A LOWER BOUND FOR THE ORDER OF TELESCOPERS FOR A HYPERGEOMETRIC TERM

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Abstract. We present an algorithm to compute a lower bound for the order of the minimal telescoper for a given hypergeometric term. We also describe a Maple implementation of the algorithm and show the efficiency improvement it provides to Zeilberger's algorithm in the construction of the telescopers.

Résumé. Cet article présente un algorithme de calcul d'une borne inférieure pour le téléscopeur minimal associé à un terme hypergéométrique. Une implantation en Maple est également décrite, qui permet d'observer le gain de notre méthode par rapport à l'algorithme de Zeilberger lors de la construction des téléscopeurs.

1. Preliminaries

Let K be an algebraically closed field of characteristic 0, the variables n, k be integer-valued, and E_n, E_k be the corresponding shift operators, acting on functions of n and k, by $E_n f(n,k) = f(n+1,k)$, $E_k f(n,k) = f(n,k+1)$. A K-valued function $t(k)$ is a *hypergeometric term* of k over K if the consecutive term ratio $R = E_k t/t$ is a rational function of k over K. This rational function is the *certificate* of $t(k)$. A K-valued function $T(n,k)$ is a hypergeometric term of two variables n and k if the two consecutive term ratios $R_1 = E_n T/T$, and $R_2 = E_k T/T$ are rational functions of n and k over K. They are called the n-certificate and the k-certificate of T, respectively. Given a hypergeometric term $T(n,k)$ as input, Zeilberger's algorithm [13, 15, 16] (which we name hereafter as $\mathcal{Z}$) constructs for $T(n,k)$ a *Z-pair* (L,G), provided that such a pair exists. The computed Z-pair consists of L, a linear recurrence operator with coefficients which are polynomials of n over K

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \qquad (1)$$

and a hypergeometric term $G(n,k)$ such that

$$LT(n,k) = (E_k - 1)G(n,k). \qquad (2)$$

The k-free operator L is called a *telescoper*. It is noteworthy that the problem of establishing a necessary and sufficient condition for the applicability of $\mathcal{Z}$ to $T(n,k)$ is solved and presented in [1] (the well-known *fundamental theorem* [15, 16] only provides a sufficient condition). It is proven in [16] that if there exists a Z-pair for $T(n,k)$, then $\mathcal{Z}$ terminates with one of the Z-pairs and the telescoper L in the returned Z-pair is of minimal order. The computed telescoper L is unique up to a left-hand factor $P(n) \in K[n]$, and we name it *the minimal telescoper.*

$\mathcal{Z}$ has a wide range of applications which include finding closed forms of definite sums of hypergeometric terms, verification of combinatorial identities, and asymptotic estimation [13, 16, 12].

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The algorithm uses an *item-by-item examination* on the order ρ of the operator L in (1). It starts with the value of 0 for ρ and increases ρ until it is successful in finding a $\mathcal{Z}$-pair (L, G) for T. As a consequence, we waste resources trying to compute without success a telescoper of $\operatorname{ord} L < \rho$ where ρ is the order of the minimal telescoper.

In this paper, we present an algorithm that computes a lower bound for the order of the telescopers. The general approach of the algorithm can be described as follow. If the given hypergeometric term $T(n, k)$ is not k-summable, i.e., there does not exist a hypergeometric term T_1 such that $T = (E_k - 1)T_1$, then we represent T as $(E_k - 1)T_1 + T_2$ where the hypergeometric term T_2 has some specific features each of which ensures that T_2 is not k-summable. It is then easy to show that a telescoper for T exists iff a telescoper for T_2 exists, and the sets of telescopers for T and T_2 are equal. We consider recurrence operators, called crushing operators, with the distinguishing property that if $M \in K[n, E_n]$ is a crushing operator for T_2, then MT_2 does not have at least one of the specific features that T_2 does (this does not guarantee that MT_2 is k-summable, though). It follows that the order of the minimal telescoper for T_2 is always greater than or equal to that of the minimal crushing operator for T_2. We then describe an algorithm to compute a lower bound μ for the order of the crushing operators for T_2. This value is automatically also a lower bound for the order of the telescopers for T.

When the algorithm is used in conjunction with the algorithm to determine the applicability of $\mathcal{Z}$ to $T(n, k)$ [1], it allows one to use $\mathcal{Z}$ to compute a $\mathcal{Z}$-pair only if the existence of such a pair is guaranteed; and in this case, one can use μ as the starting value for the order of L, instead of 0. Additionally, the computation of a lower bound is much less expensive than the construction of a telescoper using $\mathcal{Z}$, especially when the order of the minimal telescoper is high.

Note that for the case where $T(n, k)$ is also a rational function, there exists a direct algorithm to compute the minimal telescoper for T efficiently without using item-by-item examination [8].

The paper is organized in the following manner. In Sections 2 we discuss some known results which are needed in subsequent sections. They include a description of the additive decomposition problem of hypergeometric terms [2, 3], and a criterion for the applicability of $\mathcal{Z}$ [1]. The main result of Section 3 is a theorem to compute a lower bound for the order of the minimal crushing operator M. An algorithm which realizes this theorem is presented in Section 4. We conclude the paper with a description of an implementation of the algorithm in Section 5. Various examples are used to show the advantages of this implementation over other implementations of the original $\mathcal{Z}$.

Throughout the paper, K is an algebraically closed field of characteristic 0, and $\mathbb{N}$ denotes the set of nonnegative integers. Following [13], we write $T_1(n, k) \sim T_2(n, k)$ if two hypergeometric terms $T_1(n, k)$ and $T_2(n, k)$ are *similar*, i.e., their ratio is a rational function of n and k.

2. The Additive Decomposition Problem and the Existence of a Telescoper

We begin this section with the notion of *Rational Normal Forms* (RNF) of rational functions [4]. This concept plays an important role in the follow-up algorithms.

Definition 1. *Let Λ be a field of characteristic 0. Let $R \in \Lambda(x)$ be a nonzero rational function. If there exist nonzero polynomials $f_1, f_2, v_1, v_2 \in \Lambda[x]$ such that*

(i) $R = F \cdot \frac{EV}{V}$ *where* $F = \frac{f_1}{f_2}$, $V = \frac{v_1}{v_2}$, *and* $\gcd(v_1, v_2) = 1$,

(ii) $\gcd(f_1, E^h f_2) = 1$ *for all* $h \in \mathbb{Z}$,

then $F \cdot \frac{EV}{V}$ is an RNF of R.

Note that every rational function has an RNF [3, Thm. 1] which in general is not unique, and the rational function F in (i) with property (ii) is called the *kernel* of the RNF.

2.1. **The Additive Decomposition Problem.** For a hypergeometric term $T(k)$ of k over $K(n)$, the algorithm to solve the additive decomposition problem [2, 3] constructs two hypergeometric terms $T_1(k)$, $T_2(k)$ similar to $T(k)$ such that

$$T(k) = (E_k - 1)\, T_1(k) + T_2(k), \tag{3}$$

and either $T_2 = 0$ or the k-certificate of T_2 has an RNF

$$\frac{f_1}{f_2} \frac{E_k(v_1/v_2)}{(v_1/v_2)} \tag{4}$$

with v_2 of minimal possible degree. Note that any RNF of the k-certificate of T_2 has $v_2 \in K(n)[k]$ of the same (minimal possible) degree.

Lemma 1. [2, 3] *Let $T(k)$ be a hypergeometric term over $K(n)$. If (3) is an additive decomposition of $T(k)$, then for any RNF of the form (4) of the k-certificate of $T_2(k)$, and for each irreducible p from $K(n)[k]$ such that $p \,|\, v_2$, the following three properties hold:*

$$\mathbf{Pa}: E_k^h p \,|\, v_2 \Rightarrow h = 0, \quad \mathbf{Pb}: E_k^h p \,|\, f_1 \Rightarrow h < 0, \quad \mathbf{Pc}: E_k^h p \,|\, f_2 \Rightarrow h > 0. \tag{5}$$

If the hypergeometric term $T_2(k)$ in (3) vanishes, then $T(k)$ is said to be k-*summable*. Otherwise, each irreducible factor p of v_2 has properties **Pa, Pb, Pc**, and T is k-non-summable.

Proposition 1. [2, 3] *Let an RNF of the k-certificate of a given hypergeometric term $T(n, k)$ be of the form (4). If there exists at least one irreducible factor p of v_2 such that all three properties* **Pa, Pb, Pc** *hold, then $T(n, k)$ is k-non-summable.*

Proposition 2. *Let the similar hypergeometric terms $T(n, k)$, $T_1(n, k)$, and $T_2(n, k)$ be as defined in (3). (The algorithm to solve the additive decomposition problem is applied to $T(n, k)$ w.r.t. k over $K(n)$.) Then*

(i) *A* Z*-pair for $T(n, k)$ exists iff a* Z*-pair for $T_2(n, k)$ exists;*
(ii) *The minimal telescopers for T and T_2 are the same.*

Proof:
(i): Let (L, G) be a Z-pair for T_2. It follows from (3) that $LT = (E_k - 1)\,(LT_1 + G)$. Since $T_1 \sim T_2, T_2 \sim G$, and $\sim$ is an equivalence relation, $LT_1 + G$ is a hypergeometric term [13, Prop. 5.6.2]. Consequently, $(L, LT_1 + G)$ is a Z-pair for T. On the other hand, let (L, G) be a Z-pair for T. By following the same argument, one can easily show that $(L, G - LT_1)$ is a Z-pair for T_2.
(ii): Let L be the minimal telescoper for T_2. It follows from (i) that L is a telescoper for T. Suppose there exists a telescoper $\tilde{L}$ for T and $\operatorname{ord} \tilde{L} < \operatorname{ord} L$. Then it follows from (i) then $\tilde{L}$ is a telescoper for T_2 and $\operatorname{ord} \tilde{L} < \operatorname{ord} L$. Contradiction. ■

Definition 2. *A polynomial $p(n, k) \in K[n, k]$ is* integer-linear *if it has the form*

$$\alpha n + \beta k + \gamma \ \text{ where } \ \alpha, \beta \in \mathbb{Z} \text{ and } \gamma \in K. \tag{6}$$

Theorem 1. [5, Thm. 8] *For a hypergeometric term $T(n, k)$, let $F, V \in K(n, k)$ be such that*

$$F \frac{E_k V}{V}$$

is an RNF over $K(n)$ of the k-certificate of T. Then there exists $D \in K(n, k)$ so that the n-certificate of T can be written as

$$D \frac{E_n V}{V}, \ D = \frac{d_1}{d_2}, \ \gcd(d_1, d_2) = 1, \tag{7}$$

and F, D both factor into constants and integer-linear polynomials.

2.2. **The Existence of a Telescoper.** Recall that the fundamental theorem [15, 16] provides only a sufficient condition for the termination of $\mathcal{Z}$. It states that if $T(n,k)$ is a *proper* hypergeometric term (see the definition from [15, 16]), then a telescoper for $T(n,k)$ exists. However, it is well-known that the set $\mathcal{S}$ of hypergeometric terms on which $\mathcal{Z}$ terminates is a proper subset of the set of all hypergeometric terms, but a super-set of the set of proper hypergeometric terms. The following theorem [1] gives a complete description of $\mathcal{S}$. It provides a necessary and sufficient condition for the termination of $\mathcal{Z}$.

Theorem 2. *(Criterion for the existence of a telescoper). Let $T(n,k)$ be a hypergeometric term of n and k, and (3) be an additive decomposition of $T(n,k)$. Let (4) be an RNF w.r.t. k over $K(n)$ of the k-certificate of $T_2(n,k)$ with $v_2 \in K[n,k]$. Then a telescoper for $T(n,k)$ exists iff each factor of $v_2(n,k)$ irreducible in $K[n,k]$ is an integer-linear polynomial, i.e., iff $T_2(n,k)$ is proper.*

3. A Lower Bound for the Order of Telescopers for a Minimal k-Non-summable Term

Definition 3. *A* minimal k-non-summable *hypergeometric term $T(n,k)$ is a hypergeometric term where the k-certificate of T has an RNF of the form (4), and for each irreducible p such that $p \mid v_2$, all three properties* **Pa**, **Pb**, **Pc** *hold.*

For the remainder of this section, we assume $T(n,k)$ to be a minimal k-non-summable hypergeometric term. Let us now introduce the notion of *crushing operators*.

Definition 4. *Let $M \in K[n, E_n]$ be such that $MT \neq 0$, and there exists an RNF*

$$F' \frac{E_k V'}{V'}, \; V' = \frac{v_1'}{v_2'} \tag{8}$$

of the k-certificate of the hypergeometric term MT such that each of the irreducible factors of v_2' does not have at least one of the three properties **Pa, Pb, Pc**. *Then M is a* crushing operator *for T. The minimal crushing operator is a crushing operator of minimal possible order.*

Proposition 3. *If L is a telescoper for T, then L is a crushing operator for T.*

Proof: The claim follows from Proposition 1. ■

Corollary 1. *If there does not exist any crushing operator for T of order less than μ, $\mu \geq 1$, then there does not exist any telescoper for T of order less than μ.*

Hence, the problem of computing a lower bound for the order of the telescopers for T is reduced to the problem of computing a lower bound for the order of the minimal crushing operator for T.

Theorem 3. *Let the k-certificate of T has an RNF $F(E_k V)/V$ of the form (4). Let the n-certificate A of T $(E_n T)/T = D(E_n V)/V$ be as defined in Theorem 1. Suppose that the polynomial $v_2 \in K[n,k]$ factors into a constant and integer-linear polynomials. Let $M \in K[n, E_n]$ be a crushing operator for $T(n,k)$, $\operatorname{ord} M = \rho$. Let p be an integer-linear factor of v_2, $\deg_k p = 1$. Then*

(i) *There exists an integer h such that*

$$E_k^h \, p \mid E_n v_2 \cdot E_n^2 v_2 \cdots E_n^{\rho} v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2; \tag{9}$$

(ii) *Let ρ_p be the minimal value of ρ in (i) such that (9) is satisfied. Then the order of the minimal crushing operator for T is not less than $\mu = \max_{p \mid v_2} \rho_p$.*

Proof:
(i): Let

$$M = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n + a_0(n),\ a_i(n) \in K[n].$$

Then

$$MT = \left(\sum_{m=0}^{\rho} a_m(n)A \cdot E_nA \cdots E_n^{m-1}A\right)T.$$

Therefore, the k-certificate of MT is

$$F\frac{E_kR}{R}, \tag{10}$$

where

$$\begin{aligned} R &= V\sum_{m=0}^{\rho} a_m(n)A \cdot E_nA \cdots E_n^{m-1}A \\ &= V\sum_{m=0}^{\rho} a_m(n)\frac{E_n^mV}{V}D \cdot E_nD \cdots E_n^{m-1}D \\ &= \sum_{m=0}^{\rho} a_m(n)\frac{E_n^mv_1 \cdot d_1 \cdot E_nd_1 \cdots E_n^{m-1}d_1}{E_n^mv_2 \cdot d_2 \cdot E_nd_2 \cdots E_n^{m-1}d_2}. \end{aligned}$$

Rewrite R as

$$R = \frac{r_1}{r_2},\ r_1,\ r_2 \in K[n,k],$$

$$r_2 = v_2 \cdot E_nv_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_nd_2 \cdots E_n^{\rho-1}d_2,\ r_1 = s_1 + v_2\, s_2,$$

where s_2 is a polynomial from $K[n,k]$, and $s_1 = a_0(n) \cdot E_nv_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_nd_2 \cdots E_n^{\rho-1}d_2$.

If p is *not* a factor of the denominator r_2 of R, then since v_2 is a factor of r_2, p must divide the numerator r_1 of R, i.e.,

$$p \,|\, (s_1 + v_2\, s_2).$$

Since p is a factor of v_2, this implies $p \,|\, s_1$. Additionally, p does not divide $a_0(n)$ since $\deg_k p = 1$. Therefore,

$$p \,|\, E_nv_2 \cdot E_n^2v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_nd_2 \cdots E_n^{\rho-1}d_2. \tag{11}$$

If p is a factor of the denominator r_2, then since M is a crushing operator for T, at least one of the three properties **Pa**, **Pb**, **Pc** does not hold for p. Notice that the k-certificates of T in (4) and MT in (10) have the same kernel F. It follows from Lemma 1 that for the integer-linear factor p of v_2, properties **Pb** and **Pc** *always* hold. Consequently, property **Pa** does *not* hold, i.e., there exists an $h \in \mathbb{Z} \setminus \{0\}$ such that E_k^hp divides r_2. Additionally, since T is a minimal k-non-summable hypergeometric term, it follows from property **Pa** that there does not exist an $h \in \mathbb{Z} \setminus \{0\}$ such that $E_k^hp|v_2$. This gives

$$E_k^h\, p \,|\, E_nv_2 \cdot E_n^2v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_nd_2 \cdots E_n^{\rho-1}d_2. \tag{12}$$

It follows from (11) and (12) that (i) is satisfied.
(ii): The claim follows from the fact that for each factor p of v_2, there does not exist any crushing operator for T of order less than ρ_p. ■

4. A General Algorithm

For a given hypergeometric term $T(n,k)$ of n and k, an algorithm to compute a lower bound μ for the order of the telescopers for T consists of two steps. A check to determine the existence of a telescoper for T is performed in the first step. This is attained by first applying to $T(n,k)$ the algorithm to solve the additive decomposition problem w.r.t. k to construct two hypergeometric terms $T_1(n,k)$, $T_2(n,k)$ such that

$$T(n,k) = (E_k - 1)\,T_1(n,k) + T_2(n,k), \tag{13}$$

and the k-certificate of T_2 has an RNF of the form (4). If v_2 does not factor into integer-linear polynomials, then it follows from Theorem 2 that $\mathcal{Z}$ is not applicable to T, and there is no need to compute a lower bound μ. Otherwise, rewrite v_2 as a product of integer-linear polynomials each of which is of the form (6). An algorithm, based on gcd and resultant computation, to check if $v_2 \in K[n,k]$ factors into integer-linear polynomials, and if this is the case, rewrite v_2 in the desired factored form is described in [6, 7]. Without loss of generality, we can assume that $\gcd(\alpha,\beta) = 1$, and $\beta \geq 0$.

In the second step, since T_2 is a minimal k-non-summable hypergeometric term, if follows from Proposition 3 that the existence of the crushing operators for T_2 is guaranteed. Additionally, all the hypotheses required for the computation of a lower bound μ for the order of the telescopers for T_2 exist. Hence, apply Theorem 3 to $T_2(n,k)$ to compute a lower bound μ. It follows from Proposition 2 that one can use μ as a lower bound for the order of the telescopers for T.

For each integer-linear factor p of v_2, $\deg_k p = 1$, the second step requires the computation of the minimal value of ρ in the pair (ρ, h), $h \in \mathbb{Z}$, $\rho \in \mathbb{N} \setminus \{0\}$ such that

(i) $E_k^h p \,|\, E_n v_2 \cdot E_n^2 v_2 \cdots E_n^\rho v_2$, or
(ii) $E_k^h p \,|\, d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$.

Consider the following simple algorithm $C_{(i)}$:

algorithm $C_{(i)}$;
input: $p = \alpha n + \beta k + \gamma$, $\alpha,\ \beta \in \mathbb{Z}$, $\gcd(\alpha,\beta) = 1$, $\beta > 0$, $\gamma \in K$,
$v_2 = \prod_{i=1}^m (\alpha_i n + \beta_i k + \gamma_i)$, $\alpha_i,\ \beta_i \in \mathbb{Z}$, $\gcd(\alpha_i,\beta_i) = 1$, $\beta_i \geq 0$, $\gamma_i \in K$;
output: the minimal value of $\rho \in \mathbb{N} \setminus \{0\}$ such that (i) is satisfied;

$\rho_{min} := \infty$;
for $i = 1, 2, \ldots, m$ **do**
  if $\alpha = \alpha_i$ **and** $\beta = \beta_i$ **and** $\gamma - \gamma_i \in \mathbb{Z}$ **then**
    find the minimal $\rho \in \mathbb{N} \setminus \{0\}$ and $h \in \mathbb{Z}$ such that
      $\alpha\rho - \beta h = \gamma - \gamma_i$;
    $\rho_{min} := \min\{\rho_{min}, \rho\}$;
  fi;
od;
return ρ_{min}.

For a given integer-linear factor p of v_2, $\deg_k p = 1$, the algorithm $C_{(i)}$ simply iterates through each integer-linear polynomial q of v_2. If $p - q = \sigma \in \mathbb{Z}$, then the algorithm solves the diophantine equation $\alpha\rho - \beta h = \sigma$, and chooses the minimal positive value of ρ. (Note that since $\gcd(\alpha,\beta) = 1$, the solution is guaranteed to exist.)

An algorithm $C_{(ii)}$ which finds the minimal value of ρ such that (ii) is satisfied can be described in a very similar manner. Note that it follows from Theorem 1 that the polynomial $d_2 \in K[n,k]$ in (7) factors into integer-linear polynomials.

By iterating through each factor p of v_2, we obtain the requested lower bound μ. This leads to the following algorithm which computes a lower bound for the order of the telescopers for a given hypergeometric term $T(n,k)$.

`algorithm` *LowerBound*;
`input:` a hypergeometric term $T(n,k) \in K[n,k]$;
`output:` a lower bound μ for the order of the telescopers for T;

apply the algorithm to solve the additive decomposition problem
    w.r.t. k to obtain $T_1(n,k), T_2(n,k)$ in (13);
if $T_2 = 0$ **then return** 0; **fi**;
at this point, T_2 has an RNF of the form (4);
if the polynomial $v_2(n,k)$ in (4) is written as
    $v_2 = \prod_{i=1}^{s} p_i$ where $p_i = (\alpha_i n + \beta_i k + \gamma_i)$,
    $\alpha_i,\ \beta_i \in \mathbb{Z}$, $\gcd(\alpha_i, \beta_i) = 1$, $\beta_i \geq 0$, $\gamma_i \in K$ **then**
    if $s = 0$ **then return** 1; **fi**;
    $\mu := -\infty$;
    Rewrite d_2 as
        $d_2 = \prod_{j=1}^{t} q_j$ where $q_j = (\alpha_j n + \beta_j k + \gamma_j)$,
        $\alpha_j,\ \beta_j \in \mathbb{Z}$, $\gcd(\alpha_j, \beta_j) = 1$, $\beta_j \geq 0$, $\gamma_j \in K$;
    for $i = 1, 2, \ldots, s$ **do**
        if $\deg_k p_i = 1$ **then**
            $\mu_{min} := C_{(i)}(p_i, v_2)$;
            $\mu_{min} := \min\{\mu_{min}, C_{(ii)}(p_i, d_2)\}$;
            $\mu := \max\{\mu, \mu_{min}\}$;
        fi;
    od;
    return μ;
else
    return "Zeilberger's algorithm is not applicable";
fi;

Note that instead of rewriting d_2 as a product of integer-linear polynomials, and using it in the call $C_{(ii)}(p_i, d_2)$ in `LowerBound`, it is possible to use a simpler polynomial which is a divisor of d_2. For a given $f \in K[n,k]$ and $c \in \mathbb{Q}$, there exists an algorithm [7] (called wc) to extract the maximal factor $w \in K[n,k]$ from f where w can be written in the form

$$\prod_i (k + c\,n + \gamma_i),\ \gamma_i \in K.$$

Hence, for each factor $p = (\alpha n + \beta k + \gamma)$ of v_2, we call wc with d_2 and α/β as input. This also helps reduce the number of integer-linear factors of d_2 to be compared with p.

Example 1 Consider the hypergeometric term

$$T = \frac{1}{(5\,n + 2\,k + 1)\,(-3\,n + 5\,k + 5)}.$$

(T is also a rational function of n and k.) Applying the algorithm to solve the additive decomposition problem yields two hypergeometric terms $T_1(n,k) = 0$ and $T_2(n,k) = T(n,k)$ in (13). Since T is a rational function, the polynomial v_2 in (4), and subsequently d_2 in (7) can be readily rewritten as

$$v_2 = (5\,n + 2\,k + 1)(-3\,n + 5\,k + 5),\ \ d_2 = 1.$$

Since v_2 can be written as a product of integer-linear polynomials, it follows from algorithm *LowerBound* that $\mathcal{Z}$ is applicable to T, and the two possible values for the integer-linear factor p are

$$p_1 = 5n + 2k + 1, \ \ p_2 = -3n + 5k + 5.$$

When $p = p_1 = 5\,n + 2\,k + 1$, the diophantine equation to be solved is $5\,\rho - 2\,h = 0$, which yields $(\rho_1, h_1) = (2, 5)$ as the solution. When $p = p_2 = -3\,n + 5\,k + 5$, the diophantine equation to be solved is $-3\,\rho - 5\,h = 0$, which yields $(\rho_2, h_2) = (5, -3)$ as the solution. Therefore, a lower bound μ for the order of the telescopers for T is $\mu = \max\{2, 5\} = 5$. Note that invoking $\mathcal{Z}$ on T results in the minimal telescoper L of order 6 where

$$L = (31\,n + 181)\,{E_n}^6 + (31\,n + 150)\,{E_n}^5 - (31\,n + 26)\,E_n - (31\,n - 5).$$

Example 2 Consider the class of hypergeometric terms of the form

$$T = \frac{1}{(a_1 n + b_1 k + c_1)\,(a_2 n + b_2 k + c_2)!}, \tag{14}$$

where $a_1, b_1, a_2, b_2 \in \mathbb{Z}$, $\gcd(a_1, b_1) = 1$, $b_1 \neq 0$, $a_1 \neq a_2$ or $b_1 \neq b_2$. Without loss of generality, we can assume that $b_1 > 0$. Applying the algorithm to solve the additive decomposition problem yields two hypergeometric terms $T_1(n, k) = 0$ and $T_2(n, k) = T(n, k)$ in (13), and the polynomial v_2 in (4) is

$$a_1 n + b_1 k + c_1,$$

which is also the only possible value of p. Subsequently, the value of d_2 in (7) is

$$\begin{cases} d_2 = (a_2 n + b_2 k + c_2 + 1) \cdots (a_2 n + b_2 k + a_2 + c_2) & \text{if } a_2 > 0, \\ d_2 = 1 & \text{if } a_2 = 0, \\ d_2 = (a_2 n + b_2 k + c_2 + a_2 + 1) \cdots (a_2 n + b_2 k + c_2) & \text{if } a_2 < 0. \end{cases}$$

Since $a_1 \neq a_2$ or $b_1 \neq b_2$, there does not exist any integer h such that $E_k^h p \,|\, d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$. When $p = a_1 n + b_1 k + c_1$, the diophantine equation to be solved is $a_1 \rho - b_1 h = 0$, which yields $(\rho_1, h_1) = (b_1, a_1)$ as the solution. Therefore, a lower bound μ for the order of the telescopers for T is $\mu = b_1$.

In summary, for the class of hypergeometric terms of the form (14), the polynomial factor $(a_1 n + b_1 k + c_1)$ is the *dominant* factor. It determines the lower bound (which is b_1) for the order of the minimal telescoper for T. As an example, the computed lower bound for the minimal telescoper for

$$T = \frac{1}{(n - 9\,k - 2)\,(2\,n + k + 3)!}$$

is 9, while the order of the minimal telescoper for T is 10. By first computing this lower bound, we can safely avoid the computation of a telescoper of order less than 9 (in addition to the assurance that the telescopers for T do exist). On the other hand, if $b_1 = 1$, then the computed lower bound μ equals 1, i.e., the lowest possible value for μ. As an example, the computed lower bound for the minimal telescoper for

$$T = \frac{1}{(n + k + 1)(n + 5\,k + 2)!}$$

is 1, while the order of the minimal telescoper for T is 6.

Notice that when the factorial term $(a_2 n + b_2 k + c_2)!$ in (14) equals 1, we have b_1 as a lower bound for the order of the minimal telescoper for T. This lower bound also equals the order of the minimal telescoper for T (see [8]).

5. Implementation

The algorithm to compute a lower bound for the order of the telescopers and related functionalities are implemented in Maple 7 [11]. These functions are merged into the module `HypergeometricSum` [9]. They include:

(1) `AdditiveDecomposition` solves the additive decomposition problem;
(2) `IsZApplicable` determines the applicability of Zeilberger's algorithm;
(3) `LowerBound` computes a lower bound for the order of the telescopers.

The function `LowerBound` has the calling sequence

$$\text{LowerBound}(T, n, k, E_n, Zpair);$$

where T is a hypergeometric term of n and k, and E_n denotes the shift operator w.r.t. n. (E_n and $Zpair$ are optional arguments.) If the non-existence of a Z-pair (L, G) for T is guaranteed, then `LowerBound` returns the conclusive error message "Zeilberger's algorithm is not applicable." Otherwise, the output is a non-negative integer μ denoting the value of the computed lower bound for the order of L. In this case, if the optional arguments E_n and $Zpair$ (each of which can be any unassigned name) are given, then the function `Zeilberger` [9] is invoked starting with μ as a lower bound for the order of L, and $Zpair$ will be assigned to the computed Z-pair (L, G).

Note that there exist different Maple implementations of $\mathcal{Z}$ such as `zeil` in the EKHAD package [13], and `sumrecursion` in the `sumtools` package. A Mathematica implementation is presented in [12]. Since the terminating condition that allows a hypergeometric term to have a Z-pair is unknown at the time these functions were implemented, an upper bound for the order of the recurrence operator L in the Z-pair (L, G) needs to be specified in advance (for instance, the default values are 6 for the parameter `MAXORDER` in `zeil`, and 5 for the global parameter `‘sum/zborder‘` in `sumrecursion`). As a consequence, when given a hypergeometric term $T(n, k)$ as input, (1) these programs might fail even if a Z-pair exists, i.e., the maximum order of L is not set "high enough", or (2) they simply "waste" CPU time trying to find a Z-pair when no such Z-pair exists. The function `LowerBound`, on the other hand, first determines the applicability of $\mathcal{Z}$ to $T(n, k)$. If the existence of a Z-pair is guaranteed, then it computes a lower bound μ for the order of L, and if requested, calls $\mathcal{Z}$ using μ as the starting value for the order of L, instead of 0. Since the existence of a Z-pair is guaranteed, there is no need to set an upper bound for the order of L.

Example 3 Consider the hypergeometric term

$$T(n, k) = \frac{1}{(2k-1)(n-8k+1)} \binom{2n-2k}{n-k} \binom{2k}{k}.$$

We first apply `LowerBound` to T. The optional arguments are provided so that the minimal Z-pair can be computed. The time and space required are recorded [1].

```
> T := binomial(2*n-2*k,n-k)*binomial(2*k,k)/
>      ((2*k-1)*(n-8*k+1)):
> t1 := time(): b1 := kernelopts(bytesused):
> LowerBound(T,n,k,En,'Zpair');
```

$$8$$

```
> printf(‘time taken: %a seconds, memory used: %a bytes\n‘,
          time()-t1, kernelopts(bytesused)-b1);
```

time taken: 30.740 seconds, memory used: 124111132 bytes

[1] All the reported timings were obtained on a 400Mhz SUN SPARC SOLARIS with 1Gb RAM.

In this example the computed lower bound equals the order of the minimal telescoper L for T, and the function `Zeilberger` is called using this lower bound as the starting value for the order of L. We now apply `Zeilberger` directly to T.

```
> Zeilberger(T,n,k,En):
```

Error, (in Zeilberger) No recurrence of order 6 was found

The function `Zeilberger` tries to compute the minimal Z-pair (L, G) for T starting with the value of 0 for the order ρ of L. It reaches the default value for the upper bound for ρ, and returns the above inconclusive error message. If one sets the upper bound to a "high enough" value, then `Zeilberger` will succeed in computing the minimal Z-pair.

```
> t1 := time(): b1 := kernelopts(bytesused):
> _MAXORDER := 8:
> Zeilberger(T,n,k,En):
> printf(`time taken: %a seconds, memory used: %a bytes\n`,
         time()-t1, kernelopts(bytesused)-b1);
```

time taken: 45.260 seconds, memory used: 174678848 bytes

Example 4 Consider the hypergeometric term

$$T(n,k) = \frac{1}{nk+1}\binom{2\,n}{2\,k}.$$

It takes `LowerBound` 0.62 seconds and 3,045 kilobytes to return the error message "Error, (in LowerBound) Zeilberger's algorithm is not applicable". The function recognizes that the polynomial $v_2(n,k)$ in (4) is $(nk+1)$ which does not factor into a product of integer-linear polynomials, and returns the conclusive answer quickly. On the other hand, it takes `Zeilberger` 33.95 seconds and 166,396 kilobytes to return the error message "Error, (in Zeilberger) No recurrence of order 6 was found". The function does not know if a Z-pair (L, G) for T exists. It tries to compute one and returns an inconclusive answer. Since there does not exist a Z-pair for T, the higher the value of the upper bound for the order of L is set, the more time and memory are wasted.

In this example T is not a proper term, and $\mathcal{Z}$ is not applicable to T.

Example 5 Consider the hypergeometric term

$$T(n,k) = \frac{(n+k+2)!}{(n^2+k+2)(k+3)!} - \frac{(n+k+1)!}{(n^2+k+1)(2+k)!} + \frac{(n+k)!}{(n+7\,k-2)\,k!}.$$

```
> T := 1/(n^2+2+k)*(n+2+k)!/(3+k)!-1/(n^2+k+1)*(n+1+k)!/(2+k)!+
>      1/(n+7*k-2)*(n+k)!/k!:
```

We first compute an RNF of the k-certificate of T :

```
> IsHypergeometricTerm(T,k,'Rk'):
> (z,f1,f2,v1,v2) := RationalCanonicalForm[1](Rk,k):
> v2;
```

$$\left(\frac{1}{7}\,n + k - \frac{2}{7}\right)\left(n^2+k+2\right)\left(n^2+k+1\right)$$

Note that the polynomial v_2 has irreducible factors that are not integer-linear. We now apply `LowerBound` to T. The optional arguments are provided so that the computation of the minimal Z-pair is carried out. `infolevel` is used to show the main steps of the function.

```
> t1 := time(): b1 := kernelopts(bytesused):
> LowerBound(T,n,k,En,'Zpair');
```

LowerBound: "check for the applicability of Zeilberger's algorithm"
LowerBound: "Zeilberger's algorithm is applicable"

```
LowerBound: “apply Theorem 3 to compute a lower bound”
LowerBound: “v2 = (n+7*k-2)”
LowerBound: “the candidate set for p is {n+7*k-2}”
LowerBound: “p = n+7*k-2”
LowerBound: “find the minimal positive integer r and integer h such that
        E_k^h p divides E_n v_2 . E_n^2 v_2 ... E_n^r v_2”
LowerBound: “find the minimal positive integer r and integer h such that
        E_k^h p divides d_2. E_n d_2 ...E_n^{r-1} d_2”

                                    7

> printf(‘time taken: %a seconds, memory used: %a bytes\n‘,
        time()-t1, kernelopts(bytesused)-b1);
  time taken: 18.070 seconds, memory used: 69908612 bytes
```

It is shown above that the additive decomposition of T yields T_2 in (13) where the polynomial v_2 in (4) is integer-linear $(n + 7k - 2)$. Finally, we apply `Zeilberger` directly to T. Note the difference in the time and space required to complete each function.

```
> t1 := time(): b1 := kernelopts(bytesused):
> _MAXORDER := 7:
> Zeilberger(T,n,k,En):
> printf(‘time taken: %a seconds, memory used: %a bytes\n‘,
        time()-t1, kernelopts(bytesused)-b1);
  time taken: 3294.680 seconds, memory used: 5936292572 bytes
```

In this example T is not a proper term. However, $\mathcal{Z}$ is applicable to T.

Example 6 For a given hypergeometric term $T(n, k)$, instead of applying $\mathcal{Z}$ to T, we suggest that $\mathcal{Z}$ be applied to the minimal k-non-summable hypergeometric term $T_2(n, k)$ in the decomposition (13). Following Proposition 2, the required Z-pair for $T(n, k)$ can then be easily obtained from the computed Z-pair for $T_2(n, k)$. This in general helps reduce the size of the problem to be solved. As an example, for $b \in \mathbb{N} \setminus \{0\}$, $j \in \{1, 3\}$, let

$$T_1(n, k) = \frac{1}{(nk - 1)(n - bk - 2)^j (2n + k + 3)!},$$

$$T_2(n, k) = \frac{1}{(n - bk - 2)(2n + k + 3)!}.$$

Consider the hypergeometric term

$$T(n, k) = (E_k - 1)\, T_1(n, k) + T_2(n, k).$$

Since $T_1 \sim T_2$, T is a hypergeometric term. We apply the functions `Zeilberger` (`Z`) and `LowerBound` (`LB`) to T. `LowerBound` is called with the optional arguments so that the minimal Z-pair for T can be computed (it follows from Example 2 that the computed lower bound is $|b|$.) Table 1 shows the time and space requirement. As one can easily notice, as b and/or j increase, the relative performance of `Zeilberger` (compared to `LowerBound`) quickly worsens.

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TABLE 1. Time and space requirements for `LowerBound` and `Zeilberger`.

		Timing (seconds)		Memory (kilobytes)	
j	b	LB	Z	LB	Z
1	1	6.49	5.35	27,838	24,702
	2	8.34	34.64	33,066	142,889
	3	11.13	124.53	44,233	535,736
	4	14.46	570.02	56,410	1,882,730
	5	25.79	2999.22	97,506	6,536,309
3	1	14.64	16.40	62,566	73,830
	2	17.24	228.59	68,304	770,529
	3	20.15	1,286.51	78,701	3,074,051
	4	24.08	8,771.08	91,844	10,766,646
	5	38.60	77,663.68	139,823	33,423,168

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