A SIMPLE BIJECTION BETWEEN LECTURE HALL PARTITIONS AND PARTITIONS INTO ODD INTEGERS

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ABSTRACT. We give a simple bijection between lecture hall partitions, having at most n parts, and integer partitions, using only odd numbers less than or equal to $2n-1$. This reproves the lecture hall theorem of Bousquet-Mélou and Eriksson. We also use this bijection to find a recursion for the generating functions of the number of lecture hall partitions which use only the k back rows of a lecture hall of length n and some other special cases.

RÉSUMÉ. Nous considérons une simple bijection entre les partitions d'un amphithéatre (lecture hall partitions), ayant au maximum n éléments, et les partitions d'entiers, utilisant seulement les nombres impairs inférieurs ou égaux a $2n-1$. Ceci redemontre le théorème de l'amphithéatre de Bousquet-Mélou et Eriksson. Nous utilisons également cette bijection pour trouver une récursivité dans les fonctions génératrices du nombre de partitions de l'amphithéatre qui utilisent seulement les k derniers rangs de l'amphithéatre de longueur n, et d'autres cas particuliers.

1. INTRODUCTION

Lecture hall partitions were presented for the first time by Bousquet-Mélou and Eriksson in 1997 [2]. Their definition is the following.

Definition 1.1. *A lecture hall partition into n parts is a partition* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *such that*

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n}.$$

The name stems from the interpretation of the partition as a design for lecture halls (see Figure 1). The conditions imposed on the parts are sufficient to ensure that each student can see the teacher. We will henceforth use the name **rows** for the parts.

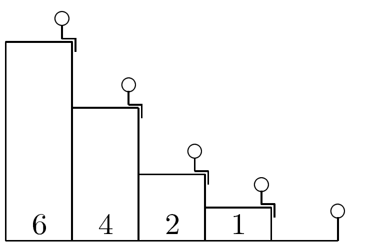

FIGURE 1. Students with teacher in a lecture hall. Thanks to the clever design, any sleeping student will be spotted by the teacher

There is a remarkable connection between lecture hall partitions into n parts and integer partitions with odd parts less than $2n$.

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Theorem 1.2 (Bousquet-Mélou and Eriksson [2]). *For fixed n, the generating function for the number of lecture hall partitions of N into n parts, $LH(N,n)$ is given by*

$$\sum_{N=0}^{\infty} LH(N,n)\, q^N = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}},$$

that is it equals the generating function of integer partitions into odd parts less than $2n$.

This theorem may be seen as a finite analogue of Euler's famous theorem stating that the number of partitions of k with odd parts equals the number of partitions of k with distinct parts; indeed, if we let n approach infinity, Theorem 1.2 becomes Euler's theorem.

Some notation: We will denote the set of lecture hall partitions into n parts $\mathcal{L}_n$ and the set of integer partitions into odd parts less than or equal to $2n-1$ will be denoted $\mathcal{O}_n$. We will prese= nt functions $\Phi_n : \mathcal{L}_n \to \mathcal{O}_n$ and $\Psi_n : \mathcal{O}_n \to \mathcal{L}_n$ such that $\Phi_n \circ \Psi_n = id_{\mathcal{O}_n}$ and $\Psi_n \circ \Phi_n = id_{\mathcal{L}_n}$.

Over the past years, several proofs of this theorem have been presented. The first paper [2] gave two proofs — one via Bott's formula for the Poincaré series of the affine Coxeter group $\widetilde{C}_n$ and one direct proof. The latter proof actually proved a refined version of the formula, keeping track of odd and even weights of the lecture hall partition as well.

In the sequel [3], Bousquet-Mélou and Eriksson introduced generalised lecture hall partitions as partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$0 \le \frac{\lambda_1}{a_1} \le \frac{\lambda_2}{a_2} \le \ldots \le \frac{\lambda_n}{a_n}$$

for some non-decreasing sequence of positive integers $a = (a_1, a_2, \ldots, a_n)$. The set of such sequences is denoted $\mathcal{L}_a$. Some results are derived for these partitions, but quite a few conjectures are also presented.

In a third paper [4], these authors provide another refinement of the theorem and two more proofs. The second one is claimed to be the first truely bijective proof of the lecture hall theorem. In fact, the proof is a bijection between lecture hall partitions with n parts and integer partitions with distinct parts between 1 and n and arbitrarily many parts between $n+1$ and $2n$. This is of course easily extendible to a bijection between lecture hall partitions and partitions with odd parts.

Apart from the original authors', there have been two contributions. The first one is by Andrews [1], who proves the theorem using MacMahon's 'partition analysis'. The latest contribution to this subject has been made by Yee [5], who also presents a combinatorial bijection.

With all these proofs, is there any *raison d'être* for another one? We claim, of course, that there is. The first reason is that the functions Φ_n and Ψ_n are, in all essentials, *independent* of n. For instance, the partition $\mu = (5,3,3)$ will give the lecture hall partition $\Psi_n(\mu) = (0, \ldots, 0, 4, 7)$ for any $n \ge 3$ (of course, the number of initial zeroes will differ). This contrasts with both previous bijections, although Yee's bijection has the property that for each μ there exists an N such that $\Psi_n(\mu)$ is independent of n if $n \ge N$.

Another reason to present this new proof is that the bijection gives new information on generalised lecture hall partitions. In fact, we are able to present a recursion for the generating function for the number = of lecture hall partitions of n with at least k empty front rows. We also present a recursion for the generating functions in the special case where the sequence $a = (a_1, a_2, \ldots, a_n)$ is increasing and $a_{n-2k} - a_{n-2k-1} = 1$ for $k \ge 0$.

We will start this paper by presenting the bijection in an intuitively clear way. We then proceed to state some technical definitions, which are followed by a proof that the map is bijective. Finally, we apply the bijection to the generalised lecture hall partitions as described above.

2. The bijection

We will describe the bijection by presenting the function Ψ_n that given a partition into odd integers will produce a lecture hall partition. This is done by building a lecture hall using components, which are determined by the odd number partition.

Here follows a short description of the steps involved.

- To each odd number n we associate a building block, B_n, consisting of n bricks. These are the basic parts of our lecture hall.
- The building blocks will be grouped into modules, M_k, according to rules defined below.
- Starting with the "smallest" module, we now add the modules, one at a time. Each time we add a module, we get a lecture hall partition, LH_k. When all modules have been added, we are done.

2.1. **The building blocks.** To each odd number n, we associate a building block of n bricks, B_n. First, put $\frac{n+1}{2}$ bricks at positions $(i, \frac{n+1}{2} - i + 1), 1 \le i \le \frac{n+1}{2}$, and then the remaining $\frac{n-1}{2}$ bricks at positions $(i, \frac{n-1}{2} - i + 1), 1 \le i \le \frac{n-1}{2}$. We thus obtain building blocks as in Figure 2.

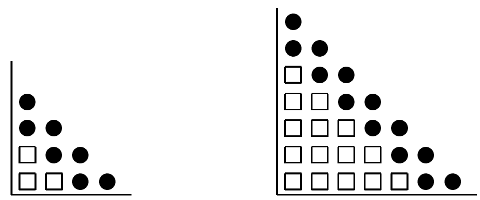

Figure 2. The building blocks B_7 and B_{13}. The dots are bricks and the boxes are empty spaces

2.2. **The modules.** First, we need to explain which building blocks go into which module. We will then show how to build the modules.

Let O be a partition into odd parts and let $2n-1$ be the largest part of O. We associate with O a matrix $A = (a_{i,j})_{i,j}$ with n rows and $\lceil n/2 \rceil$ columns. The top right half of this matrix is easy to fill: $a_{i,j} = 0$ if $2j > i + 1$. We shall describe below how to construct the other half. In the resulting matrix, the sum of the entries in the ith row will be the number of parts of O equal to $2(n-i)+1$. For $1 \le k \le \lceil n/2 \rceil$, we shall define the module M_k as the sum of the blocks encoded by the kth column of A: more precisely, $M_k = \sum_i a_{i,k} B_{2(n-i)+1}$.

We define the **sequence** $\text{seq}(M_k)$ of a module M_k as $\{a_{i+2(k-1),k}\}_{i=1}^{n-2(k-1)}$. Order the parts of O in decreasing order. A part $2l - 1$ should go into row $n - l + 1$. We do this by increasing one element in the designated row of the first part in the decreasing order. In doing this, we should choose the rightmost column such that the sequences $\text{seq}(M_k)$ are always lexocographically ordered. Then we remove the first element and iterate. An example will clarify this explanation.

Example 2.1. *The partition* $(17, 11, 11, 9, 7, 3, 3, 1)$ *corresponds to the building blocks* $\{B_{17}, B_{11}, B_{11}, B_9, B_7, B_3, B_3, B_1\}$*, which we will now divide into modules.* B_{17} *should go into row* $n - l + 1 = 9 - 9 + 1 = 1$ *and column* 1*, since it is the leftmost column allowed. Next, we have* B_{11}*, which goes into row* $9 - 6 + 1 = 4$*. We may add one to the element in column*

2, *since we then get* $seq(M_2) = (0, 1, 0 \ldots) \leq (1, 0, \ldots) = seq(M_1)$. *Column* 3, *however, is forbidden, so column* 2 *is chosen. The second* B_{11} *goes into the same position.*

As for the next block, B_9, *putting it in column* 3 *would give* $seq(M_3) > seq(M_2)$, *but column 2 will do. We then continue in this fashion until we get the matrix in Figure 3. From this we conclude that* $M_1 = B_{17}$, $M_2 = B_{11} + B_{11} + B_9$, $M_3 = B_7 + B_3 + B_1$ *and* $M_4 = B_3$.

	1	2	3	4	5
17	1				
15	0				
13	0	0			
11	0	2			
9	0	1	0		
7	0	0	1		
5	0	0	0	0	
3	0	0	1	1	
1	0	0	1	0	0

FIGURE 3. Putting each building block into the right module. Observe that the sequences are in lexicographic order.

Given the building blocks of a module, assembling the modules is easy. Just put the building block on top of each other. The empty spaces are heavier than the bricks and will sink to the bottom. This is shown in Figure 4. Since this operation has all the nice properties like associativity, commutativity and such, we will use the + sign for this operation.

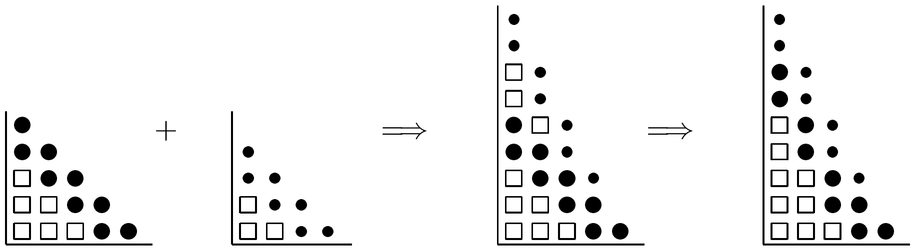

FIGURE 4. Adding B_9 and B_7

2.3. **The lecture hall partitions.** We assemble the modules, one by one, starting with the module with the highest index, k. The lecture hall partitions obtained from modules M_i to M_k, $i \leq k$, will be named LH_i. This will be denoted $LH_i = LH_{i+1} \oplus M_i$.

Given LH_{i+1} and M_i, we construct LH_i as follows. First insert LH_{i+1} into M_i from below. If any bricks in LH_{i+1} collide with bricks in M_i, the bricks in M_i will slide up. Then we push from the right, until there are no holes in the lecture hall partition.

Example 2.2. *An example of such an assembly can be viewed in Figure 5. The spaces in the module are no longer drawn. We are given the set* $\{B_{11}, B_5, B_5, B_1\}$ *of building blocks and get the modules* $M_1 = B_{11}, M_2 = B_5 + B_5$ *and* $M_3 = B_1$. *We get* $LH_3 = \{0, 0, 0, 0, 0, 0\} \oplus M_3 = \{0, 0, 0, 0, 0, 1\}$ *and* $LH_=2 = LH_3 \oplus M_2 = \{0, 0, 0, 1, 4, 6\}$. *These assemblies were quite trivial, but here comes the tricky part, which can be viewed in the figure. We start with* LH_2 *(black) and* $M_1 = B_{11}$ *(white) (a).* LH_2 *is pushed into* M_1 *(b) and then we compress from the right to obtain (c).*

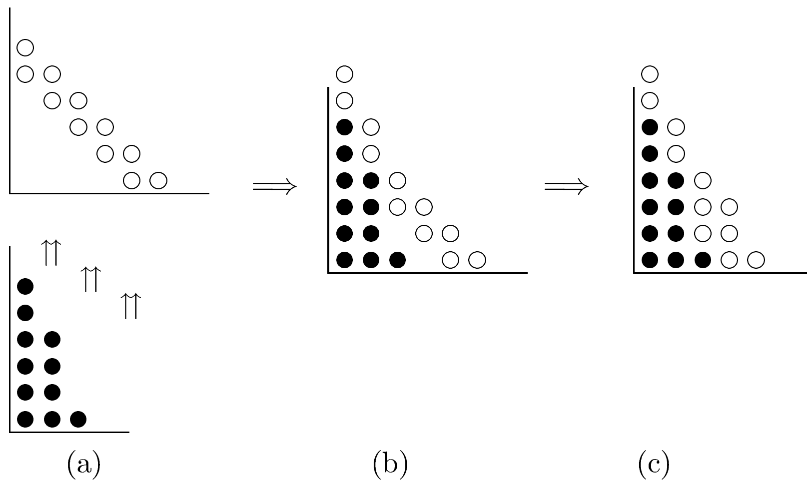

FIGURE 5. Assembling $M_1 = B_{11}$ (white) with $LH_2 = (B_5 + B_5) \oplus B_1$ (black).

3. Mathematical tools for lecture hall partitions

Some useful measures on the building blocks, as well as on the modules and the lecture halls, are the height, the ceiling and the range.

Definition 3.1. *The* **height** *of an object (building block, module, lecture hall) at a certain row, is the y-coordinate of the highest brick in that row. The* **ceiling** *of an object at a certain row is one less than the y-coordinate of the lowest brick in that row. If the row contains no bricks, both maps are zero. These are written as $h(A, k)$ and $c(A, k)$, respectively, for an object A and row k.*

Observation 3.2. *We observe that*

$$h(B_n, k) = \begin{cases} \frac{n+3}{2} - k, & 1 \le k \le \frac{n+3}{2}; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$c(B_n, k) = \begin{cases} h(B_n, k) - 2, & 1 \le k \le \frac{n-1}{2}; \\ 0, & \textit{otherwise} \end{cases}$$

Definition 3.3. *The* **range** *of an object A and a row k is given by*

$$r(A, k) = \frac{h(A, k)}{h(A, k) - h(A, k+1)} + k,$$

provided that $h(A, k) > 0$, if A is a building block, and $h(A, k+1) > 0$ otherwise. For larger k, $h(A, k) = 0$. The range of an object A is the maximal range over all rows.

The range of a lecture hall partition tells at which position the teacher must be placed if the partition should be a valid lecture hall partition (the teacher must stand to the right of the range).

Example 3.4. *In Figure 5, the ranges are $r(LH, 1) = 5, r(LH, 2) = 5$, $r(LH, 3) = 7, r(LH, 4) = 5.5$ and $r(LH, 5) == 0$. We then get $r(LH) = 7$. The teacher must be placed at $x = 7$ to be visible to all students.*

For modules, we can easily calculate the height and the ceiling:

$$h(M,k)=\sum_{j=1}^{s} a_j h(B_{n_j},k)$$

and

$$c(M,k)=\sum_{j=1}^{s} a_j c(B_{n_j},k)$$

for all rows k.

We find that the range of a row in a module is the arithmetical mean of the ranges of the building blocks in the module, except for possibly some of the smallest ones.

Lemma 3.5. *Consider a module*

$$M=a_1B_{n_1}+a_2B_{n_2}+\ldots+a_sB_{n_s},$$

where $n_1>n_2<\ldots n_s$, and let $A_k=\{i:n_i\geq k\}$. Then, for $k<\frac{n_1+1}{2}$, the range is given by

$$r(M,k)=\frac{\sum_{A_{2k-1}} a_i r(B_{n_i},k)}{\sum_{A_{2k-1}} a_i}.$$

Proof. This follows from Observation 3.2 and the definition of range. We have

$$r(M,k)=\frac{h(M,k)}{h(M,k)-h(M,k+1)}+k$$

$$=\frac{\sum_{A_{2k-3}} a_i\left(\frac{n_i+3}{2}-k\right)}{\sum_{A_{2k-3}} a_i\left(\frac{n_i+3}{2}-k\right)-\sum_{A_{2k-1}} a_i\left(\frac{n_i+3}{2}-k-1\right)}+k$$

$$=\frac{\sum_{A_{2k-1}} a_i\left(\frac{n_i+3}{2}-k\right)}{\sum_{A_{2k-1}} a_i}+k$$

$$=\frac{\sum_{A_{2k-1}} a_i\left(\frac{n_i+3}{2}\right)}{\sum_{A_{2k-1}} a_i}=\frac{\sum_{A_{2k-1}} a_i r(B_{n_i},k)}{\sum_{A_{2k-1}} = a_i}.$$

□

We will call a module M_k **simple** if seq(M_k) contains no non-zero elements that are not adjacent. Otherwise, the module is **complex**. In Example 2.1, M_1, M_2 and M_4 are simple, but M_3 is complex.

Lemma 3.6. *For each lecture hall partition LH and row k such that the height $h(LH,k)$ is positive, there exists a unique set of numbers $a>0$, $b>0$, $n=2m+1\geq 1$ such that height of the module $M=aB_n+bB_{n-2}$ equals the height of LH at rows k and $k+1$. For a fixed $n=2m+1\geq 1$, if there exists number $a\geq 0$, $b>0$, these are unique as well.*

Proof. We have three unknowns but only two equations:

$$a\left(\frac{n+3}{2}-k\right)+b\left(\frac{n+1}{2}-k\right)=h(LH,k)$$

and

$$a\left(\frac{n+3}{2}-k-1\right)+b\left(\frac{n+1}{2}-k-1\right)=h(LH,k+1).$$

This can, however, be rewritten into

$$a+b=h(LH,k)-h(LH,k+1)$$

and

$$(h(LH,k)-h(LH,k+1))\left(\frac{n+3}{2}-k\right)-b=h(LH,k).$$

We may vary b freely between 0 and $h(LH,k)-h(LH,k+1)-1$, so we see that there is exactly one solution to the last equation. This also gives a uniquely.

For fixed n, it is still clear that if such numbers exist, they are unique. □

This paves the way for the following definition.

Definition 3.7. *Given a lecture hall partition LH and a row k, we define the* **characteristic triple** (a,b,n)*, which is a set of numbers such that $a>0, b\geq 0, n=2m+1\geq 1$ and such that the height of the module $M=aB_n+bB_{n-2}$ equals the heights of LH at rows k and $k+1$. The* **relaxed characteristic triple** (a,b,n) *is defined similarily, except for that we allow $a=0$.*

Definition 3.8. *We introduce the* **step operator** $^+$ *that operates on characteristic triples. We have $(a,b,n)^+=(a+1,b-1,n)$ if $b\geq 1$ and $(a,0,n)^+=(1,a-1,n+2)$. If a row k has characteristic triple (a,b,n), applying the step operator corresponds to increasing the height of rows k and $k+1$ by one.*

Definition 3.9. *The* **top range row** *of a lecture hall partition LH is the rightmost row k such that its characteristic triple (a,b,n) has $n=2\lceil r(LH)\rceil-3$, maximal a, and maximal b, given the value of a.*

Example 3.10. *The top range row of LH in Figure 5 is of course 3, since it has the highest range. The characteristic triple associated with row 3 is* $(1,0,11)$*.*

Observation 3.11. *Assume that we are constructing a lecture hall partition using the procedure described above. The height of the rows in LH_i is then given as follows.*

- *If $h(LH_{i+1},k)>c(M_i,k)$, then $h(LH_i,k)=h(M_i,k)-c(M_i,k)+h(LH_{i+1},k)$.*
- *If $h(LH_{i+1},k)\leq c(M_i,k)$ and either $k\in\{1,2\}$ or $h(LH_{i+1},k-j)>c(M_i,k-j)$ for $j\in\{1,2\}$, then $h(LH_i,k)=h(M_i,k)$.*
- *Otherwise, $h(LH_i,k)=h(LH_{i+1},k-2)$.*

Definition 3.12. *If $h(LH_{i+1},k)>c(M_i,k)$ for any row k, we say that the M_i is* **disturbed**.

Remark 3.13. *Each time we add a module, we increase the number of rows by two, except for the case with only B_1s in the module. This fact will be used in the last section to calculate the number of lecture hall partitions with the k front rows empty.*

4. Getting back

We have described how to generate a well-defined partition from a partition into odd integers. What remains is to verify that we indeed obtain a lecture hall partition, and that two different partitions of odd integers do not produce the same lecture hall partition. The first question will be answered implicitly, since we will show that there is a limit on the range of the partition obtained, thus implying that the partition really fulfills the conditions imposed on a lecture hall partition. The other will be adressed by producing a way to obtain the partition into odd numbers from any lecture hall partition. Then the LHP generator is clearly bijective.

We will show that given any partition $LH_1 = \Psi_n(O)$, we can reveal the contents of M_1. We can then remove M_1 and iterate, to find the contents of all modules. This will give the odd part partition $O = \Phi_n(LH_1)$ and also show that LH_1 is indeed a lecture hall partition, since the range is limited from above by $n + 1$.

We must, at this stage, warn sensitive readers that the next page contains some ugly mathematics. The beauty of this bijection lies in its simple construction, not in the ease with which we show that it is bijective.

Theorem 4.1. *Given a partition $LH_1 = \Psi_n(O)$, where $O \in \mathcal{O}_n$, we are able to read off M_1 and remove it. The module M_1 is read as follows. First, find the top range row k and let (a, b, n) be its characteristic triple. Set $M_1 = aB_n + bB_{n-2}$. We then continue with row $k - 2l$, for $l = 1, 2, \dots$, in that order. In each step, we determine the smallest index p of any building block in M_1. Then, if there exists a relaxed characteristic triple $(a_l, b_l, p - 2)$ or (a_l, b_l, p) on row $k - 2l$ using heights $h(LH, k - 2l) - h(M_1, k - 2l)$ and $h(LH, k - 2l + 1) - h(M_1, k - 2l + 1)$, then we add $a_lB_{p-2} + b_lB_{p-4}$ (or $a_lB_p + b_lB_{p-2}$, respectively), to M_1 and continue. Otherwise, we are done and we can remove M_1. As a special rule when we get to the leftmost rows, we should note that we will never accept the relaxed characteristic triple $(0, s, 3)$ at row 0 and we also note that if $LH = (0, \dots, 0, m)$, this corresponds to the module mB_1.*

Proof. This theorem will be shown by induction. We will always assume that M_2 can be read from LH_2 as described, and look at how LH_1 appears. It should be noted that since $\mathrm{seq}(M_1) > \mathrm{seq}(M_2)$, $r(LH_1) \geq r(LH_2) + 2$.

It is easy to see that the first module added will be a lecture hall partition with only one well-defined range. We then find the characteristic triple and are done.

Now assume that LH_2 is non-empty. We will, in turn, cover the four possible cases: M_1 is simple or complex, disturbed or not disturbed.

- M_1 **is simple and undisturbed:** It is trivial to see that $h(LH_1, k) = h(M_1, k)$ for $k = 1, 2$ and $h(LH_1, k) = h(LH_2, k - 2)$ otherwise. Since M_1 is simple, we will find the top range at row 1, and the characteristic triple will tell the tale on M_1.
- M_1 **is complex and undisturbed:** Again, the heights are distributed as in the simple case. However, the top range is no longer found at row 1. Rather, if the top range of LH_2 is found at row $k - 2$, we now find it at row k. Since we got valid characteristic triples for LH_2, we also get them for LH_1. Finally, we reach $k = 1$ or $k = 0$, and we then read the same heights as in M_1. These are also valid and we get the right M_1.
- M_1 **is simple and disturbed:** We assume that the first k' rows of M_1 are disturbed (there can not be any other disturbed rows, since this would give too large range for LH_2). We then have $h(LH_1, k) = h(LH_2, k) + h(M_1, k) - c(M_1, k) = h(LH_2, k) + 2(a + b)$ for $k \leq k'$, $h(LH_1, k) = h(M_1, k)$ for $k = k' + 1, k' + 2$, and $h(LH_1, k) = h(LH_2, k - 2)$ otherwise. We need to show that the ranges of rows $k < k'$ are to

small to affect our construction. We start by showing that the top range is given by row $k'+1$ or $k'+2$.

It is clear that the top range row can not be found among the rows k such that $k > k'+2$. We must show that it can not be found among the first k' rows either. Let the characteristic triple of row $k'+1$ be (a, b, n). We then know that the characteristic triple of a row $k < k'$ in LH_2 either has the form $(a', b', n-l)$ for $l > 4$ or $(a', b', n-4)$ with $a' < a$ (otherwise, the modules would not be sorted alphabetically). We also know that $a'+b' > a+b$ (or row k would not be disturbed). Now, the corresponding row in LH_1 has height that is $2(a+b)$ larger. We apply the step operator $2(a+b)$ times to the characteristic triple. Since $a'+b' > a+b$, we never get a characteristic triple that is greater than (a, b, n).

We have shown that the top range can be found only on rows $k'+1$ and $k'+2$. We must now show that we can not continue reading another characteristic triple. We assume that the top range row is $k'+1$ and that the characteristic triple of $k'-1$ in LH_2 is $(a', b', n-4)$, with $a' < a$ and $d = a'+b'-(a+b) > 0$. If we apply the step operator $2(a+b)$ times, we get the characteristic triple $(a'-2d, b'+2d, n)$ or, if d is sufficiently high, $(a'-2d+a'+b', b'+2d-a'-b', n-2)$ or $(a'+2(a+b), b'-2(a+b), n-4)$. From these triples, we must break loose a triple (a, b, n). If we do this to the first, we get $(a'-2d, b'+2d, n) = (a'-2d, b+2d+a-a', n) + (d, 0, n-2) = (a, b, n) + (d-(a-a'), a-a', n-6)$ (in the last equality, we have increased the first term by $2d + (a-a')$ and reduced the second term by an equal amount). We see that the characteristic triple $(d-(a-a'), a-a', n-6)$ is not valid, since $a-a' > 0$. Similar calculations show that the other cases do not produce valid triples either.

Had the top range been row $k'+2$, we should have look at the characteristic triple of k' instead. But then we still have that the heights of rows k' and $k'+1$ in LH_1 are given by the corresponding height in LH_2 increased by $2(a+b)$, and the result follows from the analysis above.

M_1 is complex and disturbed: As for the undisturbed case, we are able to read off all valid building blocks. The only case where we can go wrong is when we enter the disturbed zone. However, since we then have removed (in thought) from the heights all building blocks that belong to M_1, we have a case similar to the simple case. From that analysis, we find that we can not continue to read valid characteristic triples.

□

Example 4.2. *Let us exemplify this by looking at the lecture hall partition* $(0, 1, 3, 4, 6, 8)$ *found in Figure 5. According to the theorem, we should start by finding the top range row and its characteristic triple. We lready now that this is row* 3 *and triple* $(1, 0, 11)$*. We thus set* $M_1 = B_{11}$*. We then look at row* $3-2=1$*. We should use the heights* $h(LH, 1) - h(M_1, 1) = 8-6 = 2$ *and* $h(LH, 2) - h(M_1, 2) = 6-5 = 1$*. The corresponding characteristic triple would be* $(1, 0, 3)$ *or, if we allow the relaxed version,* $(0, 1, 5)$*. In neither case, the last coordinate is* 11 *or* $11-2=9$*, which we need in order to add more building blocks to* M_1*. Since we found something that we wished to add, but could not,* M_1 *will not contain any more building blocks, and we can remove* M_1 *from* LH*.*

We now get $LH_2 = (0, 0, 0, 1, 4, 6)$*. It is easy to see that the top range row is* 1 *and that the corresponding chracteristic triple is* $(2, 0, 5)$*. We then set* $M_2 = B_5 + B_5$*. Since we are already at row* 1*, we can not continue to the left and are done. Removing* M_2 *will give* $LH_3 = (0, 0, 0, 0, 0, 1)$*, and it is not hard to see that this gives* $M_3 = B_1$*. From this we conclude that* $\Phi_n(0, 1, 3, 4, 6, 8) = (11, 5, 5, 1)$ *for* $n \geq 6$*.*

5. The distant teacher and other generalisations

We now turn to generalised lecture hall partitions. For starters, we take a closer look at real world lecture halls. Usually, there is some distance between the teacher and the students. The following theorem will give the generating function for this case. The function will be defined recursively.

Theorem 5.1. *Let $P_{LH}(N, n, k)$ be the number of ways to partition N into $\lambda = [\lambda_1, \ldots, \lambda_k)$ such that we have*

$$0 \leq \frac{\lambda_1}{n-k+1} \leq \frac{\lambda_2}{n-k+2} \leq \ldots \leq \frac{\lambda_k}{n}.$$

Then the generating function is

$$\sum_N P_{LH}(N, n, k)\; q^N = Q(n, k)$$

where $Q(n, k)$ is given recursively by

$$Q(n, k) = Q(n-1, k) + \frac{q^{2n-1}}{(1-q^{2n-1})(1-q^{2n-3})} Q(n-2, k-2), \quad \text{for } 2 \leq k < n,$$

$$Q(n, 0) = 1,\; Q(n, 1) = \frac{1}{1-q},\; Q(n, n) = \frac{1}{\prod_{i=1}^{n}(1-q^{2i-1})}.$$

Proof. First look a the boundary conditions. For $k = n$ we have the lecture hall theorem and for $k = 1$ and $k = 0$, the results are trivial.

Let us now take a closer look at the proof of the bijection above. For each module we create, we will use two more rows in the lecture hall (unless the module contains only ones). Thus, if we wish to partition λ, we can either use the number $2n-1$, thereby creating a new module, or not use the number $2n-1$. In the first case, we add the factor $\frac{q^{2n-1}}{(1-q^{2n-1})(1-q^{2n-3})}$ to acknowledge the fact that $2n-1$ is used at least once and that $2n-3$ may be used freely, and to this we multiply $Q(n-2, k-2)$ for the rest of the modules, which can only use number strictly less then $2n-3$ and may only use the remaining $k-2$ rows. On the other hand, not using $2n-1$ will not reduce the available number of rows.

□

Using the same line of thinking, the following theorem follows naturally.

Theorem 5.2. *Let a be an increasing sequence of n positive integers such that $a_{n-2i} - a_{n-2i-1} = 1, 0 \leq i < \lfloor \frac{n}{2} \rfloor$ and $P_{LH}(N, a)$ be the number of ways to partition N into $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that we have*

$$0 \leq \frac{\lambda_1}{a_1} \leq \ldots \leq \frac{\lambda_n}{a_n}.$$

Then the generating function is

$$\sum_N P_{LH}(N, a)\; q^N = Q(a_n, a)$$

where $Q(m, a)$ is given recursively by

$$Q(m, a) = Q(m-1, a) + \frac{q^{2m-1} Q(m-(a_n - a_{n-2}), (a_1, \ldots, a_{n-2}))}{(1-q^{2m-1})(1-q^{2m-3})}, \quad \text{for } n > 1,\; m > 1,$$

$$Q(m, ()) = 1,\; Q(m, (a_1)) = \frac{1}{1-q},$$

$$Q(1, a) = \frac{1}{1-q},\; Q(2, a) = \frac{1}{(1-q)(1-q^3)}.$$

It should be noted that the building blocks and the way the modules are put together change somewhat in this case. An example will clarify this better than any formal definition.

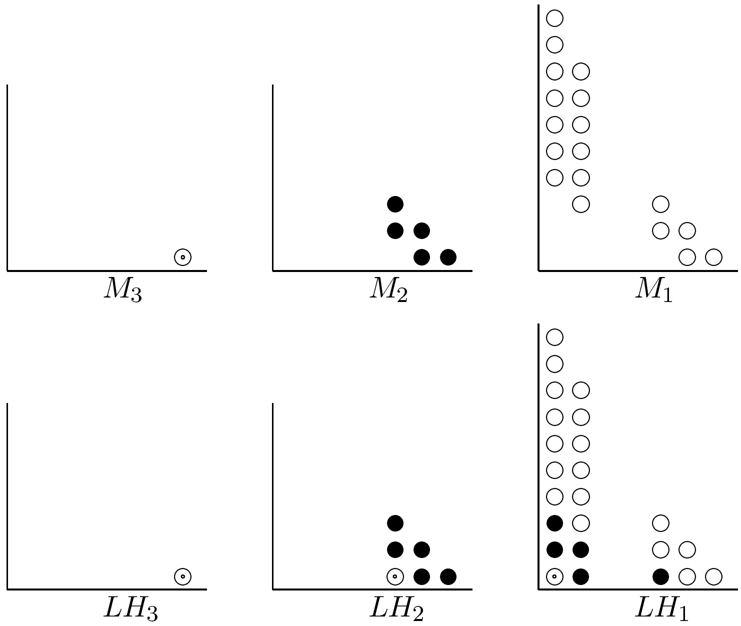

FIGURE 6. We have marked the bricks from M_3 with a dot and the bricks from M_2 are marked black, to make it easier to follow what is happening. We see that $M_3 = B_1$ and $M_2 = B_5$ have their usual appearances, although they are shifted somewhat to the right, but $M_1 = B_{13} + B_5$ looks differently from the standard appearance, since some rows are kept empty.

Example 5.3. *We wish to put* $\{13, 5, 5, 1\}$ *in an* $(1, 2, 3, 6, 7)$*-lecture hall partition. This gives the matrix*

$$\begin{array}{r|ccc|} & 1 & 2 & 3 \\ 13 & 1 & & \\ 11 & 0 & & \\ 9 & 0 & & \\ 7 & 0 & & \\ 5 & 1 & 1 & \\ 3 & 0 & 0 & \\ 1 & 0 & 0 & 1 \end{array}$$

The second module now starts 4 *steps lower than the first module in the matrix. The reason is that* $a_n - a_{n-2} = a_5 - a_3 = 7 - 3 = 4$ *and not* 2*, as we get in the standard case. However, we still demand that the sequences are ordered lexicographically.*

We get $M_1 = B_{13} + B_5, M_2 = B_5$ *and* $M_3 = B_1$*. The modules and the lecture hall partitions can be found in Figure 6. In essence, we may say that the we get the same modules as before, but those bricks that occupy rows that should be empty are moved to the closest rows to the left. By this procedure, the building blocks will not occupy any rows that must be empty. We also build* LH_k *at row* a_{n-2k} *instead of row* a_n*. It is then slided to the left before we add the next module.*

Discussion

In [3], Bousquet-Mélou and Eriksson conjecture that all sequences $a = (a_1, \ldots, a_n)$ that give generating functions of the form

$$\frac{1}{(1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n})}$$

have $a_1 | a_i$ for all a_i. The results obtained in the previous section allows for calculations that could falsify their conjecture, but so far, no counterexample has been found. In the same article, several other conjectures were made, and we intend to take a closer look at them to see if we can use this new bijection to verify or falsify them.

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