GENERALIZED DYCK EQUATIONS AND MULTICOLOR TREES

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ABSTRACT. New topological operations are introduced in order to recover in another way the generalized Dyck equation presented by D. Arquès and J.F. Béraud, for the generating function of maps, by decomposing maps topologically and bijectively. Applying repeatedly the operations which allowed to reveal the generalized Dyck equation to the successive transformed maps, a one-to-one correspondence is obtained between maps on any surface and trees whose vertices can be colored with several colors, following rules. This bijection provides us with a coding of these maps.

RÉSUMÉ. De nouvelles opérations topologiques sont introduites afin de nous permettre de retrouver l'équation de Dyck généralisée aux cartes de genre quelconque donnée par D. Arquès et J.F. Béraud, par des méthodes topologiques et bijectives de décomposition des cartes. En appliquant plusieurs fois les opérations qui nous ont permis de retrouver l'équation de Dyck généralisée aux cartes successives obtenues, on obtient une bijection entre cartes de genre quelconque et des arborescences où les sommets peuvent être coloriés de plusieurs couleurs suivant des règles que nous définirons. Cette bijection nous fournit un codage de ces cartes.

1. INTRODUCTION

The enumerative study of maps starts in 1962 with W. Tutte [17, 18], who enumerates rooted planar maps with n edges. Maps can also be described as combinatorics objects [13]. In 1975, R. Cori [8] studies planar maps in this perspective and extends these results with A. Machi [9] to orientable maps. In 1987, D. Arquès [1] determines functional relations satisfied by generating functions of rooted maps on the torus and obtains closed formulas to enumerate these maps by vertices and faces. Several studies follow on maps of greater genus, orientable or not, as for example the papers of E. Bender and E. Canfield [7] and also D. Arquès and A. Giorgetti [4].

The study of rooted maps independently of their genus begins with T.R.S. Walsh and A. Lehman [19]. They give a recursive relation on the number of rooted maps with respect to the number of edges, which does not lead to an explicit enumeration formula of these maps. In 1990, D.M. Jackson and I.T. Visentin [12] use an algebraic approach and obtain a closed formula for the generating functions of orientable rooted maps with respect to the number of edges and vertices.

More recently, D. Arquès and J.F. Béraud [2, 3] determine a functional equation satisfied by the generating functions of rooted maps with respect to the number of edges and vertices, that generalizes the Dyck equation on trees, and expresse the solution in a continued fraction form. This continued fraction reveals an interesting bijection, since it also enumerates connected fixed-point free involutions. P. Ossona de Mendez and P. Rosenstiehl [16] describe this bijection. From the combinatorial structure they give for rooted maps, they deduce a code for each map with a connected fixed-point free involution.

Topological operations applied to a map such as the removal or the addition of an edge, the fusion of two vertices, modify sometimes the genus of the map. These operations can not therefore be carried out in a systematic way when one works with fixed genus. However these elementary operations make it possible to find new functional equations on maps studied independently of genus and to establish bijections between families of maps.

In Section 2, we recall general definitions on maps. New topological operations are introduced in Section 3, in order to establish in Section 4, a bijection between maps of indifferent genus, and maps of indifferent genus with a root bridge edge, in which a subset of their vertices has been selected. This bijection provides us with a new proof of the generalized Dyck equation on orientable rooted maps given by D. Arquès and J.F. Béraud. They obtain this equation by an analytic resolution of a differential equation satisfied by the generating function of rooted maps. We here present a new proof of this equation, without any transformation on the generating function, but only by transcription of the given bijection. P. Flajolet [11] moreover showed that many continued fractions having integer coefficients can be explained in a purely combinatorial way, and here is another instance of this assertion.

We then give in Section 5, a bijection between orientable rooted maps and a family of trees whose vertices can be colored by several colors according to certain rules, which is deduced from the one presented in Section 4 by successive applications of this bijection. A generalization of the bijection between planar maps and well labelled trees [10], to maps of genus g and well labelled g-trees [14], allowed M. Marcus and B. Vauquelin to obtain a code for maps of genus g by words product of a shuffle of Dyck words with constraints and of a sequence of integers. The bijection enables us to determine a new language coding the maps of indifferent genus.

2. Definitions

Let us recall some definitions used in the sequel (for further details, see for example [8, 9]).

A *topological map* C in an orientable surface Σ of $\mathbb{R}^3$ is a partition of Σ in three finite sets of cells:

(1) The set of vertices of C, which is a finite set of dots;
(2) The set of edges of C, which is a finite set of open Jordan arcs, pairwise disjoint, whose extremities are vertices;
(3) The set of faces of C. Each face is simply connected and its border is the union of vertices and edges.

The *genus* of the map C is the genus of Σ. A cell is *incident* to another cell if one is contained in the boundary of the other. A *bridge* is an edge incident on both sides to the same face. We call half-edge an oriented edge of the map.

Let B be the set of half-edges of the map. With each half-edge, one can associate its initial vertex, its final vertex and its underlying edge. α (resp. σ) is the permutation in B associating to each half-edge b its opposite half-edge (resp. the first half-edge met when turning round the initial vertex of b in the positive way of the surface). The cycles of α (resp. σ) represent the edges (resp. the vertices) of the map. The cycles of $\bar{\sigma} = \sigma \circ \alpha$ are the oriented borders of the faces of the map. (B, σ, α) is the *combinatorial definition* of the topological orientable map associated C.

A map $C = (B, \sigma, \alpha)$ is *rooted* if a half-edge $\tilde{b}$ is distinguished. The half-edge $\tilde{b}$ is called the *root half-edge* of C, and its initial vertex is the *root vertex*. C is then defined as the triplet $(\sigma, \alpha, \tilde{b})$. Face $\bar{\sigma}^*(\tilde{b})$ is called the *exterior face* of C. By convention, the one vertex map (one vertex, no edge) is said to be rooted.

Two orientable maps of the same genus are isomorphic if there is a homeomorphism of the surfaces, preserving its orientation, mapping vertices, edges and faces of one map onto vertices, edges and faces respectively of the other map. An isomorphic class of orientable rooted maps of genus g will simply be called an orientable rooted map.

Let us denote by $\{p\}$ the one vertex map, $\mathcal{M}$ the set of orientable rooted maps, $\mathcal{I}$ the subset of $\mathcal{M}$ of maps with a bridge root edge, and for any map $I \in \mathcal{I}$, $Right(I)$ (resp. $Left(I)$) the maximal submap of I incident to the root vertex (resp. the final vertex of $\tilde{b}$) such that the root half-edge $\tilde{b}$ (resp. $\alpha(\tilde{b})$) of I does not belong to $Right(I)$ (resp. $Left(I)$) (see Figure 3).

3. Preliminaries

In Section 3.1, we describe two algorithms of half-edges and vertices numbering of a map. Numbering induces an order relation on half-edges and vertices that allows us to define in Section 3.2, new topological operations on maps. These operations will be useful to prove Theorem 1. These two operations are reciprocal, and they are interesting since the derivation allows to gather in one vertex a subset of vertices of a map, and the integration allows to get back to this subset of vertices.

3.1. Order relations in a rooted map.

Order relations on half-edges and vertices of a map are introduced in this Section. We show an algorithm that explains how to traverse a map along its half-edges: they are numbered beginning with the root half-edge and in their order of appearance in the oriented circuit given by the algorithm (see map C in Figure 1. Each number appears near the initial vertex of the half-edge). Half-edges are then naturally ordered by their number.

The root half-edge $\tilde{b}$ gets number 0, then the other half-edges of its face, $\bar{\sigma}^*(\tilde{b})$, are numbered. Afterwards while there still are numberless half-edges:

- Among numbered half-edges, the smallest half-edge b is chosen with a numberless opposite half-edge.
- Along the face $\bar{\sigma}^*(\alpha(b))$, beginning with $\alpha(b)$, half-edges are numbered.

Definition 1. Order relation on vertices. Let C be a rooted map and s_1, s_2 two vertices of C. The vertex s_1 is smaller than s_2 if the smallest half-edge of s_1 is smaller than the smallest half-edge of s_2.

Vertices are numbered by this order relation. Number 1 is affected to the root vertex and other vertices are numbered in an ascending order such that if vertex v_1 is encountered in the traversal of the map earlier than vertex v_2, its number must be smaller than the number of v_2 (see numbers in bold on map C of Figure 1).

A map is *ordered* when its half-edges and vertices are numbered by the algorithms given above.

Definition 2. Path and subpath of a map. The *path* of an ordered map C corresponds to the increasing ordered sequence of the half-edges of C, starting from its root half-edge. A *subpath* of C is defined as an increasing subsequence of ordered and successive half-edges of C.

Property 1. On the smallest half-edges of a face and of a vertex of an ordered map. *The smallest half-edge b_s of a vertex s different from the root vertex, of an ordered map $C = (\sigma, \alpha, \tilde{b})$, is not the smallest half-edge of its face $\bar{\sigma}^*(b_s)$.*
The smallest half-edge b_f of a face f different from the exterior face, of an ordered map $C = (\sigma, \alpha, \tilde{b})$, is not the smallest half-edge of its initial vertex.

Proof of property 1. If b_s belongs to the exterior face of C, as s is different from the root vertex, we have $\tilde{b} < b_s$ and b_s cannot be the smallest half-edge of its face.
If b_s does not belong to the exterior face of C, half-edges of face $\bar{\sigma}^*(\alpha(b_s))$ have been

numbered before b_s (see the algorithm above). Thus $\alpha(b_s)$ is smaller than b_s. Then $\bar{\sigma}(\alpha(b_s)) = \sigma(b_s)$, which belongs to vertex s, is smaller than b_s.

Since b_f is the smallest half-edge of face f, the half-edge $\alpha(b_f)$ is smaller than b_f and $\bar{\sigma}(\alpha(b_f)) = \sigma(b_f)$, which belongs to the initial vertex of b_f, is smaller than b_f $\diamond$

3.2. **Topological and bijective operations on maps.** In Section 3.2.1, we define the *derivation* operation, that gathers a subset of vertices of a map and the root vertex of a second map, in one vertex. These vertices can be recovered by applying the inverse operation, called *integration* operation and defined in Section 3.2.2, which uses the order properties on a map to get back all the gathered vertices. These operations are the main tools used in the proof of Theorem 1.

Let us denote by $\mathcal{M}_2$ the subset of maps of $\mathcal{M}$ that have at least two distinct vertices.

3.2.1. *Derivation of maps.* In this Section, we define a derived map of a pair of maps (C, R) of $\mathcal{M}_2 \times \mathcal{M}$ with respect to certain vertices of C. To derive a pair of maps with respect to vertices $s_1, \ldots, s_m$ of C means to collect these vertices in one vertex while respecting an order and afterwards to glue this vertex to the root vertex of R, as described in definition below.

Definition 3. Derived map. Let $C = (\sigma, \alpha, \tilde{b})$ be a map of $\mathcal{M}_2$, with root vertex $\tilde{s}_C$ and $R = (\sigma_R, \alpha)$ be a map of $\mathcal{M}$, with root vertex $\tilde{s}_R$ and if $R \neq \{p\}$, let $(b_{\tilde{s}_R,1}, b_{\tilde{s}_R,2}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}})$ be the half-edges of $\tilde{s}_R$ and $b_{\tilde{s}_R,1}$ be the root half-edge of R. Let $\mathscr{S} = \{s_1, \ldots, s_m\}$ be a set of distinct vertices of C such that $\tilde{s}_C < s_1 < s_2 < \ldots < s_m$. Forall i in $[1, m]$, let $(b_{s_i,1}, \ldots, b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$, be the half-edges of initial vertex s_i, in which $b_{s_i,1}$ is the smallest half-edge of s_i.

The *derived map* $C'_{\mathscr{S},R} = (\sigma', \alpha, \tilde{b})$ of $(C, \mathscr{S}, R)$ is then the map obtained from C and R after the gathering in a unique vertex s, of the vertices of $\mathscr{S} \bigcup \{\tilde{s}_R\}$ in the following way (see Figure 1):

$$s = (\underbrace{b_{s_1,1}, \ldots, b_{s_1,l_{s_1}}}_{s_1}, \underbrace{b_{s_2,1}, \ldots, b_{s_2,l_{s_2}}}_{s_2}, \ldots, \underbrace{b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}}_{s_m}, \underbrace{b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R}) = \sigma'^*(b_{s_1,1}).$$

In terms of permutation, it means: $\sigma' = \tau_{1R}\tau_{1m}\ldots\tau_{12}\sigma = \gamma\sigma$ with $\tau_{1i} = (b_{s_1,1}b_{s_i,1})$, $\tau_{1R} = (b_{s_1,1}b_{\tilde{s}_R,1})$ and $\gamma = (b_{s_1,1}\ldots b_{s_m,1}b_{\tilde{s}_R,1})$.

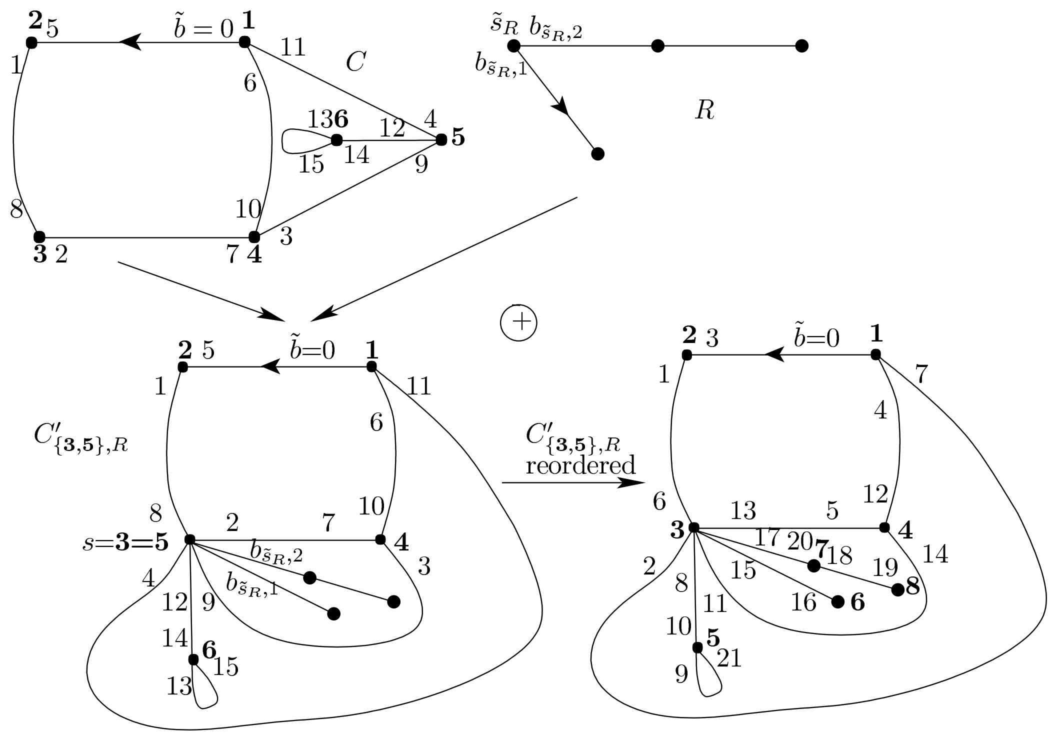
Property 2. Orders of $C'_{\mathscr{S},R}$, of C and of R

(1) *In $C'_{\mathscr{S},R}$, if $R \neq \{p\}$, $b_{\tilde{s}_R,1}$ is the smallest half-edge among the half-edges of R (see Figure 1 in which $b_{\tilde{s}_R,1} = 15$).*
(2) *The half-edges smaller than or equal to $\alpha(b_{s_1,l_{s_1}})$ have the same numbering in the ordered maps $C'_{\mathscr{S},R}$ and C.*

Proof of property 2. (1) By construction, R is recovered if in $C'_{\mathscr{S},R}$, the subset of half-edges belonging also to R, i.e. $\{b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}\}$, is unglued from vertex s. Thus in the traversal of $C'_{\mathscr{S},R}$, starting from its root half-edge, $\tilde{b}$, to reach any half-edge of R, one has to cross s. It implies that there exists i, $1 \leq i \leq l_{\tilde{s}_R}$ such that $b_{\tilde{s}_R,i}$ is the smallest half-edge of the half-edges of R in $C'_{\mathscr{S},R}$. If $l_{\tilde{s}_R} > 1$, let us prove that $b_{\tilde{s}_R,1}$ is the smallest half-edge of the half-edges of R in $C'_{\mathscr{S},R}$.

$b_{\tilde{s}_R,i}$ cannot be the smallest half-edge of its face, $\overline{\sigma'}^*(b_{\tilde{s}_R,i})$, otherwise $\alpha(b_{\tilde{s}_R,i})$, which belongs to R and has been previously numbered to the face $\overline{\sigma'}^*(b_{\tilde{s}_R,i})$, is smaller than $b_{\tilde{s}_R,i}$.

If $i > 1$, $b_{\tilde{s}_R,i} = \sigma'(b_{\tilde{s}_R,i-1}) = \overline{\sigma'}(\alpha(b_{\tilde{s}_R,i-1}))$, so that $\alpha(b_{\tilde{s}_R,i-1})$, which belongs to



$$
\begin{array}{lllllll}
 & (b_{s_2,1} & , \; b_{s_2,2} & , \; b_{s_2,3} = b_{s_2,l_{s_2}} & , \; b_{s_1,1} & , \; b_{s_1,2} = b_{s_1,l_{s_1}} &) \\
= & (4 & , \; 12 & , \; 9 & , \; 2 & , \; 8 &)
\end{array}
$$

FIGURE 1. Derived map with respect to vertices **3** and **5** of a pair of maps.

R, is smaller than $b_{\tilde{s}_R,i}$ (as $b_{\tilde{s}_R,i}$ is not the smallest half-edge of its face), which contradicts definition of $b_{\tilde{s}_R,i}$. Thus $i = 1$.

(2) In C, $s_1 < s_2 < \ldots < s_m$ implies that $b_{s_1,1} < b_{s_2,1} < \ldots < b_{s_m,1}$.
Furthermore, forall i in $[1, m]$, $\bar{\sigma}(\alpha(b_{s_i,l_{s_i}})) = b_{s_i,1}$ and $b_{s_i,1}$ is not the smallest half-edge of its face (see Property 1), so that $\alpha(b_{s_i,l_{s_i}})$ precedes $b_{s_i,1}$ in the ordered map C.
One then has in C, $\tilde{b} < \alpha(b_{s_1,l_{s_1}}) < b_{s_1,1} < \alpha(b_{s_2,l_{s_2}}) < b_{s_2,1} < \ldots < \alpha(b_{s_m,l_{s_m}}) < b_{s_m,1}$.
Thus in C, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not cross any half-edge $\alpha(b_{s_i,l_{s_i}})$.
If one proves that in $C'_{\mathscr{S},R}$, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not cross $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$, then one will conclude from what precedes that in $C'_{\mathscr{S},R}$, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not cross any of the half-edges $\alpha(b_{s_i,l_{s_i}})$. As

$$
\overline{\sigma'}(a) = \begin{cases} b_{s_{i+1},1} & \text{if } a = \alpha(b_{s_i,l_{s_i}}) \;\; \forall 1 \leq i < m \\ b_{\tilde{s}_R,1} & \text{if } a = \alpha(b_{s_m,l_{s_m}}) \\ b_{s_1,1} & \text{if } a = \alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}) \\ \bar{\sigma}(a) & \text{if } a \in C, a \neq b_{s_i,l_{s_i}} \;\; \forall 1 \leq i \leq m \\ \overline{\sigma_R}(a) & \text{if } a \in R, a \neq b_{\tilde{s}_R,l_{\tilde{s}_R}} \end{cases},
$$

it means that the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ in $C'_{\mathscr{S},R}$ is unchanged.
Let us then prove that the subpath of $C'_{\mathscr{S},R}$ from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not cross the half-edge $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$.

Since $\overline{\sigma'}(\alpha(b_{s_m,l_{s_m}})) = b_{\tilde{s}_R,1}$ and $b_{\tilde{s}_R,1}$ is not the smallest half-edge of its face (see item 1 of this proof), $\alpha(b_{s_m,l_{s_m}})$ precedes $b_{\tilde{s}_R,1}$ in the path of $C'_{\mathscr{S},R}$.
Furthermore, from Property 2.1, $b_{\tilde{s}_R,1} < \alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$ as $\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}) \in R$ and $b_{\tilde{s}_R,1}$ is the smallest half-edge of R in $C'_{\mathscr{S},R}$. Thus in $C'_{\mathscr{S},R}$, $\alpha(b_{s_1,l_{s_1}}) < \alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}})$ $\diamond$

The following technical lemma gives us the way to recover vertices $s_1,\ldots,s_m,\tilde{s}_R$, which compose vertex s, as will be showed in Lemma 2.
Notations of Definition 3 are used here.

Lemma 1. *In $C'_{\mathscr{S},R}$, $\sigma'(b_{s_1,l_{s_1}}) = \begin{cases} b_{s_2,1} & \text{if } m>1 \\ b_{\tilde{s}_R,1} & \text{if } R\neq\{p\} \text{ and } m=1 \\ b_{s_1,1} & \text{if } R=\{p\} \text{ and } m=1 \end{cases}$ is the smallest half-edge among half-edges of vertex s.*

Proof of lemma 1. (1) If $R=\{p\}$ and $m=1$ then $C=C'_{\mathscr{S},R}$, $s=s_1$ and thus, $\sigma'(b_{s_1,l_{s_1}}) = b_{s_1,1}$ is the smallest half-edge among the half-edges of s.

(2) Let us assume that $R\neq\{p\}$ or $m\neq 1$. One has

$$\overline{\sigma'}(a) = \begin{cases} b_{s_{i+1},1} & \text{if } a=\alpha(b_{s_i,l_{s_i}}) \ \ \forall 1\le i<m \\ b_{\tilde{s}_R,1} & \text{if } a=\alpha(b_{s_m,l_{s_m}}) \\ b_{s_1,1} & \text{if } a=\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}) \\ \bar{\sigma}(a) & \text{if } a\in C, a\neq\alpha(b_{s_i,l_{s_i}}) \ \ \forall 1\le i\le m \\ \overline{\sigma_R}(a) & \text{if } a\in R, a\neq\alpha(b_{\tilde{s}_R,l_{\tilde{s}_R}}) \end{cases}.$$

Let $\hat{b}$ be the smallest half-edge of face $\overline{\sigma}^*(b_{s_1,1})$ in C.

(a) In C, $b_{s_1,1}$ is the smallest half-edge of vertex s_1. From Property 1, as $s_1\neq\tilde{s}_C$, $b_{s_1,1}$ is not the smallest half-edge of its face. It implies that there exists $j>0$ such that $\bar{\sigma}^j(\hat{b}) = b_{s_1,1}$.

(b) Let us prove at last Lemma 1, that is: $\sigma'(b_{s_1,l_{s_1}})$ is the smallest half-edge of s in $C'_{\mathscr{S},R}$.
From Property 2.2, one knows that the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ in $C'_{\mathscr{S},R}$ is identical to the one in C. Thus $\alpha(b_{s_1,l_{s_1}}) = \bar{\sigma}^{j-1}(\hat{b}) = \overline{\sigma'}^{j-1}(\hat{b})$.
Furthermore, in C, the subpath from $\tilde{b}$ to $\alpha(b_{s_1,l_{s_1}})$ does not cross s as $b_{s_1,1}$ is the smallest half-edge of the half-edges of s in C and $\alpha(b_{s_1,l_{s_1}})$ is smaller than $b_{s_1,1}$ in C (see the proof of Property 2.2). It is the same in $C'_{\mathscr{S},R}$.
Thus $\sigma'(b_{s_1,l_{s_1}}) = \overline{\sigma'}(\alpha(b_{s_1,l_{s_1}}))$ is the smallest half-edge of s in $C'_{\mathscr{S},R}$ $\diamond$

$b_{s_1,1}$ is the smallest half-edge of $\mathscr{S}$ in C. Its predecessor in the path of C, is the half-edge $\alpha(b_{s_1,l_{s_1}})$ as $b_{s_1,1}$ is not the smallest half-edge of its face (see Property 1). In map $C'_{\mathscr{S},R}$, built from C and R by gluing together vertices of C and the root vertex of R in one vertex s, the successor of $\alpha(b_{s_1,l_{s_1}})$ becomes $b_{s_2,1}$, which then is the smallest half-edge of s in $C'_{\mathscr{S},R}$ reordered. If $b_{s_1,1}$ has been marked, one gets back thus vertex s_1 which is detached from s, then recursively vertices $s_2,\ldots,s_m$. Thus the pair of initial maps can be recovered from its derived map. A formal definition of this inverse operation, which will be called integration, is given in the next Section.

3.2.2. *Integration of a map.* A topological operation of opening of a vertex into two vertices is introduced in order to define the integration of a map, which consists in the splitting of a vertex into several vertices. It will then be seen that to recover a pair of maps (C,R) and the subset of vertices of C if its derived map is known, one has to integrate this last map.

Definition 4. Topological operation of opening of a map with respect to a half-edge. Let $C=(\sigma,\alpha,\tilde{b})$ be a map and b a half-edge of C. Let b_s be the smallest half-edge of a vertex $s = \sigma^*(b)$. The *opening* of C with respect to b consists in the splitting of the vertex s into two vertices s_1 and s_2 in the following way:

$$s = (b, \dots, \sigma^{-1}(b_s), b_s, \dots, \sigma^{-1}(b)) \quad \rightarrow \quad s_1 = (b, \dots, \sigma^{-1}(b_s)) \text{ and } s_2 = (b_s, \dots, \sigma^{-1}(b)).$$

It means that the following permutation $\widehat{\sigma_b}$ is applied to the half-edges of C: $\widehat{\sigma_b} = \tau\sigma$ with $\tau = (bb_s)$.

The result of the opening of C with respect to b is a map or a pair of maps:

(i) If $b_s \neq b$ and if the group generated by $(\widehat{\sigma_b}, \alpha, \tilde{b})$ acts transitively on the set of half-edges of C (i.e. $(\widehat{\sigma_b}, \alpha, \tilde{b})$ generates a map and not two disconnected maps), then a new map $\widehat{C_b} = (\widehat{\sigma_b}, \alpha, \tilde{b})$ is defined.

(ii) Otherwise, a pair of maps $(\widehat{C_b}, D)$, $\widehat{C_b} = (\widehat{\sigma_b}, \alpha, \tilde{b})$, $D = (\widehat{\sigma_b}, \alpha, b_s)$, is obtained, D being the map $\{p\}$ if $b_s = b$.

Remark 1. If $s \neq \tilde{s}$, $\widehat{C_b} \in \mathcal{M}_2$.

The next definition explains that in order to integrate a map C with respect to a given half-edge b, one has to recursively apply this topological operation of opening of C until getting a pair of maps.

Definition 5. Integration of a map. Let $C = (\sigma, \alpha, \tilde{b})$ be a map of $\mathcal{M}_2$, of root vertex $\tilde{s}$. Let $s \neq \tilde{s}$ be a vertex of C and $b \in s$. Let $\mathscr{S} = \emptyset$.

It will be said that a map C *is integrated* with respect to an half-edge b, when the operation of the opening of C is recursively applied until case (ii) of Definition 4 is reached, that is:

- Let us denote by b_s the smallest half-edge of $\sigma^*(b)$, then C is opened with respect to b (see Definition 4).
- If this operation gives a map $\widehat{C_b}$ (see Figure 2, drawing $\boxed{2}$), the vertex obtained after the opening, incident to b (the other obtained vertex is incident to b_s), is added to $\mathscr{S}$ and the opening operation starts again with $C \leftarrow \widehat{C_b}$ and $b \leftarrow b_s$.
- Otherwise, a pair of maps of $\mathcal{M}_2 \times \mathcal{M}$, $(\widehat{C_b}, D)$ is obtained (see Figure 2, drawing $\boxed{3}$), and also a set of vertices of $\widehat{C_b}$, $\mathscr{S}$ with the added vertex of $\widehat{C_b}$ which was split from the root vertex of D (vertex of $\widehat{C_b}$ to which b belongs).

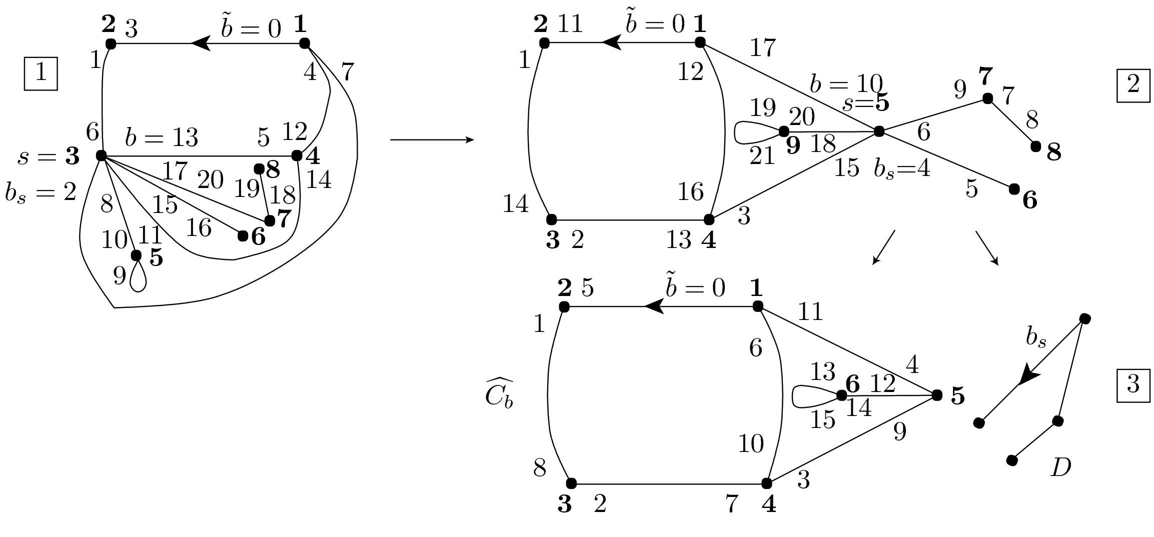


FIGURE 2. Integration of map C with respect to the half-edge b: a pair of maps $(\widehat{C_b}, D)$ of $\mathcal{M}_2 \times \mathcal{M}$ is obtained.

Lemma 2. *Let $C'_{\mathscr{S},R}$, be the derived map of a pair of maps (C,R) of $\mathcal{M}_2\times\mathcal{M}$ with respect to a set of vertices $\mathscr{S}$ of C. Let us denote by b ($= b_{s_1,1}$ of Definition 3) the smallest half-edge of $\mathscr{S}$ in C. Integration of $C'_{\mathscr{S},R}$ with respect to b gives $(C,\mathscr{S},R)$.*

Proof of lemma 2. With notations of Definitions 3 and 5, the map $C'_{\mathscr{S},R}=(\sigma',\alpha,\tilde{b})$ is integrated with respect to the half-edge $b_{s_1,1}$: $b=b_{s_1,1}$ and $b_s=b_{s_2,1}$ (from Lemma 1). The opening operation of vertex s unglues vertex s_1 from s, and one gets the map $(\widehat{C'_{\mathscr{S},R}})_b=(\widehat{\sigma'}_b,\alpha,\tilde{b})$:

$$s=(\underset{\substack{\uparrow\\ b}}{b_{s_1,1}},\dots,b_{s_1,l_{s_1}},\underset{\substack{\uparrow\\ b_s}}{b_{s_2,1}},\dots,b_{s_2,l_{s_2}},\dots,b_{s_m,1},\dots,b_{s_m,l_{s_m}},b_{\tilde{s}_R,1},\dots,b_{\tilde{s}_R,l_{\tilde{s}_R}})$$

Two vertices are obtained, a vertex $s_1=(b_{s_1,1},\dots,b_{s_1,l_{s_1}})$ and a vertex $s=(b_{s_2,1},\dots,b_{s_2,l_{s_2}},\dots,$ $b_{s_m,1},\dots,b_{s_m,l_{s_m}},b_{\tilde{s}_R,1},\dots,b_{\tilde{s}_R,l_{\tilde{s}_R}})$. One has: $\widehat{\sigma'}_b=\tau_{12}\sigma'$.
Thus, $(\sigma_2=\widehat{\sigma'}_b,\alpha,\tilde{b})=C'_{\{s_2,\dots,s_m\},R}$ and $\mathscr{S}=\{s_1\}$. One successively obtains maps $C'_{\{s_i,\dots,s_m\},R}=(\sigma_i=\tau_{i-1i}\sigma_{i-1},\alpha,\tilde{b})$ for $\tau_{i-1i}=(b_{s_{i-1},1}b_{s_i,1})$, and $\mathscr{S}=\{s_1,\dots,s_{i-1}\}$, with $3\le i\le m$. Applying for the last time to $C'_{\{s_m\},R}$ the topological operation of opening of $s=(b_{s_m,1},\dots,b_{s_m,l_{s_m}},b_{\tilde{s}_R,1},\dots,b_{\tilde{s}_R,l_{\tilde{s}_R}})$, two disconnected maps, $C=(\sigma,\alpha,\tilde{b})$ and $R=(\sigma,\alpha,b_{\tilde{s}_R,1})$, are recovered and also $\mathscr{S}=\{s_1,\dots,s_m\}$. One has: $\sigma=\tau_{Rm}\tau_{mm-1}\dots\tau_{12}\sigma'=\delta\sigma'$ with $\delta=\gamma^{-1}$ (see Definition 3) $\diamond$

4. Generalized Dyck equation on maps of indifferent genus

The well-known Dyck equation on trees, is based on a one-to-one correspondence between rooted planar trees $\mathcal{A}$ deprived of the one vertex tree, and $\mathcal{A}^2$. In Section 4.1, an equation generalizing the Dyck equation to rooted maps studied independently of genus, is given. This equation is equivalent to an equation on sets which is determined. A proof of the equation on sets is given in Section 4.2. Topological operations introduced in Section 3.2 will be used for this demonstration.

4.1. **Generalized Dyck equations.** The equation on sets is given as a bijection between the set of rooted maps of indifferent genus, $\mathcal{M}$, and the set of pairs of maps of $\mathcal{M}$, where in one of these maps a subset (possibly empty) of its vertices is selected. Equation (2) is then a translation with generating functions of this bijection.

For any map of $\mathcal{M}$, let us denote by $\mathcal{V}_M$ the set of vertices of M and $\mathcal{P}(\mathcal{V}_M)$ the set of all subsets of $\mathcal{V}_M$.

Theorem 1. Equation on sets

$$\mathcal{M}\leftrightarrow\{p\}\bigcup\left[\bigcup_{M\in\mathcal{M}}M\times\mathcal{P}(\mathcal{V}_M)\right]\times\mathcal{M}\tag{1}$$

The proof of this theorem is given in Section 4.2.

The translation of this bijection with generating functions provides a generalized Dyck equation generalizing the Dyck equation on trees.

Let us denote by y the variable which exponent enumerates the vertices of a map of $\mathcal{M}$, and by z the variable which exponent enumerates the edges of a map of $\mathcal{M}$ and $M(y,z)$ the generating function of rooted maps of indifferent genus.

One gets the following corollary:

Corollary 1. Generalized Dyck equation

$$M(y,z) = y + zM(y,z)M(y+1,z) \tag{2}$$

4.2. **Proof of Theorem 1.** A bijection between maps of $\mathcal{M}$, different from the one vertex map and $\left(\bigcup_{M\in\mathcal{M}} M\times\mathcal{P}(\mathcal{V}_M)\right)\times\mathcal{M}$ is described, which means between maps of $\mathcal{M}$ and maps of $\mathcal{I}$ in which for each map I of $\mathcal{I}$, a set $\mathscr{S}$ of vertices of the submap incident to the final vertex of the root half-edge, $Left(I)$, has been selected. As a matter of fact, $\mathcal{I}$ is in one-to-one correspondence with $\mathcal{M}^2$, as to each map I of $\mathcal{I}$, a pair of maps of $\mathcal{M}^2$, $(Left(I), Right(I))$, can be associated. Furthermore the set of pairs $(Left(I),\mathscr{S})$ is the set $\bigcup_{M\in\mathcal{M}} M\times\mathcal{P}(\mathcal{V}_M)$.

Lemma 3. Bijection of theorem 1. *There is a one-to-one correspondence between $\mathcal{M}$ and the set of pairs $(I,\mathscr{S})$, in which I is a map of $\mathcal{I}$ and $\mathscr{S}$ a set of vertices of $Left(I)$, possibly empty.*

Proof of lemma 3. Integration of a map with respect to an half-edge b allows to recover a pair of maps as well as a set of vertices of one of the obtained maps. Thus when a derived map I' is obtained, to have the possibility of going back, one has to memorize the half-edge b. To do this, if the root vertex of I' is only incident to the root half-edge, then it is sufficient to glue the root half-edge just before b in order to obtain a map M of $\mathcal{M}$.

Starting with a map I of $\mathcal{I}$ in which a set $\mathscr{S}$ of vertices of $Left(I)$ has been selected, we will first see how to obtain a map M of $\mathcal{M}$, and then how to recover map I and its set of vertices $\mathscr{S}$ from M.

Let $I = (\sigma,\alpha,\tilde{b})$ be a map of $\mathcal{I}$ of root vertex $\tilde{s}_I$ (see Figure 3). Let us denote by I_L, the map I deprived of $Right(I)$, with the same root half-edge than I and $\tilde{s}_R = \tilde{s}_I\setminus\{\tilde{b}\}$ the root vertex of $Right(I)$.

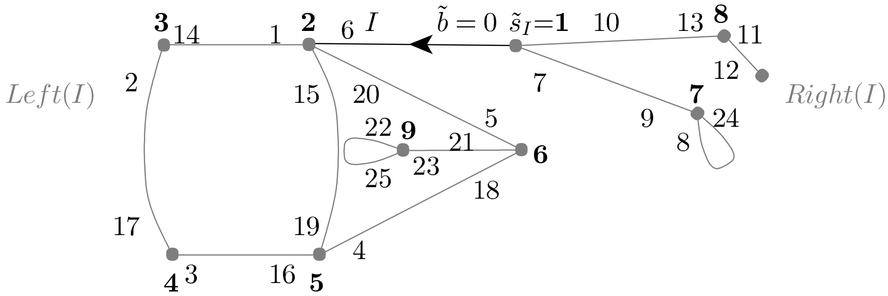


FIGURE 3. Map of $\mathcal{I}$

Stage 1: Derivation of $(I_L,\mathscr{S},Right(I))$. Let $\mathscr{S}$ be a subset of vertices of $Left(I_L) = Left(I)$.

If $\mathscr{S}$ is not empty, let $\{s_1,\ldots,s_m\}$ be m distinct vertices of $\mathscr{S}$ such that $s_1<\ldots<s_m$. Forall i in $[1,m]$, let $(b_{s_i,1},\ldots,b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$, be the half-edges of initial vertex s_i, in which $b_{s_i,1}$ is the smallest half-edge of s_i. Let $(b_{\tilde{s}_R,1},\ldots,b_{\tilde{s}_R,l_{\tilde{s}_R}}) = \tilde{s}_R$, with $b_{\tilde{s}_R,1} = \sigma(\tilde{b})$. Let $I' = (I_L)'_{\mathscr{S},Right(I)} = (\sigma',\alpha,\tilde{b})$ be the derived map of $(I_L, Right(I))$ with respect to $\mathscr{S}$. Let us recall that the vertices of $\mathscr{S}\bigcup\{\tilde{s}_R\}$ are joined into one vertex s_d in the following way (see Figure 4):

$$s_d = (\underbrace{b_{s_1,1},\ldots,b_{s_1,l_{s_1}}}_{s_1},\underbrace{b_{s_2,1},\ldots,b_{s_2,l_{s_2}}}_{s_2},\ldots,\underbrace{b_{s_m,1},\ldots,b_{s_m,l_{s_m}}}_{s_m},\underbrace{b_{\tilde{s}_R,1},\ldots,b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R}) = \sigma'^*(b_{s_1,1}).$$

If $\mathscr{S}$ is empty, i.e. $m = 0$, then $I' = I$.

Stage 2 : Labelization of $b_{s_1,1}$, getting of a map of $\mathcal{M}$.

- If $\mathscr{S} = \emptyset$, $I' = I$ ($(I_L, Right(I))$ has been derived with respect to no vertex) and $M = I' = I$.
- Otherwise a map $M = (\sigma_M, \alpha, \tilde{b})$ is built (see Figure 4), gluing the root vertex of $\tilde{b}$ to the vertex s_d in the following way:
$$(\underbrace{b_{s_1,1}, \ldots, b_{s_1,l_{s_1}}}_{s_1}, \underbrace{b_{s_2,1}, \ldots, b_{s_2,l_{s_2}}}_{s_2}, \ldots, \underbrace{b_{s_m,1}, \ldots, b_{s_m,l_{s_m}}}_{s_m}, \underbrace{b_{\tilde{s}_R,1}, \ldots, b_{\tilde{s}_R,l_{\tilde{s}_R}}}_{\tilde{s}_R}, \widetilde{\mathbf{b}})$$
$$= \sigma_M{}^*(\tilde{b})$$
The following permutation is then applied to the half-edges of I': $\sigma_M = (\tilde{b} b_{s_1,1})\sigma'$.

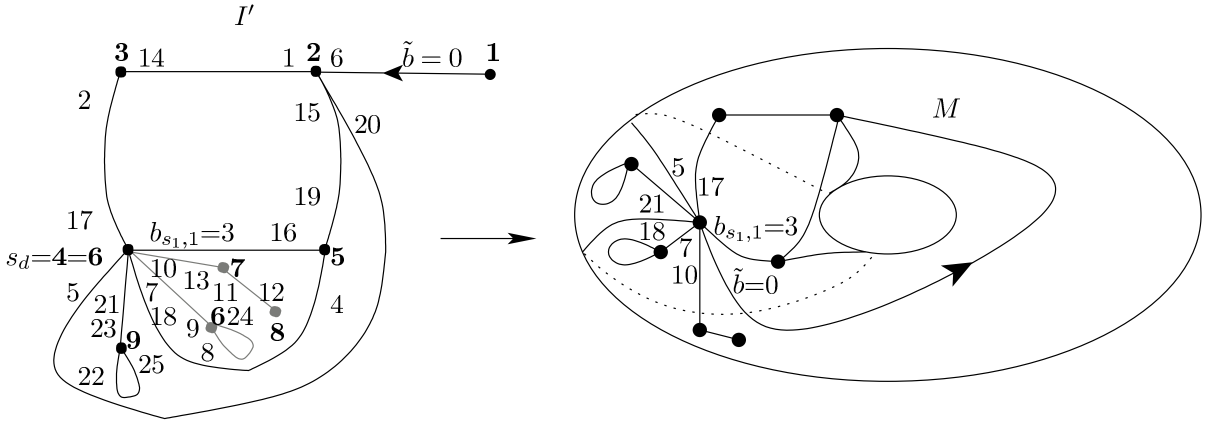


FIGURE 4. Getting of map $I' = (I_L)'_{\{\mathbf{4},\mathbf{6}\},Right(I)}$ and of a map M of $\mathcal{M}$ from map I' and its half-edge $b_{s_1,1}$ (I' and M have not been reordered)

Remark 2. If $\mathscr{S} = \emptyset$, $M = I$ and if I is a tree, then M is a tree.

Recovering of $(I, \mathscr{S})$ from M. If the map M obtained belongs to $\mathcal{I}$, it means that $\mathscr{S}$ was empty and then $M = I$. Thus to get back I from M, nothing has to be done. Let us remark that, thanks to Remark 2, when we restrict ourselves to the case of trees, one recovers the decomposition induced by the Dyck equation on trees.

Let us assume that M does not belong to $\mathcal{I}$. Then $\sigma_M(\tilde{b}) = b_{s_1,1}$. In map I, $b_{s_1,1}$ is the smallest half-edge among the half-edges incident to the vertices of $\mathscr{S}$. Conditions of Lemma 2 are satisfied, and one can apply this lemma. Thus, to recover $(I, \mathscr{S})$ from M, one has to:

- unglue $\tilde{b}$ from the root vertex,
- integrate this new map M_1 with respect to $b_{s_1,1} = \sigma_M(\tilde{b})$. From Lemma 2, one gets back I_L and $Right(I)$, respectively rooted in $\tilde{b}$ and $b_{\tilde{s}_R,1}$, and $\mathscr{S}$.
- Then $Right(I)$ is glued to the root vertex of I_L, such that $\sigma(\tilde{b}) = b_{\tilde{s}_R,1}$ $\diamond$

5. Bijection between maps of indifferent genus and multicolor trees

The operation that allowed to prove Theorem 1 transforms a map of $\mathcal{M}$ into a map with a bridge root edge in which a subset of its vertices has been selected. If this operation is iterated on the successive submaps incident to the two vertices incident to the bridge half-edge, and if the subset of vertices associated with each map is colored (one distinct color for each subset), the initial map is transformed into a tree whose vertices can be colored with several colors, following repartition rules. One then obtains what we will call a *multicolor tree.*

In Section 5.1, we give the definition of a multicolor tree and in Section 5.2, we prove the one-to-one correspondence between maps of $\mathcal{M}$ and multicolor trees. This bijection leads to a coding of maps by words of language, as shown in Section 5.3.

5.1. **Multicolor trees.** We give the definition of a multicolor tree, called hereafter simply multicolor tree. We then define a one-to-one correspondence in Section 5.2 between multicolor trees and maps of indifferent genus. These multicolor trees are trees whose vertices can be colored with several colors, following repartition rules that will be defined. Order relations given in Section 3.1 are applied to multicolor trees. An order on half-edges and vertices is thus established in a classical in-depth descent of the tree. Let us notice that the smallest half-edge of a vertex is also its left son in the tree structure, since a tree has only one face.

Definition 6. Multicolor tree. Let $T = (\sigma, \alpha, \tilde{b})$ be a rooted tree. Let $\mathcal{W} = \{w_1, \dots, w_n\}$ be a set of n distinct colors, eventually empty ($n \geq 0$). Each vertex of T can be colored by 0 to n colors. forall i in $[1, n]$, let us denote by s_i the smallest vertex of T of color w_i.

T is a multicolor tree (see T in Figure 5) if T complies with the following rules:

(1) each color of W is assigned to at least two distinct vertices from T;
(2) let $(b_{s_i,1}, \dots, b_{s_i,l_{s_i}}) = \sigma^*(b_{s_i,1})$ be the half-edges of initial vertex s_i, where $b_{s_i,1}$ is the smallest half-edge of $\sigma^*(b_{s_i,1})$, i.e. the left son of s_i. The half-edges $b_{s_i,j}, 1 \leq j < l_{s_i}$ are the half-edges, sons of s_i, and $b_{s_i,l_{s_i}}$ is the half-edge which goes up towards the father of s_i. Let $T_{s_i,j}$, be the subtree of T incident to the final vertex of $b_{s_i,j}$, rooted in $\bar{\sigma}(b_{s_i,j})$ and $\overline{T}_{s_i,j}$, the tree composed of $T_{s_i,j}$ and of the half-edge $b_{s_i,j}$ which is its root half-edge. Then:
   (a) there is a single j_i such that in T, w_i colors s_i and exclusively vertices of T_{s_i,j_i}. Let us denote this subtree by $T_{s_i,w_i} = T_{s_i,j_i}$, $\tilde{b}_{s_i} = b_{s_i,j_i}$ its root half-edge and $\overline{T}_{s_i,w_i} = \overline{T}_{s_i,j_i}$;
   (b) forall k in $[1, n]$, $k \neq i$, if $s_i = s_k$ then $\overline{T}_{s_i,w_i} \cap \overline{T}_{s_k,w_k} = \emptyset$.
(3) For all distinct colors w_i and w_j, if there is a vertex s of colors w_i and w_j where s_i is smaller than s_j, then $s = s_j$ and s is the only vertex of color w_j which is also of color w_i.

We will say that two multicolor trees are isomorphic if one can be obtained from the other by a permutation on its colors. A class of isomorphism of multicolor trees will simply be called *multicolor tree.*

Let $\mathcal{T}$ be the set of multicolor trees.

Remark 3. If T is a multicolor tree with m vertices and n distinct colors, then $n < m$.

5.2. **Bijection between $\mathcal{M}$ and $\mathcal{T}$.**

Theorem 2. *The set of rooted maps of indifferent genus with n edges is in bijection with the family of multicolor trees with n edges.*

Proof of theorem 2. In order to simplify our notations, a map whose vertices can be colored (by several colors) will also be called a map.

Let M be a map of $\mathcal{M}$ not reduced to the one vertex map and w_1 be a color. Let $w = w_1$. A map T of $\mathcal{T}$ is obtained from map M (see Figure 5), while proceeding in the following way:

(1) (a) If M deprived of its colors does not belong to $\mathcal{I}$, one applies to M the decomposition induced by Lemma 3, which transforms bijectively a map of $\mathcal{M}$ into

a pair $(I, \mathscr{S})$, where $I \in \mathcal{I}$ and $\mathscr{S}$ is a set of vertices of $Left(I)$. Then one assigns color w to the vertices resulting from the partition of the root vertex, $\tilde{s}_M$, of M, i.e. to the vertices of $\mathscr{S} \bigcup \{\tilde{s}_I\}$, where $\tilde{s}_I$ is the root vertex of the obtained map I. If colors are not taken into account, $I \in \mathcal{I}$. Let $\mathcal{W}_{\tilde{s}_M}$ be the set of colors which color the root vertex $\tilde{s}_M$ of M. Then in I, the set $\mathcal{W}_{\tilde{s}_I}$ of colors assigned to $\tilde{s}_I$ is equal to $\mathcal{W}_{\tilde{s}_M} \bigcup \{w\}$. Colors of $\mathcal{W}_{\tilde{s}_M}$ are not deferred to the other vertices resulting from $\tilde{s}_M$.

(b) Otherwise M is renamed I.

(2) If $I \notin \mathcal{T}$, let w_{left} and w_{right} two distinct colors, also distinct from all the colors already coloring I. One begins again at stage 1a with $M = Left(I)$, $w = w_{left}$ and $M = Right(I)$, $w = w_{right}$.

From Lemma 3, one gets that each stage of the transformation of a map of $\mathcal{M}$ into a multicolor tree is bijective.

By construction, T follows all the rules of Definition 6, and $T \in \mathcal{T}$, as:

- *1* is checked since a color w is assigned to the vertices resulting from the same vertex s.
- *2a* is checked since if M deprived of its colors does not belong to $\mathcal{I}$, the root half-edge of M is unglued from the root vertex and a map M_1 is obtained. Thus all the vertices to be colored belong to $Left(M_1)$, the root vertex excluded.
- *2b* is checked since after application of the transformation induced by Lemma 3 to a map of $\mathcal{M}$, its root half-edge becomes a bridge.
- *3* is checked according to item 1a above $\diamond$

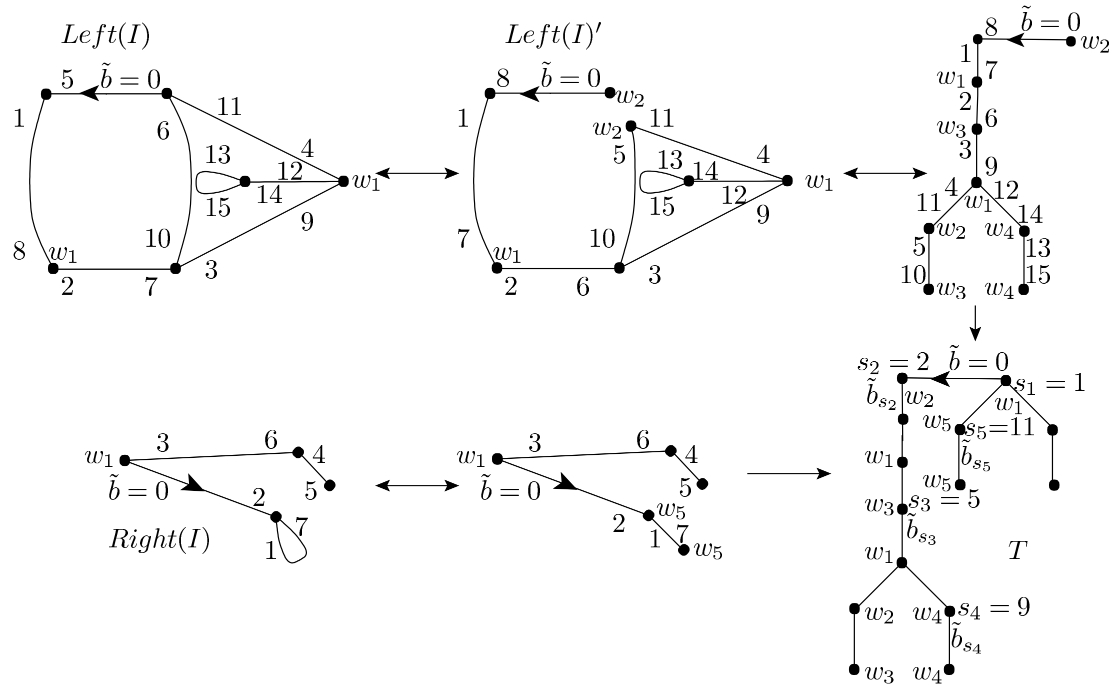


FIGURE 5. The multicolor tree associated with the map M of Figure 4

5.3. **Application: a language coding maps of indifferent genus.** In this Section we present a language coding rooted maps. The equation defining this language is a generalization of the well-known equation on Dyck words. In fact this language codes multicolor trees and thus by bijection rooted maps.

In order to clarify the signification of each letter of the alphabet of the language that we present, we need to give a definition.

Definition 7. Twin colors. Two colors w and w' of a tree of $\mathcal{T}$ are *twin* if there is a vertex of T colored by these two colors or if there is a subsequence of colors of T, $w_1 = w, w_2, \ldots, w_n = w'$ such that forall i in $[1, n]$, w_i and w_{i+1} color the same vertex. One thus defines classes of equivalence of colors, where two colors are in the same class if they are twin.

Let us denote by c the variable coding a half-edge, whose opposite half-edge is not coded, $\bar{c}$ the variable coding an half-edge, whose opposite half-edge is coded, y the variable coding a vertex in case of a map or in case of a multicolor tree, a vertex not colored or the smallest vertex among the vertices having the same or a twin color, and y_i, $i \geq 1$, the variable coding a vertex of color w_i (with $w_i \neq w_j$ if $i \neq j$) of a multicolor tree. In a rooted map, y_i, codes the half-edges belonging to a subset of the set of half-edges of initial vertex s_i, for a given vertex s_i of arity strictly superior to 1 (s_i can be equal to s_j if $i \neq j$).

Theorem 3. *The set of rooted maps is coded by the language* $L_\infty = \lim_{n\to\infty} L_n$, *where* L_n *represents the language coding rooted maps with at most* n *edges and is defined in the following way:*

$$\begin{aligned} L_n(y, y_1, \ldots, y_n, c, \bar{c}) &= y + c\ L_{n-1}(y + y_n, y_1, \ldots, y_{n-1}, c, \bar{c}) \\ &\qquad \bar{c}\ L_{n-1}(y, y_1, \ldots, y_{n-1}, c, \bar{c})\ (1 - \epsilon_n + y_n\epsilon_n)\delta_{c,n} \end{aligned} \tag{3}$$

$$L_0(y, c, \bar{c}) = y \tag{4}$$

where for every word m_1 *of* $L_{n-1}(y + y_n, y_1, \ldots, y_{n-1}, c, \bar{c})$*:*

- $\epsilon_n = \begin{cases} 1 & \text{if } y_n \in m_1 \\ 0 & \text{otherwise} \end{cases}$
- *for every word* m_2 *of* $L_{n-1}(y, y_1, \ldots, y_{n-1}, c, \bar{c})$*:*
  $\delta_{c,n} = \begin{cases} 1 & \text{if the number of occurrences of } c \text{ in } c\ m_1\ \bar{c}\ m_2 \leq n \\ & \text{and } \nexists 1 \leq k \leq n / y_k \in m_1 \text{ and } y_k \in m_2 \\ 0 & \text{otherwise} \end{cases}$

Conclusion

The bijection determined in Section 4 which leads to a generalized Dyck equation on maps can easily be extended to n-colored orientable rooted maps [15] and gives us a direct bijective proof of the generalized Dyck equation on n-colored maps precedently determined in an analytic way [5, 6].

The one-to-one correspondence between maps and multicolor trees raises many questions. Can new equations on generating functions of families of maps be determined? Could it lead to new enumeration formulas of these families? This bijection can be specialized to planar maps [15] and it would be interesting to see if it can also be done to maps of genus g, $g > 0$. It would also be interesting to see what kind of informations we can get from the obtained coding of maps. It is straightforward to deduce the number of vertices and edges of a map from its associated word but we have not yet searched for other information.

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