INEQUALITIES FOR POLYTOPES AND ZONOTOPES

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ABSTRACT. We prove that the coefficients of the **cd**-index of a convex polytope are increasing when replacing $\mathbf{c}^2$ with $\mathbf{d}$. That is, for P an n-dimensional convex polytope and u and v two **cd**-monomials such that the sum of their degrees is $n-2$, we have $[u\mathbf{c}^2 v]\Psi(P) \leq [u\mathbf{d}v]\Psi(P)$. This yields $(5 \cdot n - \sqrt{5})/25 \cdot \phi^n$ linear inequalities on the flag f-vector of an n-dimensional polytope, where ϕ is the golden ratio. We prove similar but stronger results for the **cd**-index of zonotopes.

RÉSUMÉ. Nous prouvons que les coefficients de l'index **cd** d'un polytope convexe sont croissant lorsque l'on remplace $\mathbf{c}^2$ par $\mathbf{d}$. En d'autres termes, pour un polytope convexe de dimension n et deux monômes u et v tels que la somme de leurs degrés est $n-2$, on a $[u\mathbf{c}^2 v]\Psi(P) \leq [u\mathbf{d}v]\Psi(P)$. On produit $(5 \cdot n - \sqrt{5})/25 \cdot \phi^n$ inégalités linéaires pour le vecteur-f de drapeaux d'un polytope de dimension n, où ϕ est le nombre d'or. Nous prouvons des résultats similaires mais plus forts pour les index **cd** de zonotopes.

1. INTRODUCTION

The flag f-vector of a convex polytope contains all the enumerative incidence information between the faces. Thus to classify the set of all possible flag f-vectors is one of great open problems in discrete geometry. To date only partial results to this problem have been obtained. The case when the polytopes are simplicial (and dually, simple) the problem reduces to classifying the f-vectors of simplicial polytopes. This major step was solved by the combined effort of Billera and Lee [14] and Stanley [33]. Returning to the general case, the classification of flag f-vectors of three-dimensional polytopes was done by Steinitz [38] about one hundred years ago. Going up one dimension to four-dimensional polytopes, the case is still open. The article by Bayer [4] contains what is known for four dimensions.

The first step toward classifying flag f-vectors was taken by Bayer and Billera [6]. They observed that there are linear redundancies in the entries of the flag f-vector of a polytope. The relations holding between the entries of the flag f-vector are known as the generalized Dehn-Somerville relations. These relations imply that flag f-vectors of polytopes all lie in a subspace of dimension F_n, where F_n is the nth Fibonacci number ($F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$).

The next natural step is to look for linear inequalities that the flag vectors of polytopes satisfy. One such example is the toric g-vector. The entries of the toric g-vector are linear combinations of the entries of the flag f-vector. Stanley [35] proved that the toric g-vector of rational polytope is non-negative. Lately, there has been a lot of work of extending this result to non-rational polytopes; see [1, 2, 3, 19, 40]. More inequalities were obtained by Kalai by convoluting these inequalities together [27]. However, this is far from being all of the linear inequalities that the flag f-vector satisfies; see the work of Stenson [39].

A different direction of work has involved the **cd**-index. The **cd**-index $\Psi(P)$ of a polytope P is a non-commutative polynomial in two variables $\mathbf{c}$ and $\mathbf{d}$ whose coefficients are linear combinations of the entries of the flag f-vector. Thus a linear inequality among the coefficients of the **cd**-index implies a linear inequality among the entries of the flag f-vector. Moreover, the **cd**-index encodes the flag f-vector without linear redundancies. Another way to express this fact is that the set of **cd**-monomials offers a basis to the subspace of Fibonacci dimension. The existence of the **cd**-index was conjectured by Fine and proved by Bayer and Klapper [9].

Stanley [36] proved that the **cd**-index of a polytope has non-negative coefficients. This was the first important result which showed that the **cd**-index will play an important role in obtaining inequalities. The next step was taken by Billera and Ehrenborg who proved that the **cd**-index over all n-dimensional polytopes is minimized coefficientwise on the n-dimensional simplex Δ_n [10]. This gives a sharpening of the inequalities obtained by Stanley.

In this paper we will continue this vein of work. For an n-dimensional polytope P the **cd**-index is homogeneous of degree n, where the variable $\mathbf{c}$ has degree 1 and $\mathbf{d}$ has degree 2. We prove that the **cd**-index of a polytope P satisfies the family of inequalities

$$[u\mathbf{c}^2 v]\Psi(P) \;\leq\; [u\mathbf{d}v]\Psi(P), \tag{1.1}$$

where u and v are two **cd**-monomials such that $\deg(u) + \deg(v) = n - 2$. That is, when replacing a $\mathbf{c}^2$ with $\mathbf{d}$, the coefficient in $\Psi(P)$ increases. In total this yields $\sum_{i+j=n-2} F_i \cdot F_j \sim (5 \cdot n - \sqrt{5})/25 \cdot \phi^n$ linear inequalities, where ϕ is the golden ratio. Note however, that the $n-1$ inequalities when the left-hand side is the coefficient of $\mathbf{c}^n$ are surpassed by the fact that the **cd**-index is minimized on the simplex.

There are quadratic inequalities known on the entries of the flag f-vector. Two large classes of quadratic inequalities are given by Braden and MacPherson [18] and Billera and Ehrenborg [10]. However, quadratic inequalities are not as fundamental as linear inequalities. That is, the set of flag f-vectors of convex polytopes seems to have as a first good approximation the cone determined by linear inequalities. Very little is known about this issue and it deserves a deeper study.

A second question of interest is to study flag f-vectors of zonotopes. Yet again, the first step is to consider linear relations. Since zonotopes are a subclass of polytopes, they satisfy the generalized Dehn-Somerville relations. Billera, Ehrenborg and Readdy proved that zonotopes satisfy no more linear relations [12]. The next step is to consider linear inequalities. Billera, Ehrenborg and Readdy proved that among all n-dimensional zonotopes (and more generally, the dual of the lattice of regions of oriented matroids), the n-dimensional cube minimizes the **cd**-index coefficientwise [11].

We prove two improvements of the inequality (1.1) for zonotopes. For an n-dimensional zonotope Z we have

$$[u\mathbf{d}v]\Psi(Z) - [u\mathbf{c}^2 v]\Psi(Z) \;\geq\; [u\mathbf{d}v]\Psi(\square_n) - [u\mathbf{c}^2 v]\Psi(\square_n), \tag{1.2}$$

for any two **cd**-monomials u and v such that $\deg(u) + \deg(v) = n - 2$ and where $\square_n$ denotes the n-dimensional cube. That is, the increase in going from the coefficient of $u\mathbf{c}^2 v$ to $u\mathbf{d}v$ is greater than or equal to the corresponding increase in the cube. The second improvement is the following class of inequalities:

$$[\mathbf{c}^k\mathbf{d}v]\Psi(Z) - 2\cdot[\mathbf{c}^{k+2}v]\Psi(Z) \;\geq\; [\mathbf{c}^k\mathbf{d}v]\Psi(\square_n) - 2\cdot[\mathbf{c}^{k+2}v]\Psi(\square_n) \;\geq\; 0,$$

where k is a non-negative integer and v a **cd**-monomials such that $k + \deg(v) = n - 2$. This improves the inequality (1.2) when u is a power of $\mathbf{c}$, that is, $u = \mathbf{c}^k$, by inserting a factor of 2 in front of the coefficient of $\mathbf{c}^{k+2}v$.

2. Preliminaries

A *partially ordered set (poset)* P is ranked if there is a rank function $\rho : P \longrightarrow \mathbb{Z}$ such that when x is covered by y then $\rho(y) = \rho(x) + 1$. Furthermore, we call P graded of rank n if it is ranked and has a minimal element $\hat{0}$ and a maximal element $\hat{1}$ such that $\rho(\hat{0}) = 0$

and $\rho(\hat{1}) = n$. Define the interval $[x, y]$ to be the subposet $\{z \in P : x \leq z \leq y\}$. Observe that the interval $[x, y]$ is also a graded poset of rank $\rho(x, y) = \rho(y) - \rho(x)$.

Let P be a graded poset of rank $n + 1$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define f_S to be the number chains $\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}$ where the rank of the element x_i is s_i for $1 \leq i \leq k$. These 2^n values constitute the *flag f-vector* of the poset P. Define the *flag h-vector* of P by the two equivalent relations $h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T$ and $f_S = \sum_{T \subseteq S} h_T$. There has been a lot of recent work in understanding the flag f-vectors of graded posets and Eulerian posets; see [5, 8, 13].

For S a subset of $\{1, \ldots, n\}$ define the monomial $u_S = u_1 u_2 \cdots u_n$, where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Define the **ab**-*index* of a graded poset P of rank $n + 1$ to be the sum

$$\Psi(P) = \sum_S h_S \cdot u_S.$$

A poset P is Eulerian if every interval $[x, y]$, where $x \neq y$, has the same number of elements of odd rank as the number of elements of even rank. This condition states that every interval $[x, y]$ satisfies the Euler-Poincaré relation. The condition of being Eulerian is also equivalent to that the Möbius function $\mu(x, y)$ is given by $(-1)^{\rho(x,y)}$. Two examples of Eulerian posets are the strong Bruhat order and face lattices of convex polytopes.

The following result was conjectured by Fine and proved by Bayer and Klapper [9].

Theorem 2.1. *Let P be an Eulerian poset. Then the* **ab**-*index of P, $\Psi(P)$, can be written in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.*

When $\Psi(P)$ is expressed in terms of $\mathbf{c}$ and $\mathbf{d}$ it is called the **cd**-*index* of poset P. There exist several proofs of this result in the literature; see [9, 15, 21, 25, 36]. The **cd**-index has been extraordinarily useful for flag vector computations; see [7, 11, 24]. Moreover, this basis is now emerging as a key tool for obtaining linear inequalities for the entries of the flag f-vector; see [10, 22, 36].

3. Polytopes

Let P be an n-dimensional convex polytope. The face lattice of P is an Eulerian poset of rank $n+1$. We denote the **cd**-index of the face lattice of the polytope P by $\Psi(P)$. A *regular cellular ball/sphere* Γ is a finite regular CW complex such that its underlying space $|\Gamma|$ is a ball/sphere. More generally, the face poset of a regular cellular sphere Γ is an Eulerian poset and we denote its **cd**-index by $\Psi(\Gamma)$.

One of the useful techniques for proving results about polytopes is shellings. In their seminal paper Bruggesser and Mani [20] proved that all polytopes are shellable. They did this by providing a special class of shellings called *line shellings*.

Recall that a *shelling* of an n-dimensional polytope P is an ordering of its facets $F_1, \ldots, F_m$ such that for $2 \leq r \leq m$, the complex $\Lambda = (F_1 \cup \cdots \cup F_{r-1}) \cap F_r$ is pure of dimension $n - 2$ and there exists a shelling $G_1, \ldots, G_k$ of the facet F_r such that $\Lambda = G_1 \cup \cdots \cup G_s$ for some s. For a line shelling of the polytope the shelling of F_r is also a line shelling. Another important observation is that for a line shelling the set $\partial(F_1 \cup \cdots \cup F_r)$ is combinatorially equivalent to an $(n-1)$-polytope. The set $\partial(F_1 \cup \cdots \cup F_r)$ is not itself an $(n-1)$-polytope. However, it can be projected onto a hyperplane such that the image is the boundary of a polytope with the same combinatorial type. We refer the reader to the article by Bruggesser and Mani for more details on line shellings.

When Stanley studied the **cd**-index in [36], he introduced a different notion of shelling, called spherical shelling or S-shelling for short. However, the line shellings of Bruggesser and Mani are also S-shellings. Hence it is enough for us to only consider line shellings.

To a regular cellular ball Γ of dimension n there are two regular cellular spheres associated with it. Namely, the boundary and the semi-suspension. The boundary $\partial\Gamma$ is an $(n-1)$-dimensional sphere. The *semi-suspension* Γ' is the n-dimensional regular cellular sphere obtained by attaching a new cell σ to Γ such that $\partial\sigma = \partial\Gamma$.

The following lemma is essential to our argument. It was proved by Billera and Ehrenborg (see [10, Lemma 4.2]) based on results of Stanley [36].

Lemma 3.1. *Let P be a polytope with a line shelling $F_1, \ldots, F_m$ and let $1 \leq r \leq m-1$. Let Λ be given by $(F_1 \cup \cdots \cup F_{r-1}) \cap F_r$. Then we have*

$$\Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})') = (\Psi(F_r) - \Psi(\Lambda')) \cdot \mathbf{c} + \Psi(\partial\Lambda) \cdot \mathbf{d}.$$

On **cd**-polynomials there is a natural ordering by letting $z \leq w$ if and only if $w - z$ only has non-negative coefficients. We now define two stronger order relations $\preceq$ and $\preceq'$.

Definition 3.2. *Let z and w be two* **cd***-polynomials.*

(1) *Define the relation $z \preceq w$ if for all* **cd***-monomials u and v we have*

$$[u\mathbf{d}v]\, z - [u\mathbf{c}^2v]\, z \leq [u\mathbf{d}v]\, w - [u\mathbf{c}^2v]\, w.$$

(2) *Define the relation $z \preceq' w$ if for all* **cd***-monomials u and v, where v is not a power of* **c***, we have*

$$[u\mathbf{d}v]\, z - [u\mathbf{c}^2v]\, z \leq [u\mathbf{d}v]\, w - [u\mathbf{c}^2v]\, w.$$

A few observations are in order. When z and w are homogeneous of degree n, we may restrict u and v to be **cd**-monomials such that the sum of the degrees of u and v is $n-2$. Both order relations are transitive. Also, $\preceq$ is the stronger relation, that is, $z \preceq w$ implies $z \preceq' w$. Observe that both of these order relations are preserved under addition, that is, $z_1 \preceq w_1$ and $z_2 \preceq w_2$ implies $z_1 + z_2 \preceq w_1 + w_2$ and the similar addition rule for $\preceq'$ holds.

We call a **cd**-polynomial w *rising* if is satisfies $0 \preceq w$, that is, for all **cd**-monomials u and v we have that

$$[u\mathbf{c}^2v]\, w \leq [u\mathbf{d}v]\, w.$$

The rising homogeneous **cd**-polynomials of degree n thus form a cone in the linear space of homogeneous **cd**-polynomials of degree n.

Lemma 3.3. *If $z \succeq' 0$ and $w \succeq 0$ then $z \cdot \mathbf{c} + w \cdot \mathbf{d} \succeq' 0$.*

Proof. Observe that $z \succeq' 0 \Longrightarrow z \cdot \mathbf{c} \succeq' 0$ and $w \succeq 0 \Longrightarrow w \cdot \mathbf{d} \succeq 0 \Longrightarrow w \cdot \mathbf{d} \succeq' 0$. The lemma follows by adding these two conclusions. □

We are now able to present the main theorem.

Theorem 3.4. *Let P be an n-dimensional polytope and let $F_1, \ldots, F_m$ be a line shelling of the polytope P. Then*

(*a*) *The* **cd***-index $\Psi(P)$ is rising.*

(*b*) *The following string of inequalities holds:*

$$0 \preceq' \Psi(F_1') \preceq' \Psi((F_1 \cup F_2)') \preceq' \cdots \preceq' \Psi((F_1 \cup \cdots \cup F_{m-1})') = \Psi(P).$$

Proof. The proof is by induction on the dimension n. We will show the implications $(a) \Longrightarrow (b) \Longrightarrow (a)$. However, in the step $(a) \Longrightarrow (b)$ we will use the result in (a) for lower dimensional polytopes. The induction basis is $n = 0$, and it is enough to observe that the **cd**-index of a point is rising.

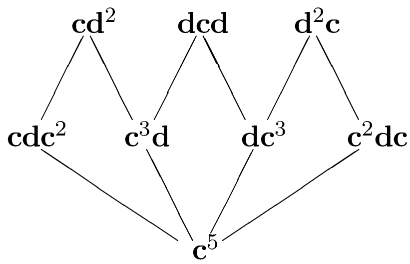

FIGURE 1. The eight **cd**-monomials of degree 5 in the partial order.

We next prove $(a) \Longrightarrow (b)$ by induction. By Lemma 3.1 we have that

$$\Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})') \quad = \quad (\Psi(F_r) - \Psi(\Lambda')) \cdot \mathbf{c} + \Psi(\partial\Lambda) \cdot \mathbf{d},$$

where $\Lambda = (F_1 \cup \cdots \cup F_{r-1}) \cap F_r$. By induction we know that $\Psi(F_r) - \Psi(\Lambda') \succeq' 0$. Now consider the set $\partial\Lambda$. We know that Λ is the union of facets of F_r that form the beginning of a line shelling. Thus $\partial\Lambda$ is combinatorially equivalent to an $(n-2)$-dimensional polytope and hence by induction $\Psi(\partial\Lambda)$ is rising. Now by Lemma 3.3 the result follows.

We prove $(b) \Longrightarrow (a)$ by three cases. The first case when v is not a power of $\mathbf{c}$ follows directly by transitivity of all the order relations in (b). For the second case when v is a power of $\mathbf{c}$ and u is not, the result follows by applying the first case to the dual polytope. Finally, the third case is when both u and v are powers of $\mathbf{c}$. However, the result is immediate from the fact that the simplex has the smallest **cd**-index coefficientwise among all polytopes and that the simplex has positive coefficients; see [10, 24]. $\square$

We remark that Stanley's proof of the non-negativity of the **cd**-index of polytopes follows the same outline as the proof of Theorem 3.4. Just replace every occurrence of the two orders $\preceq$ and $\preceq'$ with the order $\leq$ and the proof of the non-negativity appears.

Corollary 3.5. *Let P be a polytope. Then the* **cd***-monomial with the largest coefficient in $\Psi(P)$ has no consecutive* **c***'s.*

It is natural to consider the partial order on the set of **cd**-monomials of degree n, where the cover relation is to replace an occurrence of $\mathbf{c}^2$ with $\mathbf{d}$. See Figure 1 for the poset in the case $n = 5$. This poset is simplicial, that is, it is the face lattice of a simplicial complex. This simplicial complex has been studied earlier; see [16, Corollary 2]. Every **cd**-monomial corresponds to a face having dimension equal to the number of **d**'s in the monomial minus one. The **cd**-monomials with no consecutive **c**'s correspond to facets. It is easy to observe that the dimensions of facets range between $\lfloor (n+1)/3 \rfloor - 1$ and $\lfloor n/2 \rfloor - 1$. Thus the simplicial complex is pure only when $n \leq 3$ or when $n = 5$. Compare this with the slightly misleading remark before Corollary 2 in [16].

4. ZONOTOPES

In this section we will improve the main inequality for zonotopes. Let $\square_n$ denote the n-dimensional cube.

Theorem 4.1. *Let Z be an n-dimensional zonotope, or more generally, let Z be the dual of the lattice of regions of an oriented matroid. Then the* **cd***-index $\Psi(Z)$ satisfies the inequality*

$\Psi(Z) \succeq \Psi(\Box_n)$. That is,

$$[u\mathbf{d}v]\Psi(Z) - [u\mathbf{c}^2v]\Psi(Z) \ \geq\ [u\mathbf{d}v]\Psi(\Box_n) - [u\mathbf{c}^2v]\Psi(\Box_n),$$

for any two **cd***-monomials u and v such that* $\deg(u) + \deg(v) = n - 2$.

We will only prove this theorem for zonotopes. The proof for oriented matroids carries through exactly the same with the geometric language replaced with oriented matroid language.

Let ω be the linear map from $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ defined on an **ab**-monomial by replacing each occurrence of **ab** with 2**d** and then replacing the remaining variables by **c**. Now we have fundamental theorem of computing the **cd**-index of a zonotope [11].

Theorem 4.2. *Let Z be a zonotope and $\mathcal{H}$ its associated central hyperplane arrangement. Let L be the intersection lattice of the hyperplane arrangement $\mathcal{H}$ and $\Psi(L)$ the* **ab***-index of the lattice L. Then the* **cd***-index of the zonotope and the sum of the* **cd***-indices of all the vertex figures of the zonotope are given by*

$$\begin{aligned} \Psi(Z) &= \omega(\mathbf{a}\cdot\Psi(L)), \\ \sum_v \Psi(Z/v) &= 2\cdot\omega(\Psi(L)), \end{aligned}$$

where v ranges over all vertices of the zonotope Z.

The first identity is [11, Theorem 3.1]. The second identity follows from the first and using the linear map h defined in Section 8 in [11].

It remains to compute the **ab**-index of the intersection lattice L. We do this using R-labelings. For more details, see [11, Section 7] and [17, 32, 34]. Linearly order the hyperplanes in the arrangement $\mathcal{H}$, that is, $\mathcal{H} = \{H_1, \ldots, H_m\}$. Mark each edge $x \prec y$ in the Hasse diagram of the lattice L with the smallest (in the given linear order) hyperplane H such that intersecting x with H gives y. That is,

$$\lambda(x, y) = \min\{i \ : \ x \cap H_i = y\}.$$

For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}\}$ define its *descent set* $D(c)$ by

$$D(c) = \{i \ : \ \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\}.$$

Then we have the following result; see Section 7 in [11].

Theorem 4.3. *The* **ab***-index of intersection lattice L is given by*

$$\Psi(L) = \sum_c u_{D(c)},$$

where the sum ranges over all maximal chains c in the lattice L.

Lemma 4.4. *Let w and z be rising non-negative* **cd***-polynomials. Then the* **cd***-polynomial $w \cdot \mathbf{d} \cdot z$ is also rising.*

Proposition 4.5. *Let Z be an n-dimensional zonotope and let Z' be the zonotope obtained by taking the Minkowski sum of Z with a line segment in the affine span of Z. Then the difference $\Psi(Z') - \Psi(Z)$ is rising.*

Proof. Let $\mathcal{H}$ and $\mathcal{H}'$ be the associated hyperplanes arrangements and let H be the new hyperplane. Let $\mathcal{H}'$ inherit the linear order of $\mathcal{H}$ with the new hyperplane H attached at the end of the linear order. Similarly, let L and L' be the corresponding intersection lattices. Observe that every maximal chain in L is also a maximal chain in L'. Also observe that there is no maximal chain in L' whose last label is H. Hence the difference in the **ab**-indices between the two intersection lattices is

$$\begin{aligned}\Psi(L') - \Psi(L) &= \sum_{c} u_{D(c)} \\ &= \sum_{\hat{0}<x\prec y} \Psi([\hat{0},x]) \cdot \mathbf{ab} \cdot \Psi([y,\hat{1}]) + \sum_{\hat{0}=x\prec y} \mathbf{b} \cdot \Psi([y,\hat{1}]),\end{aligned}$$

where the first sum is over all maximal chains c containing the label H and the next two sums is over edges $x \prec y$ in the Hasse diagram of L' having the label H. Applying the map $w \longmapsto \omega(\mathbf{a} \cdot w)$ we obtain

$$\Psi(Z') - \Psi(Z) = \sum_{\hat{0}<x\prec y} \omega(\mathbf{a} \cdot \Psi([\hat{0},x])) \cdot 2\mathbf{d} \cdot \omega(\Psi([y,\hat{1}])) + \sum_{\hat{0}\prec y} 2\mathbf{d} \cdot \omega(\Psi([y,\hat{1}])). \tag{4.1}$$

Observe that the term $\omega(\mathbf{a} \cdot \Psi([\hat{0},x]))$ is the **cd**-index of a zonotope and hence is rising by Theorem 3.4. Similarly, the term $\omega(\Psi([y,\hat{1}]))$ is one half of the sum of **cd**-indices of the vertex figures of a zonotope and hence is also rising. Now by Lemma 4.4 and the property that being rising is preserved under addition, the result follows. □

Proof of Theorem 4.1. Observe that any n-dimensional zonotope is obtained from the n-dimensional cube $\square_n$ by Minkowski adding line segments. Thus the result follows from Proposition 4.5. □

The second improvement of the inequalities is when comparing the coefficients of $\mathbf{c}^k\mathbf{d}v$ and $\mathbf{c}^{k+2}v$, that is, when u is a power of $\mathbf{c}$. For ease in notation, we introduce a third order relation.

Definition 4.6. *Define the order relation $z \preceq'' w$ on the* **cd***-polynomials z and w by*

$$[\mathbf{c}^k\mathbf{d}v]z - 2 \cdot [\mathbf{c}^{k+2}v]z \;\leq\; [\mathbf{c}^k\mathbf{d}v]w - 2 \cdot [\mathbf{c}^{k+2}v]w.$$

We call w weakly 2-rising *if $0 \preceq'' w$.*

Theorem 4.7. *Let Z be an n-dimensional zonotope, or more generally, let Z be the dual of the lattice of regions of an oriented matroid. Then the* **cd***-index $\Psi(Z)$ satisfies the inequalities*

$$\Psi(Z) \succeq'' \Psi(\square_n) \succeq'' 0.$$

That is, for all non-negative integers k and **cd***-monomials v such that $k + \deg(v) = n - 2$, we have*

$$[\mathbf{c}^k\mathbf{d}v]\Psi(Z) - 2 \cdot [\mathbf{c}^{k+2}v]\Psi(Z) \;\geq\; [\mathbf{c}^k\mathbf{d}v]\Psi(\square_n) - 2 \cdot [\mathbf{c}^{k+2}v]\Psi(\square_n) \;\geq\; 0.$$

Observe that Theorem 4.7 gives $\sum_{j=0}^{n-2} F_j = F_n - 1$ inequalities.

The proof of Theorem 4.7 consists of the following lemma and two propositions.

Lemma 4.8. *Let w be a weakly 2-rising* **cd***-polynomial and z a* **cd***-polynomial with non-negative coefficients. Then the* **cd***-polynomial $w \cdot \mathbf{d} \cdot z$ is also weakly 2-rising.*

Proposition 4.9. *The* **cd***-index of the n-dimensional cube $\square_n$ is weakly 2-rising.*

Proof. Proof by induction on n. The induction basis is $n = 0$ which is directly true. The induction step is based on the Purtill recursion for the **cd**-index of the n-dimensional cube; see [23, 30] or [24, Proposition 4.2]:

$$\Psi(\square_{n+1}) \;=\; \Psi(\square_n) \cdot \mathbf{c} + \sum_{i=0}^{n-1} 2^{n-i} \cdot \binom{n}{i} \cdot \Psi(\square_i) \cdot \mathbf{d} \cdot \Psi(\Delta_{n-i-1}).$$

Observe that the sum is weakly 2-rising by Lemma 4.8. However, the term $\Psi(\square_n) \cdot \mathbf{c}$ is not weakly 2-rising. It does satisfy the inequality $2 \cdot [\mathbf{c}^{k+2}v]\ \Psi(\square_n) \cdot \mathbf{c} \leq [\mathbf{c}^k\mathbf{d}v]\ \Psi(\square_n) \cdot \mathbf{c}$ for $0 \leq k \leq n-2$ but not for $k = n-1$. Hence $\Psi(\square_{n+1})$ satisfies the weakly 2-rising inequalities for $k \leq n-2$. To complete the proof it is enough to verify the $k = n-1$ case for $\Psi(\square_{n+1})$. This is straightforward since this amounts to stating that the cube $\square_{n+1}$ has at least four facets, which is true for cubes in dimension two and higher. □

Proposition 4.10. *Let Z be an n-dimensional zonotope and let Z' be the zonotope obtained by taking the Minkowski sum of Z with a line segment in the affine span of Z. Assume that all zonotopes of dimension $n-1$ and less have their* **cd***-indices to be weakly 2-rising. Then the order relation $\Psi(Z) \preceq'' \Psi(Z')$ holds.*

Proof. The proof follows the same outline as the proof of Proposition 4.5. Observe that each term in equation (4.1) is weakly 2-rising by Lemma 4.8. Since the property of being weakly 2-rising is preserved under addition, the result follows. □

We now prove Theorem 4.7.

Proof of Theorem 4.7. The proof is by induction. The induction basis is $n = 0$ which is straightforward. For the induction step assume that every zonotope of dimension k less than n satisfies the inequality $\Psi(\square_k) \preceq'' \Psi(Z)$. Especially, we know that the **cd**-index of a lower dimensional zonotope is weakly 2-rising. Thus by Proposition 4.10 we know that $\Psi(Z) \preceq'' \Psi(Z')$ holds for n-dimensional zonotopes. Now the theorem follows from Propositions 4.9. □

5. Concluding remarks

Stanley conjectured that the **cd**-index over all Gorenstein* lattices of rank $n+1$ is coefficientwise minimized on the n-dimensional simplex [37]. In the light of our results for zonotopes in Section 4, it is natural to conjecture the following strengthening of Stanley's conjecture.

Conjecture 5.1. *Let L be a Gorenstein* lattice of rank $n+1$. Then the* **cd***-index $\Psi(L)$ satisfies the inequality $\Psi(L) \succeq \Psi(\Delta_n)$. That is, for all* **cd***-monomials u and v we have*

$$[u\mathbf{d}v]\Psi(L) - [u\mathbf{c}^2v]\Psi(L) \;\geq\; [u\mathbf{d}v]\Psi(\Delta_n) - [u\mathbf{c}^2v]\Psi(\Delta_n).$$

One possible method to prove this conjecture for polytopes is to use the following proposition and conjecture.

Proposition 5.2. *If the inequality $\Psi(\Delta_n) \preceq' \Psi(P)$ holds for all n-dimensional polytopes P then for all n-dimensional polytopes P we have $\Psi(\Delta_n) \preceq \Psi(P)$.*

Proof. This proof follows the exact same lines as the argument given for the implication $(b) \Longrightarrow (a)$ in the proof of Theorem 3.4. □

Conjecture 5.3. *Let P be an n-dimensional polytope and F a face of dimension k of P. Let G be a k-dimensional face of the simplex Δ_n. Let $F_1, \dots, F_r$ be the facets of P that contain the face F and let $G_1, \dots, G_{n-1-k}$ be the facets of Δ_n containing the face G. Then*

$$\Psi((G_1 \cup \cdots \cup G_{n-1-k})') \preceq' \Psi((F_1 \cup \cdots \cup F_r)').$$

When $k = 0$ this conjecture states that $\Psi(\Delta_n) \preceq' \Psi((F_1 \cup \cdots \cup F_r)')$. Thus Conjecture 5.1 follows from Theorem 3.4, Proposition 5.2 and Conjecture 5.3.

An even more daring conjecture is the following:

Conjecture 5.4. *Let L be a Gorenstein* lattice of rank $n+1$. Then the* **cd***-index $\Psi(L)$ satisfies the inequality*

$$\frac{[u\mathbf{d}v]\Psi(L)}{[u\mathbf{c}^2v]\Psi(L)} \geq \frac{[u\mathbf{d}v]\Psi(\Delta_n)}{[u\mathbf{c}^2v]\Psi(\Delta_n)}.$$

If this conjecture fails, it would be desirable to obtain a lower bound for $[u\mathbf{d}v]\Psi(L)/[u\mathbf{c}^2v]\Psi(L)$. That is, find an estimate for the value

$$c_{u,v} = \inf_L \frac{[u\mathbf{d}v]\Psi(L)}{[u\mathbf{c}^2v]\Psi(L)},$$

where u and v are two **cd**-monomials such that the sum of their degrees is $n-2$ and L ranges over all Gorenstein* lattices of rank $n+1$.

Another question is to determine for which **cd**-monomials u and v is it possible to find two polytopes P and Q such that

$$\begin{aligned} [u]\Psi(P) &< [v]\Psi(P)\,, \\ [v]\Psi(Q) &< [u]\Psi(Q)\,? \end{aligned}$$

For instance, consider $u = \mathbf{d}^2$ and $v = \mathbf{cdc}$. We have that $[\mathbf{d}^2]\Psi(\Delta_4) = 4 < 5 = [\mathbf{cdc}]\Psi(\Delta_4)$ and $[\mathbf{cdc}]\Psi(\Box_4) = 16 < 20 = [\mathbf{d}^2]\Psi(\Box_4)$. However, by considering the known inequalities among the entries of the flag f-vector for four dimensional polytopes [4], one has that $[\mathbf{dc}^2]\Psi(P) \leq [\mathbf{cdc}]\Psi(P)$ and by duality $[\mathbf{c}^2\mathbf{d}]\Psi(P) \leq [\mathbf{cdc}]\Psi(P)$. Observe that these two inequalities do not follow from Theorem 3.4.

Meisinger, Kleinschmidt and Kalai proved that every 9-dimensional rational polytope has a three-dimensional face with less than 78 vertices or 78 facets [29, Theorem 5]. Their proof used the non-negativity of the toric g-vector of rational polytopes and convolutions of these inequalities. They also prove that every 9-dimensional polytope has the three-dimensional simplex as a quotient. It would be very interesting if one could improve their result using convolutions of the linear inequalities in Theorem 3.4.

Let H be a homogeneous **cd**-polynomial H of degree k, that is, we write $H = \sum_w \alpha_w \cdot w$, where the sum is over **cd**-monomials w of degree k. We call H *inequality generating* if the following inequality holds true:

$$\sum_w \alpha_w \cdot [uwv]\Psi(P) \geq 0,$$

for all polytopes P and **cd**-monomials u and v such that the sum of k and the degrees of u and v is the dimension of P. Using this language Theorem 3.4 states that the polynomial $\mathbf{d} - \mathbf{c}^2$ is inequality generating. Stanley's result that the **cd**-index of polytopes are non-negative amounts to saying that each **cd**-monomial is inequality generating. Can we find other examples of inequality generating polynomials? More generally, can we classify the set of inequality generating polynomials?

Now returning to zonotopes, the natural conjecture is the following.

Conjecture 5.5. *Let L be the dual of the lattice of regions of an oriented matroid. Then the* **cd***-index $\Psi(L)$ satisfies the inequality*

$$\frac{[u\mathbf{d}v]\Psi(L)}{[u\mathbf{c}^2v]\Psi(L)} \geq \frac{[u\mathbf{d}v]\Psi(\Box_n)}{[u\mathbf{c}^2v]\Psi(\Box_n)},$$

where $\Box_n$ is the n-dimensional cube.

Other linear inequalities for the flag f-vector of zonotopes have been obtained by Varchenko and Liu; see [26, 28, 41].

It would interesting to continue the work of Readdy [31], who studied the question of determining the largest coefficient of the **ab**-index of certain polytopes. Thus to continue Corollary 3.5 it would be interesting to determine which coefficient of the **cd**-index is the largest for the simplex and the cube.

It can be shown that the rising property is preserved under the two linear operators Pyr, Prism corresponding to the geometric pyramid and prism operations; see [24]. Is the rising property preserved under the two bilinear operators $M(\cdot,\cdot), N(\cdot,\cdot)$ that occur in the work of [22, 24]? Moreover, in [22] it is proved that three polytopes P, Q and R satisfy

$$\Psi(P \circledvee (Q \times R)) \leq \Psi((P \circledvee Q) \times R),$$

where $\circledvee$ denotes the free join of polytopes and $\times$ the Cartesian product. Can this inequality be sharpened by replacing $\leq$ with $\preceq$?

Acknowledgements

I would like to thank the MIT Mathematics Department for their kind support where this research was done while the author was a Visiting Scholar. The author also thanks Margaret Readdy for many helpful discussions and Marge Bayer for suggesting references.

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