{p+q}$ |

When a semigroup is cancellative, its generalized Cayley graphs are forward vertex-transitive.

▶ **Fact 38.** Any generalized Cayley graph of a cancellative semigroup is propagating.

**Proof.**
Let $(M, \cdot)$ be a cancellative semigroup.
According to the proof of Fact 29, it remains to check that

$$s = s \cdot r \implies t = t \cdot r \text{ for any } r, s, t \in M.$$

Let $r, s, t \in M$ such that $s = s \cdot r$. So $s \cdot s = s \cdot r \cdot s$.
As $\cdot$ is left-cancellative, $s = r \cdot s$. Thus $t \cdot s = t \cdot r \cdot s$.
As $\cdot$ is right-cancellative, we get $t = t \cdot r$. ◀

Let us extend Theorem 35 to the cancellative semigroups in order to get another description of their Cayley graphs [1].

▶ **Theorem 39.** *A graph $G$ is a Cayley graph of a cancellative (resp. and commutative) semigroup* if and only if *it is deterministic and co-deterministic, and there is an injection $i$ from $A_G$ into $V_G$ such that*

*$G$ is accessible from $i(A_G)$,*
$i(a) \overset{u}{\longrightarrow} \overset{v}{\longleftarrow} i(b) \iff i(c) \overset{au,bv}{\longrightarrow}$ *for any* $a,b,c \in A_G$ *and* $u,v \in A_G^*$
*(resp. and* $i(a) \overset{b}{\longrightarrow} \overset{a}{\longleftarrow} i(b)$ *for any* $a,b \in A_G$*).*

**Proof.**
Let us complete the proof of Theorem 37.
$\Longrightarrow$: The monoid $M'$ remains cancellative (for instance see [Cau]).
By Fact 29, $G'$ is a propagating graph. Let $c \in A_G$. So $1 \downarrow_{G'} i(c)$.
Let $a,b \in A_G$ and $u,v \in A_G^*$. We have $1 \overset{au,bv}{\longrightarrow}_{G'} \iff i(c) \overset{au,bv}{\longrightarrow}_{G'}$.
Thus $i(a) \overset{u}{\longrightarrow}_G {}_G\overset{v}{\longleftarrow} i(b) \iff i(c) \overset{au,bv}{\longrightarrow}_G$.
$\Longleftarrow$: Let us assume that
$$i(a) \overset{u}{\longrightarrow}_G {}_G\overset{v}{\longleftarrow} i(b) \iff i(c) \overset{au,bv}{\longrightarrow}_G \quad \text{for any } a,b,c \in A_G \text{ and } u,v \in A_G^*.$$
This equivalence coincides with
$$r \overset{au,bv}{\longrightarrow}_{\widehat{G}} \iff i(c) \overset{au,bv}{\longrightarrow}_{\widehat{G}} \quad \text{for any } a,b,c \in A_G \text{ and } u,v \in A_G^*.$$
As $r$ is target of no edge, we get
$$r \overset{x,y}{\longrightarrow}_{\widehat{G}} \iff s \overset{x,y}{\longrightarrow}_{\widehat{G}} \quad \text{for any } r \longrightarrow_{\widehat{G}} s \text{ and } x,y \in A_G^*.$$
By Proposition 11 (d), $\widehat{G}$ is a propagating graph.
By Lemma 32, $*_r$ is left-cancellative on $V_{\widehat{G}} = V_G \cup \{r\}$ hence on $V_G$.
By hypothesis $G$ is co-deterministic. However $\widehat{G}$ can be not co-deterministic.
As $r$ is target of no edge, we get that $*_r$ is also right-cancellative.
Precisely let $s,s',t \in V_G$ such that $s *_r t = s' *_r t$.
There exists $u \in A_G^*$ such that $r \overset{u}{\longrightarrow}_{\widehat{G}} t$.
So $s \overset{u}{\longrightarrow}_{\widehat{G}} s *_r t$ and $s' \overset{u}{\longrightarrow}_{\widehat{G}} s' *_r t = s *_r t$.
As $r$ is target of no edge, we have $s \overset{u}{\longrightarrow}_G s *_r t$ and $s' \overset{u}{\longrightarrow}_G s *_r t$.
As $G$ is co-deterministic, we get $s = s'$. ◀

For instance, let us consider the graph $G = \{ (m,n) \overset{p}{\longrightarrow} (m+p, n+1) \mid m,n,p \in \mathbb{N} \}$.
By Theorem 39, this graph is a Cayley graph of a commutative and cancellative semigroup.
By Lemma 32, $G = \mathcal{C}[\![\mathbb{N}{\times}\mathbb{N}, \mathbb{N}{\times}\{0\}]\!]$ where $[\![(p,0)]\!] = p$ for any $p \in \mathbb{N}$, and for the path-operation $*$ defined by
$$(m,n) * (p,q) = (m+p, n+q+1) \quad \text{for any } m,n,p,q \in \mathbb{N}$$
which is indeed associative, cancellative and commutative.

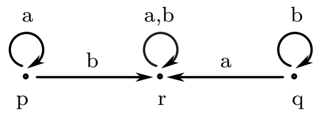
A *semilattice* $(M,\cdot)$ is a commutative semigroup which is also *idempotent*: $p \cdot p = p$ for any $p \in M$. Let us apply Lemma 32 with Theorem 37.

▶ **Theorem 40.** *A graph $G$ is a Cayley graph of a semilattice* if and only if *it is deterministic and there is an injection $i$ from $A_G$ into $V_G$ such that*
*$G$ is accessible from $i(A_G)$*
$i(a) \overset{u}{\longrightarrow} \overset{v}{\longleftarrow} i(b) \Longrightarrow s \overset{au,bv}{\longrightarrow}$ *for any* $s \in V_G$, $a,b \in A_G$ *and* $u,v \in A_G^*$
$i(a) \overset{b}{\longrightarrow} \overset{a}{\longleftarrow} i(b)$ *for any* $a,b \in A_G$
$i(a) \overset{u}{\longrightarrow} s \Longrightarrow s \overset{au}{\longrightarrow} s$ *for any* $s \in V_G$, $a \in A_G$ *and* $u \in A_G^*$.

**Proof.**
Let us complete the proof of Theorem 37.
$\Longrightarrow$: Suppose further that $s \in M$ is idempotent *i.e.* $s{\cdot}s = s$.
Let $i([\![q]\!]) \overset{u}{\longrightarrow}_G s$ with $u = [\![q_1]\!]{\cdot}\ldots{\cdot}[\![q_n]\!]$ for some $n \geq 0$ and $q, q_1, \ldots, q_n \in Q$.
As $i([\![q]\!]) = q$, we get $s = q{\cdot}q_1{\cdot}\ldots{\cdot}q_n$ thus $s \overset{[\![q]\!]u}{\longrightarrow}_G s{\cdot}q{\cdot}q_1{\cdot}\ldots{\cdot}q_n = s{\cdot}s = s$.

$\Longleftarrow$: Let $i(a) \xrightarrow{u} s$ and $s \xrightarrow{au} s$ for some $s \in V_G$, $a \in A_G$ and $u \in A_G^*$.
So $r \xrightarrow{au}_{\widehat{G}} s$. Thus $s \xrightarrow{au}_{\widehat{G}} s *_r s$.
As $r \notin V_G$ is initial (not target of an edge), $s \xrightarrow{au}_G s *_r s$.
As $G$ is deterministic, we get $s *_r s = s$. ◀

For instance, the following finite graph:




satisfies the properties of Theorem 40 with $i(a) = p$ and $i(b) = q$, hence is a Cayley graph of a semilattice. By Lemma 32, this graph is equal to $\mathcal{C}[\![\{p,q,r\},\{p,q\}]\!]$ where $[\![p]\!] = a$ and $[\![q]\!] = b$ for the semilattice $(\{p,q,r\},*)$ where $*$ is defined by the following Cayley table:

| $*$ | $p$ | $q$ | $r$ |
|---|---|---|---|
| $p$ | $p$ | $r$ | $r$ |
| $q$ | $r$ | $q$ | $r$ |
| $r$ | $r$ | $r$ | $r$ |

## 7 Cayley graphs of groups

We have already given in [1] a graph-theoretic characterization for the Cayley graphs of groups. We give here another description in a weak form. Finally, we strengthen these two characterizations for the Cayley graphs of abelian groups (see Theorem 45).

Recall that a *group* is a monoid left and right-invertible (w.r.t. the unit element). This induces the target completeness of its generalized Cayley graphs.

▶ **Fact 41.** Any generalized Cayley graph of a group is target-complete.

**Proof.**
Let $G = \mathcal{C}[\![M,Q]\!]$ for some group $M$ and $Q \subseteq M$.
We have $p{\cdot}q^{-1} \xrightarrow{[\![q]\!]}_G p$ for any $p \in M$ and $q \in Q$. Thus $G$ is target-complete. ◀

Another basic property of the generalized Cayley graphs follows from Fact 27.

▶ **Fact 42.** For any generalized Cayley graph of a group, $1$ is a chain-propagating vertex.

**Proof.**
Let $G = \mathcal{C}[\![M,Q]\!]$ for some group $M$ and $Q \subseteq M$. For any $q \in Q$, we have

$$s \xrightarrow{[\![q]\!]}_G t \iff t \xrightarrow{[\![\overline{q}]\!]}_G s \iff s = t \cdot q \iff t = s \cdot q^{-1}.$$

Let $H = \mathcal{C}[\![M, Q \cup Q^{-1}]\!]$ for $[\![\;]\!]$ extended injectively on $Q^{-1} - Q$. We have

$$s \xrightarrow{u}_G t \iff s \xrightarrow{i(u)}_H t \text{ for any } u \in (Q \cup \overline{Q})^* \text{ and } s,t \in M$$

where $i$ is the morphism on $(Q \cup \overline{Q})^*$ into $(Q \cup Q^{-1})^*$ defined by

$$i(q) = q \text{ and } i(\overline{q}) = q^{-1} \text{ for any } q \in Q.$$

By Fact 27, $1$ is a propagating vertex of $H$ *i.e.* $1$ is a chain-propagating vertex of $G$. ◀

Recall that a *Cayley graph of a group* $M$ is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ such that $M$ is generated by $Q$ *i.e.* $M$ is equal to the least subgroup $(Q \cup Q^{-1})^*$ containing $Q$ where $Q^{-1} = \{\, q^{-1} \mid q \in Q \,\}$ is the set of inverses of the elements in $Q$.

▶ **Fact 43.** A group $M$ is generated by $Q \iff \mathcal{C}[\![M,Q]\!]$ is connected.

Let us adapt Theorem 35 for groups. We extend Lemma 32.

▶ **Proposition 44.** *Let $r$ be a chain-propagating vertex of a connected, deterministic and co-deterministic graph $G$. We can define the* chain-operation $\overline{*}_r$ *for any $s,t \in V_G$ by*

$$s \xrightarrow{u}_G s \,\overline{*}_r\, t \quad \text{if} \quad r \xrightarrow{u}_G t \ \text{ for some } u \in (A_G \cup \overline{A_G})^*.$$

*Then $(V_G, \overline{*}_r)$ is a right-cancellative monoid of identity $r$.*
*If $r$ is chain-commutative then $\overline{*}_r$ is commutative.*
*If $r$ is source and target-complete then $(V_G, \overline{*}_r)$ is a group generated by $\longrightarrow_G(r)$ ;*
*moreover if $r$ is an in-simple and out-simple vertex then*

$$G = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!] \ \text{ where } [\![q]\!] = a \ \text{ for any } r \xrightarrow{a}_G q.$$

**Proof.**
**i)** The graph $\overline{G}$ is strongly connected and it remains deterministic and co-deterministic.
Furthermore $r$ is a propagating vertex of $\overline{G}$.
By Lemma 32, $(V_{\overline{G}}\,, *_r)$ is a right-cancellative monoid of identity $r$.
For any $u \in (A_G \cup \overline{A_G})^*$, the binary relation $\xrightarrow{u}_G$ on $V_G = V_{\overline{G}}$ is equal to $\xrightarrow{u}_{\overline{G}}$.
Thus ${}_{\overline{G}}*_r$ is equal to ${}_G\overline{*}_r$ . This operation $\overline{*}_r$ is illustrated as follows:


If $r$ is chain-commutative for $G$ then $r$ is commutative for $\overline{G}$,
hence by Lemma 32, ${}_{\overline{G}}*_r$ is commutative.
**ii)** Let us assume that $r$ is source-complete and target-complete.
By Fact 7, $G$ is source and target-complete.
So $\overline{G}$ remains source and target-complete.
By Lemma 32, $(V_{\overline{G}}\,, *_r)$ is a group generated by $\longrightarrow_G(r)$.
Suppose that in addition $r$ is an in-simple and out-simple vertex of $G$. However $r$ can be not an out-simple vertex of $\overline{G}$. As in the proof of Fact 42, we restrict $\overline{G}$ to the graph

$$\widehat{G} = G \cup \{\, t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a}_G t \not\longrightarrow_G s \,\}.$$

So $\widehat{G}$ is the label restriction of $\overline{G}$ to $A_{\widehat{G}}$.
Thus $\widehat{G}$ remains deterministic and $r$ is a source-complete propagating root of $\widehat{G}$.
Furthermore $r$ is an out-simple vertex of $\widehat{G}$.
By Lemma 32, $\widehat{G} = \mathcal{C}[\![V_G, \longrightarrow_{\widehat{G}}(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a}_{\widehat{G}} s$.
Precisely for any $s \in \longrightarrow_{\widehat{G}}(r) = \longrightarrow_G(r) \cup \longrightarrow_{G^{-1}}(r)$, we have

$$[\![s]\!] = \begin{cases} a & \text{if } r \xrightarrow{a}_G s \\ \overline{a} & \text{if } s \xrightarrow{a}_G r. \end{cases}$$

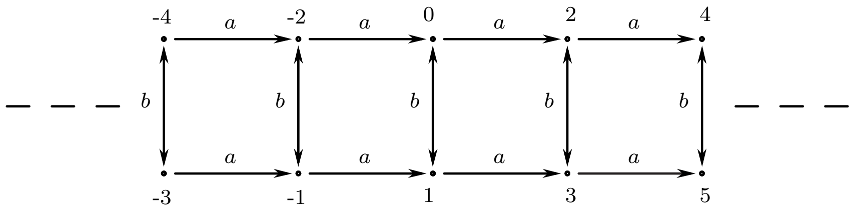
Finally $G = \widehat{G}^{|A_G} = \mathcal{C}[\![V_G, \longrightarrow_G(r)]\!]$. ◀

We can complete a graph-theoretic characterization for the Cayley graphs of groups [1].

▶ **Theorem 45.** *For any graph $G$, the following properties are equivalent:*
**a)** *$G$ is a Cayley graph of a (resp. commutative) group,*
**b)** *$G$ is connected, deterministic and co-deterministic, with a chain-propagating (resp. and locally commutative) source and target-complete in-simple and out-simple vertex,*
**c)** *$G$ is connected, simple, deterministic and co-deterministic, vertex-transitive (resp. and commutative).*

**Proof.**
$(a) \implies (b)$: Let $G$ be a Cayley graph of a group.
By Fact 17, $G$ is deterministic and source-complete.
By Fact 18, $G$ is co-deterministic and simple.
By Fact 41, $G$ is target-complete. By Fact 42, the identity $1$ is a chain-propagating vertex.
By Fact 43, $G$ is connected.
If the group is commutative then by Fact 21, the identity $1$ is also locally commutative.
$(b) \implies (a)$: By Fact 7 and Proposition 44 (resp. and Lemma 14).
$(b) \iff (c)$: By Fact 7 and Corollary 16 (e) (resp. and Lemma 14). ◀

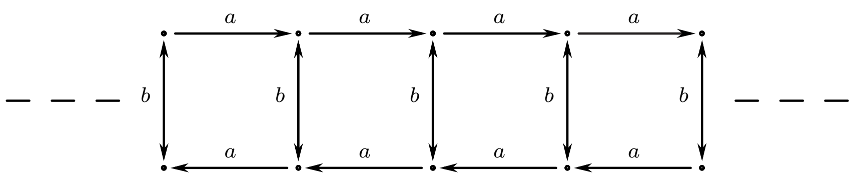
Let us generalize the graph defined after Theorem 36 into the following graph:




This graph satisfies the properties of Theorem 45 (b) or (c). Thus and according to Proposition 44, the resulting chain-operation $\overline{*}$ defined for any $p, q \in \mathbb{Z}$ by

$$p \overline{*} q \;=\; \begin{cases} p+q & \text{if } p \text{ or } q \text{ is even,} \\ p+q-2 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

makes that $(\mathbb{Z}, \overline{*})$ is a commutative (abelian) group. Similarly, the following graph:




is a Cayley graph of a (non commutative) group.

## 8 Conclusion

We obtained simple graph-theoretic characterizations for Cayley graphs of elementary algebraic structures. This is a first step in a graph description of algebraic structures.

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