WORDS RESTRICTED BY PATTERNS WITH AT MOST 2 DISTINCT LETTERS

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ABSTRACT. We find generating functions for the number of words avoiding certain patterns or sets of patterns with at most 2 distinct letters and determine which of them are equally avoided. We also find exact number of words avoiding certain patterns and provide bijective proofs for the resulting formulas.

RÉSUMÉ. On obtient les séries génératrices pour les mots qui évitent certains motifs et certains ensembles de motifs, tous contenant au plus deux lettres différentes. On détermine ainsi des motifs équitablement évités. On donne enfin des formules d'énumération explicites dans certains cas particuliers, cas pour lesquels on donne aussi des preuves bijectives.

The main goal of this note to give analogies of enumerative results on certain classes of permutations characterized by *pattern-avoidance* in the symmetric group (see [SS]), and in the words on k letters (see [B]).

Pattern avoidance in the symmetric group. Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The first paper devoted entirely to study of permutations avoiding certain patterns appeared in 1985 (see [SS]) and currently there exist more than 70 papers on this subject. These papers containing more than 5 analogies, for example: words (see [B] and references therein), generalized patterns (see [BS]), signed permutations (see [S]), and coloured permutations [M] and references therein). In the present paper we give another analogue for this problem.

Pattern avoidance in the words on k letters. Let $[k] = \{1, 2, \ldots, k\}$ be a (totally ordered) alphabet on k letters. We call the elements of $[k]^n$ *words*. Consider two words, $\sigma \in [k]^n$ and $\tau \in [\ell]^m$. In other words, σ is an n-long k-ary word and τ is an m-long ℓ-ary word. Assume additionally that τ contains all letters 1 through ℓ. We say that σ contains an *occurrence* of τ, or simply that σ *contains* τ, if σ has a subsequence *order-isomorphic* to τ, i.e. if there exist $1 \le i_1 < \ldots < i_m \le n$ such that, for any relation $\phi \in \{<, =, >\}$ and indices $1 \le a, b \le m$, $\sigma(i_a)\phi\sigma(i_b)$ if and only if $\tau(a)\phi\tau(b)$. In this situation, the word τ is called a *pattern*. If σ contains no occurrences of τ, we say that σ *avoids* τ.

Up to now, most research on forbidden patterns dealt with cases where both σ and τ are permutations, i.e. have no repeated letters. Some papers (see Atkinson et al. [AH], Burstein [B], Regev [R] and references therein) also dealt with cases where only τ is a permutation. The natural analogue of avoiding permutations in $[k]^n$ is avoiding words.

In this paper, we consider some cases where forbidden patterns τ contain repeated letters. Just like [B], this paper is structured in the manner of Simion and Schmidt [SS], which was the first to systematically investigate forbidden patterns and sets of patterns.

1. Preliminaries

Let $[k]^n(\tau)$ denote the set of n-long k-ary words which avoid pattern τ. If T is a set of patterns, let $[k]^n(T)$ denote the set of n-long k-ary words which simultaneously avoid all patterns in T, that is $[k]^n(T) = \cap_{\tau\in T}[k]^n(\tau)$.

For a given set of patterns T, let $f_T(n,k)$ be the number of T-avoiding words in $[k]^n$, i.e. $f_T(n,k) = |[k]^n(T)|$. We denote the corresponding exponential generating function by $F_T(x;k)$; that is, $F_T(x;k) = \sum_{n\geq 0} f_T(n,k)x^n/n!$. Further, we denote the ordinary generating function of $F_T(x;k)$ by $F_T(x,y)$; that is, $F_T(x,y) = \sum_{k\geq 0} F_T(x;k)y^k$. The reason for our choices of generating functions is that $k^n \geq |[k]^n(T)| \geq n!\binom{k}{n}$ for any set of patterns with repeated letters (since permutations without repeated letters avoid all such patterns). We also let $G_T(n;y) = \sum_{k=0}^{\infty} f_T(n,k)y^k$, then $F_T(x,y)$ is the exponential generating function of $G_T(n;y)$.

We say that two sets of patterns T_1 and T_2 belong to the same *cardinality class*, or *Wilf class*, or are *Wilf-equivalent*, if for all values of k and n, we have $f_{T_1}(n,k) = f_{T_2}(n,k)$.

It is easy to see that, for each τ, two maps give us patterns Wilf-equivalent to τ. One map, $r : \tau(i) \mapsto \tau(m+1-i)$, where τ is read right-to-left, is called *reversal*; the other map, where τ is read upside down, $c : \tau(i) \mapsto \ell+1-\tau(i)$, is called *complementation*. For example, if $\ell = 3$, $m = 4$, then $r(1231) = 1321$, $c(1231) = 3213$, $r(c(1231)) = c(r(1231)) = 3123$. Clearly, $c\circ r = r\circ c$ and $r^2 = c^2 = (c\circ r)^2 = id$, so $\langle r, c\rangle$ is a group of symmetries of a rectangle. Therefore, we call $\{\tau,\ r(\tau),\ c(\tau),\ r(c(\tau))\}$ the *symmetry class* of τ.

Hence, to determine cardinality classes of patterns it is enough to consider only representatives of each symmetry class.

2. Two-letter patterns

There are two symmetry classes here with representatives 11 and 12. Avoiding 11 simply means having no repeated letters, so

$$f_{11}(n,k) = \binom{k}{n} n! = (k)_n, \qquad F_{11}(x;k) = (1+x)^k.$$

A word avoiding 12 is just a non-increasing string, so

$$f_{12}(n,k) = \binom{n+k-1}{n}, \qquad F_{12}(x;k) = \frac{1}{(1-x)^k}.$$

3. Single 3-letter patterns

The symmetry class representatives are 123, 132, 112, 121, 111. It is well-known [Kn] that

$$|S_n(123)| = |S_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number. It was also shown earlier by the first author [B] that

$$f_{123}(n,k) = f_{132}(n,k) = 2^{n-2(k-2)} \sum_{j=0}^{k-2} a_{k-2,j}\binom{n+2j}{n},$$

where

$$a_{k,j} = \sum_{m=j}^{k} C_m D_{k-m}, \quad D_t = \binom{2t}{t},$$

and

$$F_{123}(x,y) = F_{132}(x,y) = 1 + \frac{y}{1-x} + \frac{2y^2}{(1-2x)(1-y) + \sqrt{((1-2x)^2 - y)(1-y)}}.$$

Avoiding pattern 111 means having no more than 2 copies of each letter. There are $0 \le i \le k$ distinct letters in each word $\sigma \in [k]^n$ avoiding 111, $0 \le j \le i$ of which occur twice. Hence, $2j + (i-j) = n$, so $j = n - i$. Therefore,

$$f_{111}(n,k) = \sum_{i=0}^{k} \binom{k}{i}\binom{i}{n-i}\frac{n!}{2^{n-i}} = \sum_{i=0}^{k} \frac{n!}{2^{n-i}(n-i)!(2i-n)!}(k)_i = \sum_{i=0}^{k} B(i, n-i)(k)_i,$$

where $(k)_i$ is the falling factorial, and $B(r,s) = \dfrac{(r+s)!}{2^s(r-s)!s!}$ is the Bessel number of the first kind. In particular, we note that $f_{111}(n,k) = 0$ when $n > 2k$.

Theorem 1. $F_{111}(x;k) = \left(1 + x + \dfrac{x^2}{2}\right)^k.$

Proof. This can be derived from the exact formula above. Alternatively, let α be any word in $[k]^n(111)$. Since α avoids 111, the number of occurrences of the letter k in α is 0, 1 or 2. Hence, there are $f_{111}(n,k-1)$, $nf_{111}(n-1,k-1)$ and $\binom{n}{2}f_{111}(n-2,k-1)$ words α with 0, 1 and 2 copies of k, respectively. Hence

$$f_{111}(n,k) = f_{111}(n,k-1) + nf_{111}(n-1,k-1) + \binom{n}{2}f_{111}(n-2,k-1),$$

for all $n, k \ge 2$. Also, $f_{111}(n,1) = 1$ for $n = 0,1,2$, $f_{111}(n,1) = 0$ for all $n \ge 3$, $f_{111}(0,k) = 1$ and $f_{111}(1,k) = k$ for all k, hence the theorem holds. □

Finally, we consider patterns 112 and 121. We start with pattern 121.

If a word $\sigma \in [k]^n$ avoids pattern 121, then it contains no letters other than 1 between any two 1's, which means that all 1's in σ, if any, are consecutive. Deletion of all 1's from σ leaves another word σ_1 which avoids 121 and contains no 1's, so all 2's in σ_1, if any, are consecutive. In general, deletion of all letters 1 through j leaves a (possibly empty) word σ_j on letters $j+1$ through k in which all letters $j+1$, if any, occur consecutively.

If a word $\sigma \in [k]^n$ avoids pattern 112, then only the leftmost 1 of σ may occur before a greater letter. The rest of the 1's must occur at the end of σ. In fact, just as in the previous case, deletion of all letters 1 through j leaves a (possibly empty) word σ_j on letters $j+1$ through k in which all occurrences of $j+1$, except possibly the leftmost one, are at the end of σ_j. We will call all occurrences of a letter j, except the leftmost j, *excess* j's.

The preceding analysis suggests a natural bijection $\rho : [k]^n(121) \to [k]^n(112)$. Given a word $\sigma \in [k]^n(121)$, we apply the following algorithm of k steps. Say it yields a word $\sigma^{(j)}$ after Step j, with $\sigma^{(0)} = \sigma$. Then Step j $(1 \le j \le k)$ is:

Step j. Cut the block of excess j's, then insert it immediately before the final block of all smaller excess letters of $\sigma^{(j-1)}$, or at the end of $\sigma^{(j-1)}$ if there are no smaller excess letters.

It is easy to see that, at the end of the algorithm, we get a word $\sigma^{(k)} \in [k]^n(112)$.

The inverse map, $\rho^{-1} : [k]^n(112) \to [k]^n(121)$ is given by a similar algorithm of k steps. Given a word $\sigma \in [k]^n(112)$ and keeping the same notation as above, Step j is as follows:

Step j. Cut the block of excess j's (which are at the end of $\sigma^{(j-1)}$), then insert it immediately after the leftmost j in $\sigma^{(j-1)}$.

Clearly, we get $\sigma^{(k)} \in [k]^n(121)$ at the end of the algorithm.

Thus, we have the following

Theorem 2. *Patterns 121 and 112 are Wilf-equivalent.*

We will now find $f_{112}(n,k)$ and provide a bijective proof of the resulting formula.

Consider all words $\sigma \in [k]^n(112)$ which contain a letter 1. Their number is

$$g_{112}(n,k) = f_{112}(n,k) - |\{\sigma \in [k]^n(112) : \sigma \text{ has no 1's}\}| = f_{112}(n,k) - f_{112}(n,k-1). \tag{1}$$

On the other hand, each such σ either ends on 1 or not.

If σ ends on 1, then deletion of this 1 may produce any word in $\bar{\sigma} \in [k]^{n-1}(112)$, since addition of the rightmost 1 to any word in $\bar{\sigma} \in [k]^{n-1}(112)$ does not produce extra occurrences of pattern 112.

If σ does not end on 1, then it has no excess 1's, so its only 1 is the leftmost 1 which does not occur at end of σ. Deletion of this 1 produces a word in $\bar{\sigma} \in \{2,\ldots,k\}^{n-1}(112)$. Since insertion of a single 1 into each such $\bar{\sigma}$ does not produce extra occurrences of pattern 112, for each word $\bar{\sigma} \in \{2,\ldots,k\}^{n-1}(112)$ we may insert a single 1 in $n-1$ positions (all except the rightmost one) to get a word $\sigma \in [k]^n(112)$ which contains a single 1 not at the end.

Thus, we have

$$\begin{aligned} g_{112}(n,k) &= f_{112}(n-1,k) + (n-1)|\{\sigma \in [k]^{n-1}(112) : \sigma \text{ has no 1's}\}| = \\ &= f_{112}(n-1,k) + (n-1)f_{112}(n-1,k-1). \end{aligned} \tag{2}$$

Combining (1) and (2), we get

$$f_{112}(n,k) - f_{112}(n,k-1) = f_{112}(n-1,k) + (n-1)f_{112}(n-1,k-1), \quad n \geq 1,\ k \geq 1. \tag{3}$$

The initial values are $f_{112}(n,0) = \delta_{n0}$ for all $n \geq 0$ and $f_{112}(0,k) = 1$, $f_{112}(1,k) = k$ for all $k \geq 0$.

Therefore, multiplying (6) by y^k and summing over k, we get

$$G_{112}(n;y) - \delta_{n0} - yG_{112}(n;y) = G_{112}(n-1;y) - \delta_{n-1,0} + (n-1)yG_{112}(n-1;y), \quad n \geq 1,$$

hence,

$$(1-y)G_{112}(n;y) = (1+(n-1)y)G_{112}(n-1;y), \quad n \geq 2.$$

Therefore,

$$G_{112}(n;y) = \frac{1+(n-1)y}{1-y}G_{112}(n-1;y), \quad n \geq 2. \tag{4}$$

Also, $G_{112}(0;y) = \dfrac{1}{1-y}$ and $G_{112}(1;y) = \dfrac{y}{(1-y)^2}$, so applying the previous equation repeatedly yields

$$G_{112}(n;y) = \frac{y(1+y)(1+2y)\cdots(1+(n-1)y)}{(1-y)^{n+1}}. \tag{5}$$

We have

$$\begin{aligned} \frac{1}{y}Numer(G_{112}(n;y)) &= (1+y)(1+2y)\cdots(1+(n-1)y) = y^n \prod_{j=0}^{n-1}\left(\frac{1}{y}+j\right) = \\ &= y^n \sum_{k=0}^{n} c(n,k)\left(\frac{1}{y}\right)^k = \sum_{k=0}^{n} c(n,k)y^{n-k} = \sum_{k=0}^{n} c(n,n-k)y^k, \end{aligned}$$

where $c(n,j)$ is the signless Stirling number of the first kind, and

$$yDenom(G_{112}(n;y)) = \frac{y}{(1-y)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k-1}{n}y^k,$$

so $f(n,k)$ is the convolution of the two coefficients:

$$f_{112}(n,k) = \left(c(n,n-k) * \binom{n+k-1}{n}\right) = \sum_{j=0}^{k} \binom{n+k-j-1}{n} c(n,n-j).$$

Thus, we have a new and improved version of Theorem 2.

Theorem 3. *Patterns 112 and 121 are Wilf-equivalent, and*

$$(6)\qquad \begin{aligned} f_{121}(n,k) = f_{112}(n,k) &= \sum_{j=0}^{k} \binom{n+k-j-1}{n} c(n,n-j), \\ F_{121}(x,y) = F_{112}(x,y) &= \frac{1}{1-y} \cdot \left(\frac{1-y}{1-y-xy}\right)^{1/y}. \end{aligned}$$

We note that this is the first time that Stirling numbers appear in enumeration of words (or permutations) with forbidden patterns.

Proof. The first formula is proved above. The second formula can be obtained as the exponential generating function of $G_{112}(n;y)$ from the recursive equation (4). Alternatively, multiplying the recursive formula (3) by $x^n/n!$ and summing over n yields

$$\frac{d}{dx}F_{112}(x;k) = F_{112}(x;k) + (1+x)\frac{d}{dx}F_{112}(x;k-1).$$

Multiplying this by y^k and summing over $k \geq 1$, we obtain

$$\frac{d}{dx}F_{112}(x,y) = \frac{1}{1-y-yx}F_{112}(x,y).$$

Solving this equation together with the initial condition $F_{112}(0,y) = \dfrac{1}{1-y}$ yields the desired formula. □

We will now prove the exact formula (6) bijectively. As it turns out, a little more natural bijective proof of the same formula obtains for $f_{221}(n,k)$, an equivalent result since $221 = c(112)$. This bijective proof is suggested by equation (3) and by the fact that $c(n,n-j)$ enumerates permutations of n letters with $n-j$ right-to-left minima (i.e. with j right-to-left nonminima), and $\binom{n+k-j-1}{n}$ enumerates nondecreasing strings of length n on letters in $\{0,1,\ldots,k-j-1\}$.

Given a permutation $\pi \in S_n$ which has $n-j$ right-to-left minima, we will construct a word $\sigma \in [j+1]^n(221)$ with certain additional properties to be discussed later. The algorithm for this construction is as follows.

Algorithm 1. (1) Let $d = (d_1,\ldots,d_n)$, where

$$d_r = \begin{cases} 0, & \text{if } r \text{ is a right-to-left minimum in } \pi, \\ 1, & \text{otherwise.} \end{cases}$$

(2) Let $s = (s_1,s_2,\ldots,s_n)$, where $s_r = 1 + \sum_{i=1}^{r} d_r$, $r = 1,\ldots,n$.
(3) Let $\sigma = \pi \circ s$ (i.e. $\sigma_r = s_{\pi(r)}$, $r = 1,\ldots,n$). This is the desired word σ.

Example 1. Let $\pi = 621/93/574/8/10 \in S_{10}$. Then $n-j = 5$, so $j+1 = 6$, $d = 0100111010$, $s = 1222345566$, so the corresponding word $\sigma = 4216235256 \in [6]^{10}(221)$.

Note that each letter s_r in σ is in the same position as that of r in π, i.e. $\pi^{-1}(r)$.

Let us show that our algorithm does indeed produce a word $\sigma \in [j+1]^n(221)$.

Since π has $n-j$ right-to-left minima, only j of the d_r's are 1s, the rest are 0s. The sequence $\{s_r\}$ is clearly nondecreasing and its maximum, $s_n = 1 + 1 \cdot j = j+1$. Thus, $\sigma \in [j+1]^n$ and σ contains all letters from 1 to $j+1$.

Suppose now σ contains an occurrence of the pattern 221. This means π contains a subsequence bca or cba, $a < b < c$. On the other hand, $s_b = s_c$, so $0 = s_c - s_b = \sum_{r=b+1}^{c} d_r$, hence $d_c = 0$ and c must be a right-to-left minimum. But $a < c$ is to the right of c, so c is not a right-to-left minimum. Contradiction. Therefore, σ avoids pattern 221.

Thus, $\sigma \in [j+1]^n(221)$ and contains all letters 1 through $j+1$. Moreover, the leftmost (and *only* the leftmost) occurrence of each letter (except 1) is to the left of some smaller letter. This is because $s_b = s_{b-1}$ means $d_b = 0$, that is b is a right-to-left minimum, i.e. occurs to the right of all smaller letters. Hence, s_b is also to the right of all smaller letters, i.e. is a right-to-left minimum of σ. On the other hand, $s_b > s_{b-1}$ means $d_b = 1$, that is b is not a right-to-left minimum of π, so s_b is not a right-to-left minimum of σ.

It is easy to construct an inverse of Algorithm 1. Assume we are given a word σ as above. We will construct a permutation $\pi \in S_n$ which has $n-j$ right-to-left minima.

Algorithm 2.

(1) Reorder the elements of σ in nondecreasing order; call the resulting string s.
(2) Let $\pi \in S_n$ be the permutation such that $\sigma_r = s_{\pi(r)}$, $r = 1, \dots, n$, given that $\sigma_a = \sigma_b$ (i.e. $s_{\pi(a)} = s_{\pi(b)}$) implies $\pi(a) < \pi(b) \Leftrightarrow a < b$). In other words, π is monotone increasing on positions of equal letters. Then π is the desired permutation.

Example 2. Let $\sigma = 4216235256 \in [6]^{10}(221)$ from our earlier example (so $j+1 = 6$). Then $s = 1222345566$, so looking at positions of 1s, 2s, etc., 6s, we get

$$\begin{aligned}
\pi(1) &= 6 \\
\pi(\{2,5,8\}) &= \{2,3,4\} &&\Longrightarrow \quad \pi(2)=2,\ \pi(5)=3,\ \pi(8)=4 \\
\pi(3) &= 1 \\
\pi(\{4,10\}) &= \{9,10\} &&\Longrightarrow \quad \pi(9)=4,\ \pi(10)=10 \\
\pi(6) &= 5 \\
\pi(\{7,9\}) &= \{7,8\} &&\Longrightarrow \quad \pi(7)=7,\ \pi(9)=8.
\end{aligned}$$

Hence, $\pi = (6,2,1,9,3,5,7,4,8,10)$ (in the one-line notation, not the cycle notation) and π has $n-j$ right-to-left minima: 10, 8, 4, 3, 1.

Note that the position of each s_r in σ is $\pi^{-1}(r)$, i.e. again the same as r has in π. Therefore, we conclude as above that π has $j+1-1 = j$ right-to-left nonminima, hence, $n-j$ right-to-left minima. Furthermore, the same property implies that Algorithm 2 is the inverse of Algorithm 1.

Note, however, that more than one word in $[k]^n(221)$ may map to a given permutation $\pi \in S_n$ with exactly $n-j$ right-to-left minima. We only need require that just the letters corresponding to the right-to-left nonminima of π be to the left of a smaller letter (i.e. not at the end) in σ. Values of 0 and 1 of d_r in Step 1 of Algorithm 1 are minimal increases required to recover back the permutation π with Algorithm 2. We must have $d_r \geq 1$ when we have to increase s_r, that is when s_r is not a right-to-left minimum of σ, i.e. when r is not a right-to-left minimum of π. Otherwise, we don't have to increase s_r, so $d_r \geq 0$.

Let $\sigma \in [k]^n(221)$, $\pi = Alg2(\sigma)$, $\tilde{\sigma} = Alg1(\pi) = Alg1(Alg2(\sigma)) \in [j+1]^n(221)$, and $\eta = \sigma - \tilde{\sigma}$ (vector subtraction). Note that $e_r = s_r(\sigma) - s_r(\tilde{\sigma}) \geq 0$ does not decrease (since $s_r(\sigma)$ cannot stay the same if $s_r(\tilde{\sigma})$ is increased by 1) and $0 \leq e_1 \leq \ldots \leq e_n \leq k-j-1$.

Since position of each e_r in η is the same as position of s_r in σ (i.e. $\eta_a = e_{\pi(a)}$, $e = e_1e_2\dots e_n$), the number of such sequences η is the number of nondecreasing sequences e of length n on letters in $\{0,\dots,k-j-1\}$, which is $\binom{n+k-j-1}{n}$.

Thus, $\sigma \in [k]^n(221)$ uniquely determines the pair (π, e), and vice versa. This proves the formula (6) of Theorem 3.

All of the above lets us state the following

Theorem 4. *There are 3 Wilf classes of multipermutations of length 3, with representatives 123, 112 and 111.*

4. Pairs of 3-letter patterns

There are 8 symmetric classes of pairs of 3-letters words, which are

$$\{111,112\},\{111,121\},\{112,121\},\{112,122\},\{112,211\},\{112,212\},\{112,221\},\{121,212\}.$$

Theorem 5. *The pairs* $\{111,112\}$ *and* $\{111,121\}$ *are Wilf equivalent, and*

$$F_{111,121}(x,y) = F_{111,112}(x,y) = \frac{e^{-x}}{1-y}\cdot\left(\frac{1-y}{1-y-xy}\right)^{1/y},$$

$$f_{111,112}(n,k) = \sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^{n-i}\binom{n}{i}\binom{k+i-j-1}{i}c(i,i-j).$$

Proof. To prove equivalence, notice that the bijection $\rho : [k]^n(121) \to [k]^n(112)$ preserves the number of excess copies of each letter and that avoiding pattern 111 is the same as having at most 1 excess letter j for each $j = 1,\dots,k$. Thus, restriction of ρ to words with ≤ 1 excess letter of each kind yields a bijection $\rho\rightharpoonup_{111}$: $[k]^n(111,121) \to [k]^n(111,112)$.

Let $\alpha \in [k]^n(111,112)$ contain i copies of letter 1. Since α avoids 111, we see that $i \in \{0,1,2\}$. Corresponding to these three cases, the number of such words α is $f_{111,112}(n,k-1)$, $nf_{111,112}(n-1,k-1)$ or $(n-1)f_{111,112}(n-2,k-1)$, respectively. Therefore,

$$f_{111,112}(n,k) = f_{111,112}(n,k-1) + nf_{111,112}(n-1,k-1) + (n-1)f_{111,112}(n-2,k-1),$$

for $n,k \ge 1$. Also, $f_{111,112}(n,0) = \delta_{n0}$ and $f_{111,112}(0,k) = 1$, hence

$$F_{111,112}(x;k) = (1+x)F_{111,112}(x;k-1) + \int xF_{111,112}(x;k-1)dx,$$

where $f_{111,112}(0,k) = 1$. Multiply the above equation by y^k and sum over all $k \ge 1$ to get

$$F_{111,112}(x,y) = c(y)e^{-x}\cdot\left(\frac{1-y}{1-y-xy}\right)^{1/y},$$

which, together with $F_{111,112}(0,y) = \dfrac{1}{1-y}$, yields the generating function.

Notice that $F_{111,112}(x,y) = e^{-x}F_{112}(x,y)$, hence, $F_{111,112}(x;k) = e^{-x}F_{112}(x;k)$, so $f_{111,112}(n,k)$ is the exponential convolution of $(-1)^n$ and $f_{112}(n,k)$. This yields the second formula. □

Theorem 6. *Let* $H_{112,121}(x;k) = \sum_{n\ge0} f_{112,121}(n,k)x^n$. *Then for any* $k \ge 1$,

$$H_k(x) = \frac{1}{1-x}H_{112,121}(x;k-1) + x^2\frac{d}{dx}H_{112,121}(x;k-1),$$

and $H_{112,121}(x;0) = 1$.

Proof. Let $\alpha \in [k]^n(112,121)$ such that contains j letters 1. Since α avoids 112 and 121, we have that for $j > 1$, all j copies of letter 1 appear in α in positions $n-j+1$ through n. When $j = 1$, the single 1 may appear in any position. Therefore,

$$f_{112,121}(n;k) = f_{112,121}(n;k-1) + nf_{112,121}(n-1,k-1) + \sum_{j=2}^{n} f_{112,121}(n-j;k-1),$$

which means that

$$\begin{aligned} f_{112,121}(n;k) &= f_{112,121}(n-1;k) + f_{112,121}(n;k-1) \\ &\quad + (n-1)f_{112,121}(n-1,k-1) - (n-2)f_{112,121}(n-2,k-1). \end{aligned}$$

We also have $f_{112,121}(n;0) = 1$, hence it is easy to see the theorem holds. □

Theorem 7. *Let $H_{112,211}(x;k) = \sum_{n\geq 0} f_{112,211}(n,k)x^n$. Then for any $k \geq 1$,*

$$H_{112,211}(x;k) = (1+x+x^2)H_{112,211}(x;k-1) + \frac{x^3}{1-x} + \frac{d}{dx}H_{112,211}(x;k-1),$$

and $H_{112,211}(x;0) = 1$.

Proof. Let $\alpha \in [k]^n(112,211)$ such that contains j letters 1. Since α avoids 112 and 211 we have that $j = 0,1,2,n$. When $j = 2$, the two 1's must at the beginning and at the end. Hence, it is easy to see that for $j = 0,1,2,n$ there are $f_{112,211}(n;k-1)$, $nf_{112,211}(n-1;k-1)$, $f_{112,211}(n-2;k-1)$ and 1 such α, respectively. Therefore,

$$f_{112,211}(n;k) = f_{112,211}(n;k-1) + nf_{112,211}(n-1,k-1) + f_{112,211}(n-2,k-1) + \delta_{n\geq 3}.$$

We also have $f_{112,121}(n;0) = 1$, hence it is easy to see the theorem holds. □

Theorem 8. *Let $a_{n,k} = f_{112,212}(n,k)$, then*

$$a_{n,k} = a_{n,k-1} + \sum_{d=1}^{n}\sum_{r=0}^{k-1}\sum_{j=0}^{n-d} a_{j,r}a_{n-d-j,k-1-r}$$

and $a_{0,k} = 1$, $a_{n,1} = 1$.

Proof. Let $\alpha \in [k]^n(112,212)$ have exactly d letters 1. If $d = 0$, there are $a_{n,k-1}$ such α. Let $d \geq 1$, and assume that $\alpha_{i_d} = 1$ where $d = 1,2,\ldots j$. Since α avoids 112, we have $i_2 = n+2-d$ (if $d = 1$, we define $i_2 = n+1$), and since α avoids 212 we have that α_a, α_b are different for all $a < i_1 < b < i_2$. Therefore, α avoids $\{112,212\}$ if and only if $(\alpha_1,\ldots,\alpha_{i_1-1})$, and $(\alpha_{i_1+1},\ldots,\alpha_{i_2-1})$ are $\{112,212\}$-avoiding. The rest is easy to obtain. □

Theorem 9.

$$f_{112,221}(n,k) = \sum_{j=1}^{k} j\cdot j!\binom{k}{j}$$

for all $n \geq k+1$,

$$f_{112,221}(n,k) = n!\binom{k}{n} + \sum_{j=1}^{n-1} j\cdot j!\binom{k}{j}$$

for all $k \geq n \geq 2$, and $f_{112,221}(0,k) = 1$, $f_{112,221}(1,k) = k$.

Proof. Let $\alpha \in [k]^n(112,221)$ and $j \leq n$ be such that $\alpha_1,\ldots,\alpha_j$ are all distinct and j is maximal. Clearly, $j \leq k$. Since α avoids $\{112,221\}$ and j is maximal, we get that the letters $\alpha_{j+1},\ldots,\alpha_n$, if any, must all be the same and equal to one of the letters $\alpha_1,\ldots,\alpha_j$. Hence, there are $j\cdot j!\binom{k}{j}$ such α if , for $j < n$ or $j = n > k$. For $j = n \leq k$, there are $n!\binom{k}{n}$ such α. Hence, summing over all possible $j = 1,\ldots,k$, we obtain the theorem. □

Theorem 10.

$$f_{121,212}(n,k)=\sum_{j=0}^{k} j!\binom{k}{j}\binom{n-1}{j-1}$$

for $k\geq 0$, $n\geq 1$, *and* $f_{121,212}(0,k)=1$ *for* $k\geq 0$.

Proof. Let $\alpha\in[k]^n(121,212)$ contain exactly j distinct letters. Then all copies of each letter 1 through j must be consecutive, or α would contain an occurrence of either 121 or 212. Hence, α is a concatenation of j constant strings. Suppose the i-th string has length $n_i>0$, then $n=\sum_{i=1}^{j} n_i$. Therefore, to obtain any $\alpha\in[k]^n(121,212)$, we can choose j letters out of k in $\binom{k}{j}$ ways, then choose any ordered partition of n into j parts in $\binom{n-1}{j-1}$ ways, then label each part n_i with a distinct number $l_i\in\{1,\dots,j\}$ in $j!$ ways, then substitute n_i copies of letter l_i for the part n_i $(i=1,\dots,j)$. This yields the desired formula. □

Unfortunately, the case of the pair $(112,122)$ still remains unsolved.

5. Some triples of 3-letter patterns

Theorem 11.

$$F_{112,121,211}(x;k)=1+\frac{(e^x-1)((1+x)^k-1)}{x},$$

$$f_{112,121,211}(n,k)=\begin{cases}\displaystyle\sum_{j=1}^{n}\frac{1}{j!}\binom{n+1}{j}\binom{k}{n+1-j}, & n\geq 1,\\ 1, \quad n=0.\end{cases}$$

Proof. Let $\alpha\in[k]^n(112,121,211)$ contain j letters 1. For $j\geq 2$, there are no letters between the 1's, to the left of the first 1 or to the right of the last 1, hence $j=n$. For $j=1$, $j=0$ it is easy to see from definition that there are $nf_{112,121,211}(n-1,k-1)$ and $f_{112,121,211}(n,k-1)$ such α, respectively. Hence,

$$f_{112,121,211}(n,k)=f_{112,121,211}(n,k-1)+nf_{112,121,211}(n-1,k-1)+1,$$

for $n,k\geq 2$. Also, $a(n,1)=a(n,0)=1$, $a(0,k)=1$, and $a(1,k)=k$. If we let $b(n,k)=f_{112,121,211}(n,k)/n!$, then

$$b(n,k)=b(n,k-1)+b(n-1,k-1)+\frac{1}{n!}.$$

Let $b_k(x)=\sum_{n\geq 0} b(n,k)x^n$, then it is easy to see that $b_k(x)=(1+x)b_{k-1}(x)+e^x-1$. Since we also have $b_0(x)=e^x$, the theorem follows by induction. □

6. Some patterns of arbitrary length

6.1. **Pattern** $11\dots1$. Let us denote by $\langle a\rangle_l$ the word consisting of l copies of letter a.

Theorem 12. *For any* $l,k\geq 0$,

$$F_{\langle 1\rangle_l}(x;k)=\left(\sum_{j=0}^{l-1}\frac{x^j}{j!}\right)^k.$$

Proof. Let $\alpha\in[k]^n(\langle 1\rangle_l)$ contain j letters 1. Since α avoids $\langle 1\rangle_l$, we have $j\leq l-1$. If α contains exactly j letters of 1, then there are $\binom{n}{j}f_{\langle 1\rangle_l}(n-j,k-1)$ such α, therefore

$$f_{\langle 1\rangle_l}(n,k)=\sum_{j=0}^{l-1}\binom{n}{j}f_{\langle 1\rangle_l}(n-j,k-1).$$

We also have $f_{\langle 1\rangle_l}(n,k) = k^n$ for $n \le l-1$, hence it is easy to see the theorem holds. □

In fact, [CS] shows that we have

$$f_{\langle 1\rangle_l}(n,k) = \sum_{i=1}^{n} M_2^{l-1}(n,i)(k)_i,$$

where $M_2^{l-1}(n,i)$ is the number of partitions of an n-set into i parts of size $\le l-1$.

6.2. **Pattern 11...121...11.** Let us denote $v_{m,l} = 11\ldots121\ldots11$, where m (respectively, l) is the number of 1's on the left (respectively, right) side of 2 in $v_{m,l}$. In this section we prove the number of words in $[k]^n(v_{m,l})$ is the same as the number of words in $[k]^n(v_{m+l,0})$ for all $m,l \ge 0$.

Theorem 13. *Let $m,l \ge 0$, $k \ge 1$. Then for $n \ge 1$,*

$$f_{v_{m,l}}(n+1,k) - f_{v_{m,l}}(n,k) = \sum_{j=0}^{m+l-1} \binom{n}{j} f_{v_{m,l}}(n+1-j,k-1).$$

Proof. Let $\alpha \in [k]^n(v_{m,l})$ contain exactly j letters 1. Since the 1's cannot be part of an occurrence of $v_{m,l}$ in α when $j \le m+l-1$, these 1's can be in any j positions, so there are $\binom{n}{j} f_{v_{m,l}}(n,k-1)$ such α. If $j \ge m+l$, then the m-th through $(j-l+1)$-st (l-th from the right) 1's must be consecutive letters in α (with the convention that the 0-th 1 is the beginning of α and $(j+1)$-st 1 is the end of α). Hence, there are $\binom{n-j+m+l-1}{m+l-1} f_{v_{m,l}}(n-j,k-1)$ such α, and hence

$$f_{v_{m,l}}(n;k) = \sum_{j=0}^{m+l-1} \binom{n}{j} f_{v_{m,l}}(n-j,k-1) + \sum_{j=m+l}^{n} \binom{n-j+m+l-1}{m+l-1} f_{v_{m,l}}(n-j,k-1).$$

Hence for all $n \ge 1$,

$$f_{v_{m,l}}(n+1,k) - f_{v_{m,l}}(n,k) = \sum_{j=0}^{m+l-1} \binom{n}{j} f_{v_{m,l}}(n+1-j,k-1).$$

□

An immediate corollary of Theorem 13 is the following.

Corollary 14. *Let $m,l \ge 0$, $k \ge 0$. Then for $n \ge 0$*

$$f_{v_{m,l}}(n,k) = f_{v_{m+l,0}}(n,k).$$

In other words, all patterns $v_{m,l}$ with the same $m+l$ are Wilf-equivalent.

Proof. We will give an alternative, bijective proof of this by generalizing our earlier bijection $\rho : [k]^n(121) \to [k]^n(112)$. Let $\alpha \in [k]^n(v_{m,l})$. Recall that α_j is a word obtained by deleting all letters 1 through j from α (with $\alpha_0 := \alpha$).

Suppose that α contains i letters $j+1$. Then all occurrences of $j+1$ from m-th through $(i-l+1)$-st, if any (i.e. if $j \ge m+l$), must be consecutive letters in α_j. We will denote as *excess* j's the $(m+1)$-st through $(i-l+1)$-st copies of j when $l > 0$, and m-th through i-th copies of j when $l = 0$.

Suppose that $m+l = m'+l'$. Then the bijection $\rho_{m,l;m',l'} : [k]^n(v_{m,l}) \to [k]^n(v_{m',l'})$ is an algorithm of k steps. Given a word $\alpha \in [k]^n(v_{m,l})$, say it yields a word $\alpha^{(j)}$ after Step j, with $\alpha^{(0)} := \alpha$. Then Step j $(1 \le j \le k)$ is as follows:

Step j.

(1) Cut the block of excess j's from ${\alpha^{(j-1)}}_{j-1}$ (which is immediately after the m-th occurrence of j), then insert it immediately after the m'-th occurrence of j if $l' > 0$, or at the end of ${\alpha^{(j-1)}}_{j-1}$ if $l' = 0$.
(2) Insert letters 1 through $j-1$ into the resulting string in the same positions they are in $\alpha^{(j-1)}$ and call the combined string $\alpha^{(j)}$.

Clearly,

$${\alpha^{(j)}}_j = {\alpha^{(j-1)}}_j = \ldots = {\alpha^{(0)}}_j \alpha_j$$

and at Step j, the j's are rearranged so that no j can be part of an occurrence of $v_{m',l'}$. Also, positions of letters 1 through $j-1$ are the same in $\alpha^{(j)}$ and $\alpha^{(j-1)}$, hence, no letter from 1 to j can be part of $v_{m',l'}$ in $\alpha^{(j)}$ by induction. Therefore, $\alpha^{(k)} \in [k]^n(v_{m',l'})$ as desired.

Clearly, this map is invertible, and $\rho_{m',l';m,l} = (\rho_{m,l;m',l'})^{-1}$. This ends the proof. □

Theorem 15. *Let $p \geq 1$ and $d_p(f(x)) = \int \ldots \int f(x)dx \ldots dx$ (and we define $d_0(f(x)) = f(x)$)). Then for any $k \geq 1$,*

$$F_{v_{p,0}}(x;k) - \int F_{v_{p,0}}(x;k)dx = \sum_{j=0}^{p-1} \left((-1)^j d_p(F_{v_{p,0}}(x;k-1)) \sum_{i=0}^{p-1-j} \frac{x^i}{i!} \right),$$

and $F_{v_{p,0}}(x;1) = e^x$, $F_{v_{p,0}}(0;k) = 1$.

Proof. By definition, we have $f_{v_{p,0}}(n,1) = 1$ for all $n \geq 0$ so $F_{v_{p,0}}(x;1) = e^x$. On the other hand, Theorem 13 yields immediately the rest of this theorem. □

Example 3. *For $p = 1$, Theorem 15 yields*

$$\sum_{n\geq 0} |[k]^n(12)| \frac{x^n}{n!} = e^x \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{x^j}{j!},$$

which means that, for any $n \geq 0$

$$|[k]^n(12)| = \binom{n+k-1}{k-1}.$$

(cf. Section 2.)

Example 4. *For $p = 2$, Theorem 15 yields*

$$F_{112}(x;k) = e^x \cdot \int (1+x)e^{-x} F_{112}(x;k-1)dx,$$

and $F_{112}(x;0) = 1$.

Corollary 16. *For any $p \geq 0$*

$$F_{v_{p,0}}(x;2) = e^x \sum_{j=0}^{p} \frac{x^j}{j!}.$$

Proof. From Theorem 15, we immediately get that

$$F_{v_{p,0}}(x;2) - \int F_{v_{p,0}}(x;2)dx = e^x \sum_{j=0}^{p-1} (-1)^j \sum_{i=0}^{p-1-j} \frac{x^i}{i!},$$

which means that

$$e^x \frac{d}{dx} \left(e^{-x} F_{v_{p,0}}(x;2) \right) = e^x \sum_{j=0}^{p-1} \frac{x^j}{j!},$$

hence the corollary holds. □

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