DIRECTED-CONVEX POLYOMINOES: ECO METHOD AND BIJECTIVE RESULTS

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Abstract. In this paper we provide bijective proofs for the number of directed-convex polyominoes having a fixed number of rows and columns, both by means of the ECO method, and of a direct mapping into the set of 2-colored Grand Motzkin paths.

Résumé. Dans cet article, nous donnons des preuves bijectives pour le nombre de polyominos dirigés convexes ayant un nombre fixé de lignes et de colonnes, en utilisant la méthodo;ogie ECO ainsi qu'une application bijective dans l'ensemble des grands chemins de Motzkin bi-colorés.

1. ECO method and directed-convex polyominoes

A *polyomino* is a finite union of elementary cells of the lattice $Z \times Z$, whose interior is connected. A polyomino is said to be *vertically convex* [*horizontally convex*] when its intersection with any vertical [horizontal] line is convex. A polyomino is *convex* if it is both vertically and horizontally convex. A polyomino is said to be *directed* when each of its cells can be reached from a distinguished cell, called the root, by a path which is contained in the polyomino and uses only north and east unitary steps. A polyomino is *directed-convex* if it is both directed and convex (see Fig. 1 (a)).

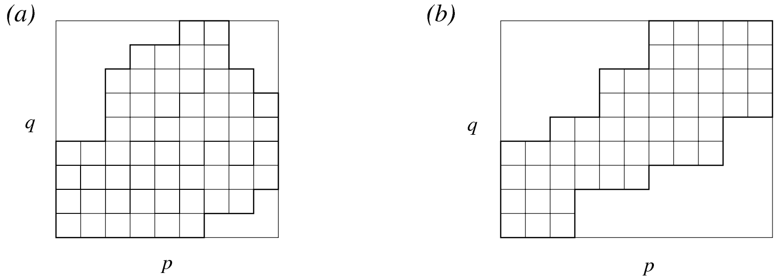

Figure 1. (a) A directed convex polyomino; (b) a parallelogram polyomino.

A *parallelogram polyomino* is a polyomino whose boundary consists of two lattice paths that intersect only initially and finally. The boundary paths, which we call upper and lower path, use the positively directed unit steps, $(1,0)$ and $(0,1)$ (see Fig. 1, (b)). Chang and Lin [3] used analytic methods to prove that the number of directed-convex polyominoes and the number of parallelogram polyominoes having q rows and p columns are equal to

$$\binom{p+q-2}{p-1}\binom{p+q-2}{q-1}, \tag{1}$$

and

$$\frac{1}{p+q-1}\binom{p+q-1}{p-1}\binom{p+q-1}{q-1}, \tag{2}$$

respectively (the second formula is originally due to Narayana, [7]). For polyominoes having $n+1$ rows and $n+1$ columns, these formulas reduce to

$$\binom{2n}{n}^2, \tag{3}$$

and

$$\frac{1}{2n+1}\binom{2n+1}{n}^2, \tag{4}$$

respectively. Furthermore, from (1) and (2) it arises that the number of directed-convex polyominoes having semiperimeter $p+q=n+2$ is equal to

$$\sum_{\substack{p,q\geq 1\\ p+q=n+2}} \binom{p+q-2}{p-1}^2 = \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}, \tag{5}$$

that is, the well known central binomial coefficient (sequence M1645 in [11]), and, analogously, the number of parallelogram polyominoes having semiperimeter $n+2$ is equal to

$$\frac{1}{n+1}\binom{2n}{n}, \tag{6}$$

the nth Catalan number (sequence M1459 in [11]).

In this paper we consider the classes:

(1) $\mathcal{D}_n$ the class of directed-convex polyominoes having semi-perimeter n;
(2) $\mathcal{P}_n$ the class of parallelogram polyominoes having semi-perimeter n;
(3) $\mathcal{D}_{p,q}$ the class of directed-convex polyominoes having p columns and q rows;
(4) $\mathcal{P}_{p,q}$ the class of parallelogram polyominoes having p columns and q rows.

In this section we use ECO method to provide a new bijective proof that the numbers of directed-convex polyominoes and the number of parallelogram polyominoes having semiperimeter $n+2$ are equal to (5) and (6), respectively, and then we obtain some statistics on directed-convex polyominoes.

1.1. **An ECO operator to construct directed-convex polyominoes.** ECO (Enumerating Combinatorial Objects) [1] is a method for the enumeration and the recursive construction of a class of combinatorial objects, $\mathcal{O}$, by means of an operator ϑ which performs "local expansions" on the objects of $\mathcal{O}$. More precisely, let p be a parameter on $\mathcal{O}$, such that $|\mathcal{O}_n| = |\{O \in \mathcal{O} : p(O) = n\}|$ is finite. An operator ϑ on the class $\mathcal{O}$ is a function from $\mathcal{O}_n$ to $2^{\mathcal{O}_{n+1}}$, where $2^{\mathcal{O}_{n+1}}$ is the power set of $\mathcal{O}_{n+1}$.

Proposition 1. *Let ϑ be an operator on $\mathcal{O}$. If ϑ satisfies the following conditions:*

1.: *for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$,*
2.: *for each $O, O' \in \mathcal{O}_n$ such that $O \neq O'$, then $\vartheta(O) \cap \vartheta(O') = \emptyset$,*

then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O) : O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$.

We refer to [1] for further details, proofs and definitions. The recursive construction determined by ϑ can be described by a *generating tree* [4], whose vertices are objects of $\mathcal{O}$. The objects having the same value of the parameter p lie at the same level, and the sons of an object are the objects it produces through ϑ. Thus a generating tree defines a non-decreasing sequence $(f_n)_{n\geq 0}$ of positive integers, f_n being the number of nodes at level n in the tree.

Let us now consider the operator:

$$\vartheta : \mathcal{D}_n \rightarrow 2^{\mathcal{D}_{n+1}},$$

working as follows on any given $P \in \mathcal{D}_n$, such that the length of its rightmost column is equal to k (see Fig. 2):

- **i):** ϑ glues a unitary cell to the right of each cell constituting the rightmost column of P;
- **ii):** ϑ glues a column of length h, $2 \leq h \leq k$ to the rightmost column of P;
- **iii):** ϑ glues a cell onto the top of the rightmost cell of the topmost row of P, one cell onto the top of the rightmost column of P, and two cells onto the top of each column between the two inserted ones (if there is any). If P is a parallelogram polyomino, then the operation **iii)** reduces to adding a cell on the top of the rightmost column of P.

The reader can check that ϑ satisfies conditions 1. and 2. of Proposition 1, meaning that ϑ constructs each polyomino $P' \in \mathcal{D}_{n+1}$ from the polyominoes in $\mathcal{DC}_n$, and each polyomino $P' \in \mathcal{DC}_{n+1}$ is obtained from one and only one $P \in \mathcal{D}_n$.

It should be clear that $|\vartheta(P)| = 2k$, then the generating tree of ϑ can be described by means of a *succession rule* of the form:

$$(7) \qquad \begin{cases} (2) \\ (2) \rightsquigarrow (2)(4) \\ (2k) \rightsquigarrow (2)^k(4)(6)\dots(2k)(2k+2), \end{cases}$$

where the power notation stands for repetitions. The expression in (7) means that the root object has 2 sons, and the $2k$ objects $O'_1, \dots, O'_{2k}$, produced by an object O are such that $|\vartheta(O'_i)| = 2$, $1 \leq i \leq k$, and $\left|\vartheta(O'_{k+j})\right| = 2(j+1)$, $1 \leq j \leq k$.

In the next paragraph we prove (5) and (6) in a bijective way, instead of determining the generating function of the rule in (7).

1.2. **A construction for Grand Dyck paths.** In the discrete plane, a Grand Dyck path is a sequence of rise $(1,1)$, and fall $(1,-1)$ steps running from $(0,0)$ to $(2n,0)$. In a Grand Dyck path, we call peak (valley, resp.) each pair of consecutive rise and fall steps (fall and rise steps, resp.). If the path ends with a fall step, we call last descent the last sequence of fall steps of the path. The number of Grand Dyck paths having semi-length n is well-known to be equal to $\binom{2n}{n}$. We define an operator ϑ' which constructs the class of Grand Dyck paths according to the succession rule (7). Let G be a Grand Dyck path of semi-length n:

- **:** - if the last step of G is a rise step, then $\vartheta'(G)$ is obtained by inserting a peak or a valley into the last point of G (Figure 3, (a)); in this case $|\vartheta'(G)| = 2$;
- **:** - otherwise, $\vartheta'(G)$ is obtained by (Figure 3, (b)):

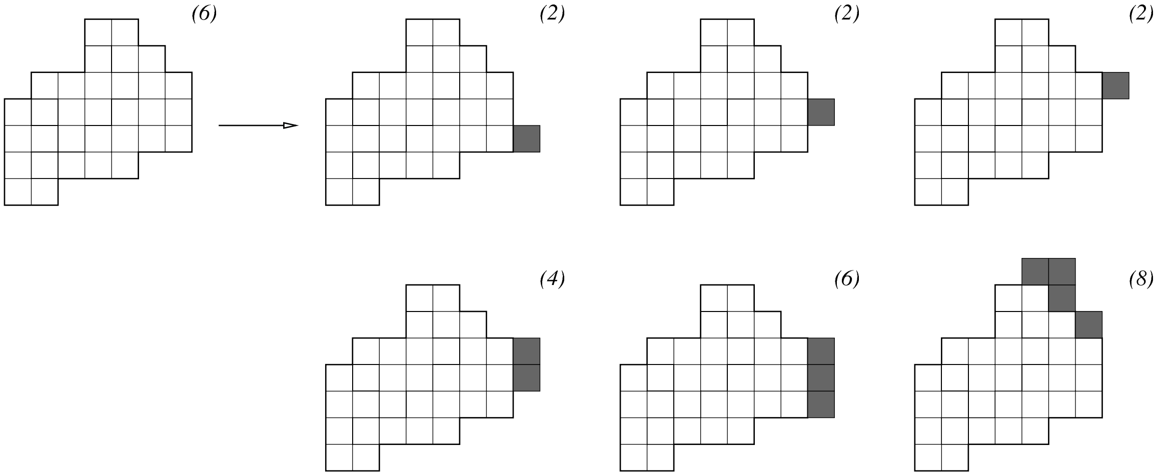

FIGURE 2. The ECO operator on directed-convex polyominoes.

A: inserting a peak into any point of the last descent of G,
B: reflecting all the paths obtained trough A (excepted the one obtained adding a peak at the end of G),
C: inserting a valley into the last point of G.

In this case, if k is the number of points in the last descent of G, then $|\vartheta'(G)| = k + (k-1) + 1 = 2k$.

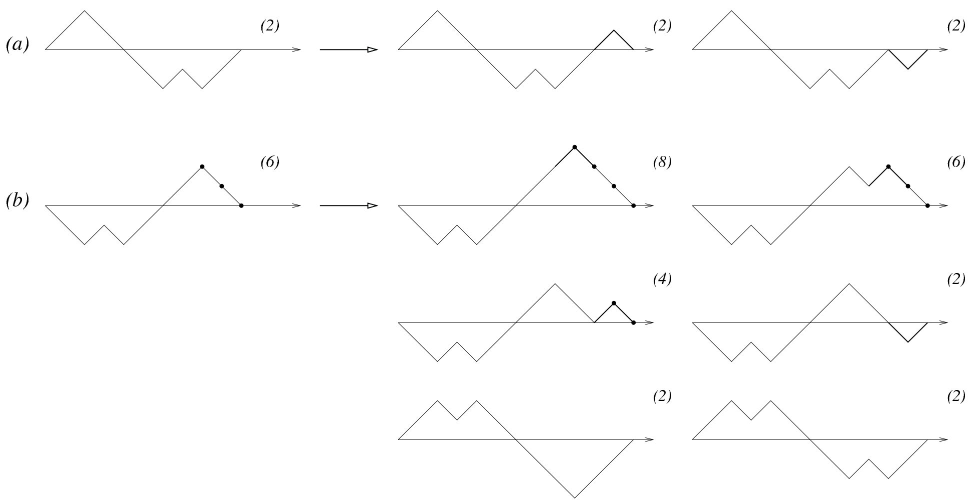

FIGURE 3. The application of the operator ϑ' to a Grand Dyck path.

It is not difficult to verify that ϑ' satisfies the conditions of Proposition 1. This fact gives a proof of (5) and establishes a bijection between the class of directed-convex polyominoes having semi-perimeter $n+2$ and the class of Grand Dyck paths having semi-length n.

1.3. **Further results.** Let $a_{n,k}$, $n \geq 0$, $k \geq 1$, be the number of labels $(2k)$ at level n of the generating tree of the succession rule (7) and consider the infinite lower triangular matrix $(a_{n,k})_{n,k\geq 0}$; in particular, for $n \geq 1$:

(8)
$$a_{n+1,1} = a_{n,1} + 2a_{n,2} + \ldots + (n+1)a_{n,n+1},$$
$$a_{n+1,k} = a_{n,k-1} + a_{n,k} + \ldots + a_{n,n+1}, \qquad k > 1$$

n	$\binom{2n}{n}$	2	4	6	8	10	12
0	1	1	0	0	0	0	0
1	2	1	1	0	0	0	0
2	6	3	2	1	0	0	0
3	20	10	6	3	1	0	0
4	70	35	20	10	4	1	0
5	252	126	70	35	15	5	1

TABLE 1. The matrix filled by the numbers $a_{n,k}$, and the row sums.

Moreover, the numbers $a_{n,k}$ have a nice closed form:

$$a_{n,k} = \binom{2n-k}{n-1}.$$

As a neat consequence of the constructions in Sections 1.1 and 1.2, we have that:

: i) There is a bijection between directed-convex polyominoes having semiperimeter $n+2$ where the right-most column is made of a single cell and Grand Dyck paths having semi-length n and ending with a rise step. The cardinality of these sets (sequence M2848 in [11]) is:

$$a_{n,1} = \frac{1}{2}\binom{2n}{n}, \quad n \geq 1.$$

: ii) There is a bijection between directed-convex polyominoes having semiperimeter $n+2$ where the right-most column is made of k cells, $k \geq 2$, and Grand Dyck paths having semi-length n where the last descent is made of $k-1$ fall steps. The cardinality of these sets is:

$$a_{n,k} = \binom{2n-k}{n-1}, \quad n \geq 1.$$

Remark 1. Let ϑ_P be the restriction of ϑ to the set of parallelogram polyominoes. Then

$$\vartheta_P : \mathcal{P}_n \to 2^{\mathcal{P}_{n+1}},$$

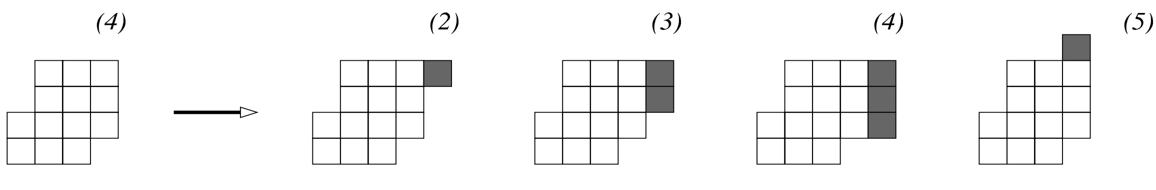

FIGURE 4. The operator ϑ_P for parallelogram polyominoes.

works as follows on a parallelogram polyomino P whose rightmost column is made of k cells (see Fig. 4):

i): it glues a column of length h, $1 \leq h \leq k$, to the rightmost column of P;
ii): it glues a cell onto the top of the rightmost column of P.

The operator $\vartheta_{\mathcal{P}}$ coincides with the classical ECO operator for parallelogram polyominoes [1], and gives rise to the well-known succession rule:

$$\begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)(4)\dots(k)(k+1), \end{cases} \tag{9}$$

defining Catalan numbers.
Remark 2. As it is known [8], also the following rule defines central binomial coefficients,

$$\begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (k) \rightsquigarrow (3)(3)(4)\dots(k)(k+1) \end{cases} \tag{10}$$

and an ECO operator exists which describes the recursive growth of Grand Dyck paths according to (10). The authors wish to point out that to find an ECO operator describing the growth of directed-convex polyominoes according to (10) is still an open problem.

2. A bijection between 2-colored Grand Motzkin paths and directed convex polyominoes

In this section we prove that:
- the class $\mathcal{DC}_{p,q}$ is enumerated by (1),
- the class $\mathcal{PP}_{p,q}$ is enumerated by (2),
by establishing a direct bijection between the class $\mathcal{D}_{p,q}$ (with $p, q \geq 1$), and a special subclass of 2-colored Grand Motzkin paths. Then we naturally extend our bijection to the class of directed-convex polyominoes with semiperimeter $p + q$.
The 2-colored Grand Motzkin paths are paths in the $Z \times Z$ plane which use four different steps: the rise step $(1, 1)$, the fall step $(1, -1)$, the α-colored horizontal step $(1, 0)$ and β-colored horizontal step $(1, 0)$. They start from $(0, 0)$ and end in $(n, 0)$ (see Fig. 5). Let us define the class of 2-colored Motzkin paths as the class of 2-colored Grand Motzkin paths which remain weakly above the x-axis.
For any $p, q \geq 1$, let $M_{p,q}$ denote the class of 2-colored Grand Motzkin paths of length $p+q-2$ where the difference between the number of α-colored and β-colored steps is $p-q$ and $H_{p,q}$ the class prefixes of 2-colored Motzkin paths of length $p+q-2$ where the difference between the number of α-colored and β-colored steps is $p-q$ and whose final point ordinate is a positive even number.

2.1. The bijection between $\mathcal{D}_{p,q}$ and $M_{p,q}$.

We first easily prove that the classes $\mathcal{D}_{p,q}$ and $M_{p,q}$ have the same cardinality.
The cardinality of $M_{p,q}$. Let us take into consideration a generic path $P \in M_{p,q}$ and code each step with a 2×1 matrix as follows:

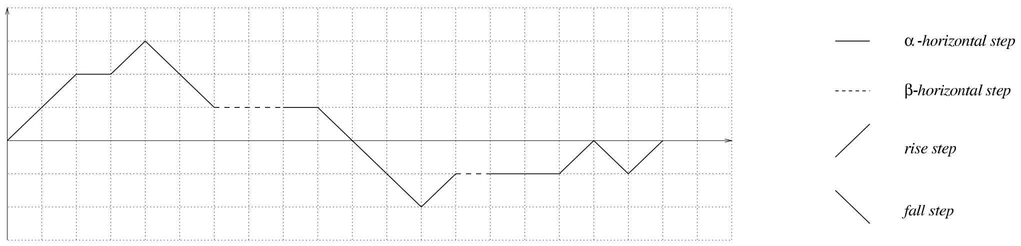

FIGURE 5. A 2-colored Grand Motzkin path.

$$\begin{array}{llll} \binom{1}{0} & \text{for a rise step,} & \binom{0}{1} & \text{for a fall step,} \\ \binom{1}{1} & \text{for a } \alpha\text{-horizontal step,} & \binom{0}{0} & \text{for a } \beta\text{-horizontal step.} \end{array} \tag{11}$$

P can be univocally represented by the $2 \times (p+q-2)$ matrix M obtained concatenating the coding of its steps as shown in Fig. 6. Let u and l be the upper and the lower row of M.

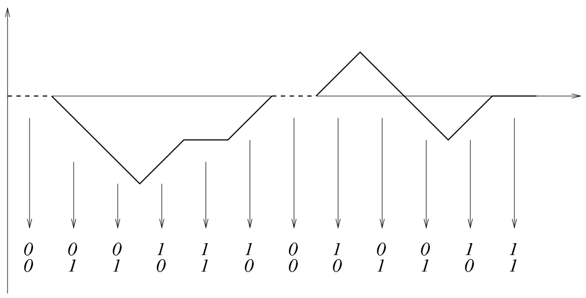

FIGURE 6. Encoding of a 2-colored Grand Motzkin path with a binary matrix.

Let j be a binary sequence, we indicate with $|j|_0$ and $|j|_1$ the number of 0 and 1 in j. We observe that:

: - $|u|_0 = |l|_0 = q-1$;
: - $|u|_1 = |l|_1 = p-1$.

From the previous two equations naturally follows:

$$|M_{p,q}| = \binom{p+q-2}{p-1}\binom{p+q-2}{q-1}.$$

The bijection between $\mathcal{D}_{p,q}$ and $H_{p,q}$. Let Q be a polyomino in $\mathcal{D}_{p,q}$ and $p \geq q$ (we can assume that with no loss of generality). Let $(p-h, q-h)$ be the last intersection point between Q and the line running from $(p-q, 0)$ to (p, q). Figure 7 shows in a simple graphical fashion that the polyomino is uniquely determined by two internal paths, both running from the initial cell to the cell identified by the points $(p-h, q-h)$, and $(p-h-1, q-h-1)$. These two paths are made up of $p+q-2$ steps: the upper, say u, is made of $(0,1)$, $(1,0)$ and $(0,-1)$ steps, and the lower, say l, is made of $(1,0)$, $(0,-1)$ and $(-1,0)$ steps.
Both u and l can be coded by a $p+q-2$ length binary vector, where 1 represents both the steps $(0,1)$ and $(0,-1)$ and 0 represents both the steps $(1,0)$ and $(-1,0)$. Then a $2 \times (p+q-2)$ binary matrix M is associated to the Q, where the first row is the coding

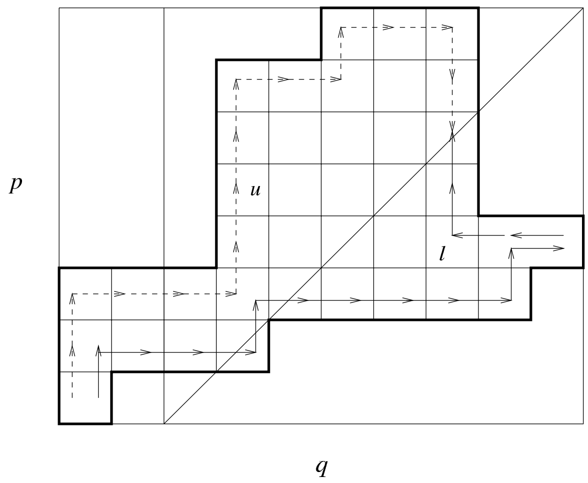

FIGURE 7. The upper and lower paths for a directed convex polyomino.

of u and the second row is the coding of l. For example, the following matrix encodes the polyomino in Fig. 7:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let us point out three properties of u and l:

(1) for every prefix s of u and every prefix v of l, having the same length, we have $|s|_1 \geq |v|_1$;
(2) the difference between the number of columns $\binom{0}{0}$ and $\binom{1}{1}$ of M is $p - q$;
(3) $|u|_1 - |l|_1 = 2h$.

Besides, the matrix M can be viewed as an array of $p + q - 2$ vectors 2×1 where each column represents a unitary step, using the code in (11). The properties 1., 2., and 3. guarantee that the obtained path is an element of $H_{p,q}$. In Figure 8 the 2-colored Motzkin prefix corresponding to the polyomino in Fig. 7 is depicted.

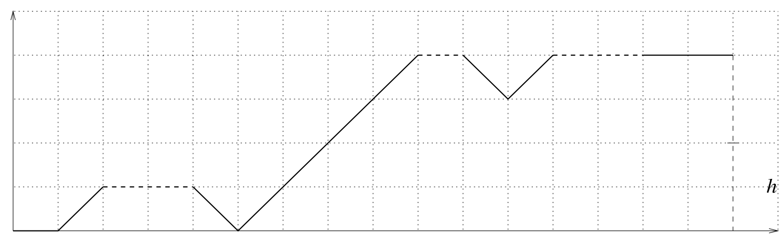

FIGURE 8. The 2-colored Motzkin prefix associated with the polyomino in Fig. 7.

To complete our bijection let us consider a generic $P' \in H_{p,q}$ and evaluate the parameters p and q. Using again the coding (11) we obtain a binary $2 \times (p + q - 2)$ matrix where the first row represents the upper paths u and the second row the lower path l of an element $Q \in \mathcal{D}_{p,q}$.
In the particular case that Q is a parallelogram polyomino we have $|u|_1 - |l|_1 = 0$ and the corresponding $P \in H_{p,q}$ is a 2-colored Motzkin path. We wish to point out that the classical bijection between parallelogram polyominoes and 2-colored Motzkin paths [5] arises naturally as a special case of our bijection.

The bijection between $H_{p,q}$ and $M_{p,q}$. Let P' be a path in $H_{p,q}$ and let $(p+q-2, 2h)$ be its final point coordinates. If $h = 0$, then P' is already a 2-colored Motzkin path. Otherwise, for every $i = 0, \ldots, h-1$ we consider the vertical side of unitary length from the point $(p+q-2, i)$ to the point $(p+q-2, i+1)$. We then draw a horizontal ray to the left from the center of this side. There are h such rays. Each ray hits for the first time a rise step in P'.

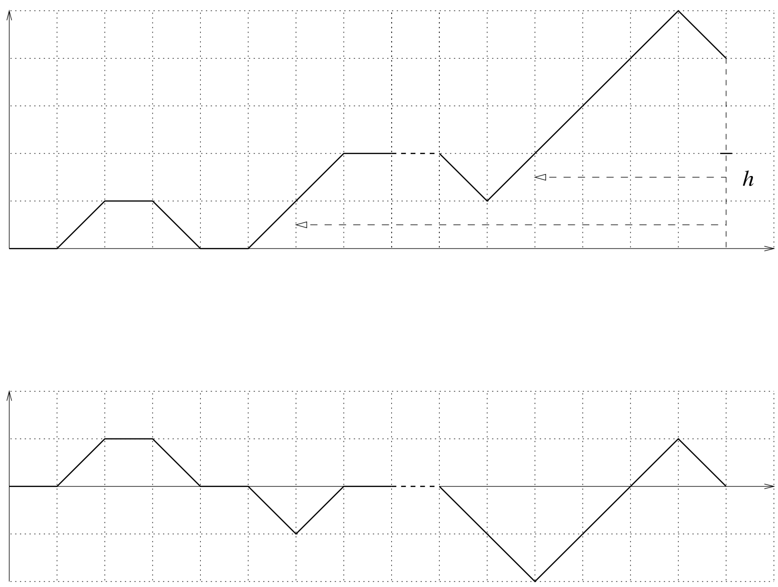

FIGURE 9. The mapping of a prefix of $H_{p,q}$ into a path of $M_{p,q}$.

We modify P' by changing the steps that are hit to fall steps. In this modified path the number of rise step is equal to the number of fall steps, while the difference between the numbers of α-horizontal and β-horizontal steps is the same as in P'. The obtained path is that corresponding to P' (see Fig. 9).

This mapping is inverted as follows (see Fig. 10). Let Q be a generic path in $M_{p,q}$, and let $-h$, $h > 0$ be the ordinate of the lowest point of Q. From each of the points $(0, -\frac{1}{2})$, $(0, -1-\frac{1}{2})$, …, $(0, -h+1-\frac{1}{2})$, we draw a ray to the right until it hits Q, necessarily at a fall step. Let Q' be the path obtained from Q in which each hit step is changed to a rise step. The path $Q' \in H_{p,q}$, and its final point ordinate is equal to $2h$.

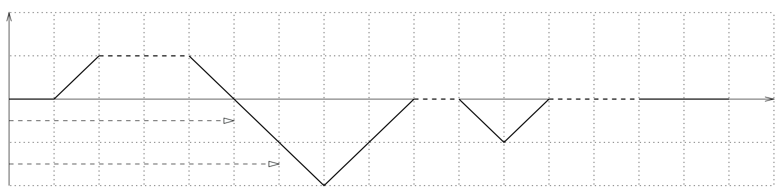

FIGURE 10. The inverse mapping of a $M_{p,q}$ path into the prefix of Fig. 8.

Remark 3. As a consequence of the previously defined bijection we have that:

-: the height of a directed convex polyomino P is equal to the sum of the numbers of rise and α-colored horizontal steps plus one, in the path corresponding to P;

-: the width of a directed convex polyomino P is equal to the sum of the numbers of rise and β-colored horizontal steps plus one, in the path corresponding to P.

Remark 4. The reader should be convinced that the previously defined bijection defines also a bijection between the set of directed-convex polyominoes having $n+1$ rows and $n+1$ columns and the set of Grand Motzkin paths of length $2n$ and having the same number of α and β-colored horizontal steps. The latter class is trivially enumerated by the numbers

$$\binom{2n}{n}^2,$$

and this proves (3). The result (4) is then a neat application of the cycle lemma.

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