THE NUMBER OF REPRESENTATIONS OF A NUMBER BY VARIOUS FORMS

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Abstract. We find formulae for the number of representations of the integer n as, for example, the sum of two triangles and two squares, or of four triangles, in terms of divisor functions. Indeed, we find sixteen formulae of this type, a few of which are known, the remainder apparently new.

Résumé. Nous trouvons des formules pour le nombre des représentations d'un entier n comme, par example, somme de deux nombres triangulaires et de deux carrés, ou de quatres nombres triangulaires, en termes des fonctions des diviseurs de n. En effet, nous trouvons seize formules de ce type, dont quelques unes sont connues, les autres apparemment nouvelles.

1. Introduction

There are several classical results which give the number of representations of a number by a quadratic form in terms of a divisor function. The object of this note is to consider four such results and from them to derive many more of the same sort.

With $d_{r,m}(n)$ denoting the number of divisors d of n with $d \equiv r (\mathrm{mod}\ m)$ amd $\sigma(n)$ the sum of the divisors of n, the results we consider are the following. Proofs of all four can be found in [2].

Theorem 1. *(Jacobi, 1828). The number of representations of $n \geq 1$ as the sum of two squares is*

$$r\{\square + \square\}(n) = 4\left(d_{1,4}(n) - d_{3,4}(n)\right). \tag{J1}$$

Theorem 2. *(Dirichlet, 1840). The number of representations of $n \geq 1$ as the sum of a square and twice a square is*

$$r\{\square + 2\square\}(n) = 2\left(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)\right). \tag{D}$$

Theorem 3. *(L.Lorenz, 1871). The number of representations of $n \geq 1$ as the sum of a square and three times a square is*

$$r\{\square + 3\square\}(n) = 2\left(d_{1,3}(n) - d_{2,3}(n)\right) + 4\left(d_{4,12}(n) - d_{8,12}(n)\right). \tag{L}$$

Theorem 4. *(Jacobi, 1829). The number of representations of $n \geq 1$ as the sum of four squares is*

$$r\{\square + \square + \square + \square\}(n) = 8 \sum_{d|n, 4\nmid d} d = 8\left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right)\right). \tag{J2}$$

We shall prove the following sixteen results of the same sort.

$$(1.1) \qquad r\{\Delta+\Delta\}(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1),$$
$$(1.2) \qquad r\{\Box+2\Delta\}(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1),$$
$$(1.3) \qquad r\{2\Box+\Delta\}(n) = d_{1,4}(8n+1) - d_{3,4}(8n+1),$$
$$(1.4) \qquad r\{\Delta+4\Delta\}(n) = \frac{1}{2}\left(d_{1,4}(8n+5) - d_{3,4}(8n+5)\right),$$
$$(1.5) \qquad r\{\Delta+2\Delta\}(n) = \frac{1}{2}\left(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3)\right),$$
$$(1.6) \qquad r\{\Box+\Delta\}(n) = d_{1,8}(8n+1) + d_{3,8}(8n+1) - d_{5,8}(8n+1) - d_{7,8}(8n+1),$$
$$(1.7) \qquad r\{\Box+4\Delta\}(n) = d_{1,8}(2n+1) + d_{3,8}(2n+1) - d_{5,8}(2n+1) - d_{7,8}(2n+1),$$
$$(1.8) \qquad r\{\Delta+3\Delta\}(n) = d_{1,3}(2n+1) - d_{2,3}(2n+1),$$
$$(1.9) \qquad r\{3\Box+2\Delta\}(n) = d_{1,3}(4n+1) - d_{2,3}(4n+1),$$
$$(1.10) \qquad r\{\Box+6\Delta\}(n) = d_{1,3}(4n+3) - d_{2,3}(4n+3),$$
$$(1.11) \qquad r\{6\Box+\Delta\}(n) = d_{1,3}(8n+1) - d_{2,3}(8n+1),$$
$$(1.12) \qquad r\{\Delta+12\Delta\}(n) = \frac{1}{2}\left(d_{1,3}(8n+13) - d_{2,3}(8n+13)\right),$$
$$(1.13) \qquad r\{2\Box+3\Delta\}(n) = d_{1,3}(8n+3) - d_{2,3}(8n+3),$$
$$(1.14) \qquad r\{3\Delta+4\Delta\}(n) = \frac{1}{2}\left(d_{1,3}(8n+7) - d_{2,3}(8n+7)\right),$$
$$(1.15) \qquad r\{\Delta+\Delta+\Delta+\Delta\}(n) = \sigma(2n+1),$$
$$(1.16) \qquad r\{\Box+\Box+\Box+\Box\}(n) = \sigma(4n+1).$$

It should be noted that not all these results are new. For instance, (1.8) is equivalent to a result of Ramanujan ([1, pp.223-224, 3, 4, p.229]).

2. Preliminary Results

As usual, let

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n\ge 0} q^{(n^2+n)/2}.$$

We shall require the easy lemmas

$$(2.1) \qquad \phi(q)\psi(q^2) = \psi(q)^2,$$

$$(2.2) \qquad \phi(q) = \phi(q^4) + 2q\psi(q^8),$$

as well as the (apparently new) result

$$(2.3) \qquad \psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}).$$

Proofs of lemmas.

$$(2.1) \qquad \phi(q)\psi(q^2) = \frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2} \cdot \frac{(q^4)_\infty^2}{(q^2)_\infty} = \frac{(q^2)_\infty^4}{(q)_\infty^2} = \psi(q)^2.$$

$$(2.2) \quad \phi(q) = \sum_{-\infty}^{\infty} q^{n^2} = \sum_{n \text{ even}} q^{n^2} + \sum_{n \text{ odd}} q^{n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{4n^2+4n+1} = \phi(q^4) + 2q\psi(q^8).$$

$$(2.3)\qquad q^4\psi(q^8)\psi(q^{24}) = \sum_{k,l=-\infty}^{\infty} q^{(4k+1)^2+3(4l+1)^2} = \sum_{k,l=-\infty}^{\infty} q^{4(k+3l+1)^2+12(k-l)^2} = \sum_{u-v\equiv 1 \pmod 4} q^{4u^2+12v^2}.$$

We now consider the two cases v even, u even. If v is even, $v = 2k$, $u = 4l+1$ or $-4l-1$, according as k is even or odd, while if u is even, $u = 2k$, $v = 4l+1$ or $-4l-1$, according as k is odd or even. Thus

$$\sum_{u-v\equiv 1 \,(\mathrm{mod} 4)} q^{4u^2+12v^2} = \sum_{k,l=-\infty}^{\infty} q^{4(4l+1)^2+12(2k)^2} + \sum_{k,l=-\infty}^{\infty} q^{4(2k)^2+12(4l+1)^2} = q^4\psi(q^{32})\phi(q^{48}) + q^{12}\phi(q^{16})\psi(q^{96}),$$

as required.

Proofs of theorems.

(J1) is equivalent to

$$(3.1)\qquad \phi(q)^2 = 1 + 4\sum_{n\geq 1}(d_{1,4}(n) - d_{3,4}(n))q^n.$$

That is, by (2.2) and (2.1),

(3.2)

$$\left(\phi(q^4) + 2q\psi(q^8)\right)^2 = \left(\phi(q^4)^2 + 4q^2\psi(q^8)^2\right) + 4q\psi(q^4)^2 = 1 + 4\sum_{n\geq 1}\left(d_{1,4}(n) - d_{3,4}(n)\right)q^n.$$

If we extract those terms in which the power of q is $1 \pmod 4$, divide by $4q$ and replace q^4 by q, we find

$$(3.3)\qquad \psi(q)^2 = \sum_{n\geq 0}\left(d_{1,4}(4n+1) - d_{3,4}(4n+1)\right)q^n,$$

from which we obtain (1.1).

By (2.1), (3.3) can be written

$$(3.4)\qquad \phi(q)\psi(q^2) = \sum_{n\geq 0}(d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n,$$

from which (1.2) follows.

Using (2.2), (3.4) can be written

$$(3.5)\qquad \psi(q^2)\left(\phi(q^4) + 2q\psi(q^8)\right) = \sum_{n\geq 0}\left(d_{1,4}(4n+1) - d_{3,4}(4n+1)\right)q^n,$$

from which (1.3) and (1.4) follow.

(D) is equivalent to

$$(3.6)\qquad \phi(q)\phi(q^2) = 1 + 2\sum_{n\geq 1}(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n))q^n,$$

or, by (2.2),

$$(3.7)\qquad (\phi(q^4)+2q\psi(q^8))(\phi(q^8)+2q^2\psi(q^{16})) = 1+2\sum_{n\geq 1}(d_{1,8}(n)+d_{3,8}(n)-d_{5,8}(n)-d_{7,8}(n))q^n,$$

from which (1.5), (1.6) and (1.7) follow.

(L) is equivalent to

$$\begin{aligned}\phi(q)\phi(q^3) &= 1+2\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n+4\sum_{n\geq1}(d_{4,12}(n)-d_{8,12}(n))q^n \\ &= 1+2\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n+4\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^{4n},\end{aligned} \tag{3.8}$$

or

$$\begin{aligned}&\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{12})+2q^3\psi(q^{24})\right) \\ &\qquad = 1+2\sum_{n\geq1}\left(d_{1,3}(n)-d_{2,3}(n)\right)q^n+4\sum_{n\geq1}\left(d_{1,3}(n)-d_{2,3}(n)\right)q^{4n}.\end{aligned} \tag{3.9}$$

If from (3.9) we extract those terms in which the power of q is 0 (mod 4) and replace q^4 by q, we find

$$\begin{aligned}&\phi(q)\phi(q^3)+4q\psi(q^2)\psi(q^6) \\ &\qquad = 1+2\sum_{n\geq1}(d_{1,3}(4n)-d_{2,3}(4n))q^n+4\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n \\ &\qquad = 1+6\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n\end{aligned} \tag{3.10}$$

where we have used the fact, which needs a little consideration, that

$$d_{1,3}(4n)-d_{2,3}(4n)=d_{1,3}(n)-d_{2,3}(n).$$

If we subtract (3.8) from (3.10), we find

$$4q\psi(q^2)\psi(q^6)=4\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n-4\sum_{n\geq1}\left(d_{1,3}(n)-d_{2,3}(n)\right)q^{4n}. \tag{3.11}$$

The right hand side of (3.11) is an odd function of q; if we divide by $4q$ and replace q^2 by q, we find

$$\psi(q)\psi(q^3)=\sum_{n\geq0}\left(d_{1,3}(2n+1)-d_{2,3}(2n+1)\right)q^n, \tag{3.12}$$

from which (1.8) follows.
If we invoke (2.3), (3.12) becomes

$$\phi(q^6)\psi(q^4)+q\phi(q^2)\psi(q^{12})=\sum_{n\geq0}\left(d_{1,3}(2n+1)-d_{2,3}(2n+1)\right)q^n, \tag{3.13}$$

so

$$\phi(q^3)\psi(q^2)=\sum_{n\geq0}(d_{1,3}(4n+1)-d_{2,3}(4n+1))q^n \tag{3.14}$$

and

$$\phi(q)\psi(q^6)=\sum_{n\geq0}(d_{1,3}(4n+3)-d_{2,3}(4n+3))q^n. \tag{3.15}$$

(1.9) and (1.10) follow.
(3.14) and (3.15) can be written respectively

$$\psi(q^2)\left(\phi(q^{12})+2q^3\psi(q^{24})\right)=\sum_{n\geq0}\left(d_{1,3}(4n+1)-d_{2,3}(4n+1)\right)q^n \tag{3.16}$$

and

$$\psi(q^6)\left(\phi(q^4)+2q\psi(q^8)\right)=\sum_{n\geq 0}(d_{1,3}(4n+3)-d_{2,3}(4n+3))q^n. \tag{3.17}$$

(1.11), (1.12), (1.13) and (1.14) follow.

(J2) is equivalent to

$$\phi(q)^4=1+8\sum_{n\geq 1}\left(\sum_{d|n,\,4\nmid d} d\right)q^n. \tag{3.18}$$

Now, the left hand side is

(3.19)

$$\begin{aligned}\phi(q)^4&=\left(\phi(q^4)+2q\psi(q^8)\right)^4\\&=\left(\phi(q^4)^4+16q^4\psi(q^8)^4\right)+8q\phi(q^4)^3\psi(q^8)+24q^2\phi(q^4)^2\psi(q^8)^2+32q^3\phi(q^4)\psi(q^8)^3\\&=\left(\phi(q^4)^4+16q^4\psi(q^8)^4\right)+8q\phi(q^4)^2\psi(q^4)^2+24q^2\psi(q^4)^4+32q^3\psi(q^4)^2\psi(q^8)^2.\end{aligned}$$

So (3.18) becomes

$$\begin{aligned}&\left(\phi(q^4)^4+16q^4\psi(q^8)^4\right)+8q\phi(q^4)^2\psi(q^4)^2+24q^2\psi(q^4)^4+32q^3\psi(q^4)^2\psi(q^8)^2\\&=1+8\sum_{n\geq 1}\left(\sum_{d|n,\,4\dagger d} d\right)q^n.\end{aligned} \tag{3.20}$$

We deduce that

$$24\psi(q)^4=8\sum_{n\geq 0}\left(\sum_{d|4n+2} d\right)q^n=24\sum_{n\geq 0}\sigma(2n+1)q^n \tag{3.21}$$

and

$$8\phi(q)^2\psi(q)^2=8\sum_{n\geq 0}\left(\sum_{d|4n+1} d\right)q^n=8\sum_{n\geq 0}\sigma(4n+1)q^n, \tag{3.22}$$

which are (1.15) and (1.16).

References

[1] Berndt, B.C., "Ramanujan's Notebooks", Part III, Springer-Verlag, New York, 1991.

[2] Hirschhorn, M.D., "Partial fractions and four classical theorems of number theory", *Amer. Math, Monthly*, **106**(2000), 268-272.

[3] Hirschhorn, M.D., "Three classical results on representations of a number", *The Andrews Festschrift, Seventeen Papers on Classical Number Theory and Combinatorics*, D.Foata, G-N.Han, eds., Springer-Verlag, (2001),159-165.

[4] Ramanujan, S., *Notebooks*, **Vol. 2**, Tata Institute of Fundamental Research, Mumbai, 1957.

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