# GRÖBNER GEOMETRY OF SCHUBERT POLYNOMIALS (EXTENDED ABSTRACT)

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Abstract. Let $w \in S_n$ be a permutation. We provide a geometric context in which both (i) polynomial representatives for the Schubert classes $[X_w]$ in the cohomology ring $H^*(\mathcal{F}\ell_n)$ of the flag manifold are uniquely singled out, with no choices other than a Borel subgroup of $GL_n$; and (ii) it is geometrically obvious that these polynomials have nonnegative coefficients. These polynomials turn out to be the Schubert polynomials $\mathfrak{S}_w(x_1, \ldots, x_n)$ of Lascoux and Schützenberger [LS82a].

Our investigations lead us to replace topology on the flag manifold with multigraded commutative algebra, by generalizing the notion of degree for subschemes of projective space. Identifying the Schubert polynomials in this context then sheds light on the algebra and geometry of determinantal ideals specified by rank conditions as in [Ful92], and especially on the combinatorics of initial ideals for certain natural term orders. These initial ideals are the Stanley–Reisner ideals for simplicial complexes whose facets are in natural bijection with the rc-graphs of Fomin and Kirillov [FK96, BB93]. We give an inductive procedure on the weak Bruhat order for listing rc-graphs. Our further analysis of rc-graphs is based on the combinatorics of words and their subwords in general Coxeter groups, which give rise to shellable simplicial balls or spheres generalizing the initial ideals constructed from rc-graphs.

Résumé. Soit $w \in S_n$ une permutation. Nous fournissons un contexte géométrique qui (i) nous permet d'exhiber de manière unique, et sans autre choix que celui d'un sous groupe de Borel de $GL_n$, des représentants polynômiaux des classes de Schubert $[X_w]$ dans l'anneau de cohomologie $H^*(\mathcal{F}\ell_n)$ de la variété des drapeaux; (ii) rend géométriquement évident le fait que les coefficients de ces polynômes sont tous non négatifs. Ces polynômes se trouvent être les polynômes de Schubert $\mathfrak{S}_w(x_1, \ldots, x_n)$ de Lascoux et Schützenberger [LS82a].

Nos investigations nous ont menés à remplacer la topologie de la variété des drapeaux par de l'algèbre commutative multigraduée, en généralisant la notion de degré aux sous-chémas de l'espace projectif. L'apparition des polynômes de Schubert dans ce contexte éclaire l'algèbre et la géométrie des idéaux déterminantaux spécifiés par des conditions de rang [Ful92], et spécialement la combinatoire des idéaux initiaux pour certains ordres de termes apparaissant naturellement. Ces idéaux sont les idéaux de Stanley–Reisner de certains complexes simpliciaux, dont les facettes sont en bijection naturelle avec l'ensemble des rc-graphes de Fomin et Kirillov [FK96, BB93]. Nous donnons une procédure inductive sur l'ordre de Bruhat faible pour lister les rc-graphes. Notre analyse ultérieure des rc-graphes se base sur la combinatoire des mots et sous-mots dans les groupes de Coxeter généraux, qui donne lieu à des boules ou sphères effeuillable, généralisant les idéaux construits à partir de rc-graphes.

## 1. Gröbner bases and multidegrees of determinantal ideals

**1.1. Matrix Schubert varieties.** Let $M_n$ be the $n \times n$ matrices over a field $\mathbf{k}$, with coordinate ring $\mathbf{k}[\mathbf{z}]$ in indeterminates $\{z_{ij}\}_{i,j=1}^n$. Throughout the paper, $q$ and $p$ will be integers with $1 \leq q, p \leq n$, and $Z$ will stand for an $n \times n$ matrix. Usually $Z$ will be the **generic matrix** of variables $(z_{ij})$, although occasionally $Z$ will be an element of $M_n$.

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Denote by $Z_{[q,p]}$ the northwest $q \times p$ submatrix of $Z$. For instance, given a permutation matrix $w^T$ (for $w \in S_n$) with 1's in row $i$ and column $w(i)$, we find that

$$\mathrm{rank}(w^T_{[q,p]}) \quad = \quad \#\{(i,j) \le (q,p) \mid w(i) = j\}$$

is the number of 1's in the submatrix $w^T_{[q,p]}$.

The following definition was made by Fulton in [Ful92].

**Definition 1.1.1.** Let $w \in S_n$. The **matrix Schubert variety** $\overline{X}_w \subseteq M_n$ consists of the matrices $Z \in M_n$ such that $\mathrm{rank}(Z_{[q,p]}) \le \mathrm{rank}(w^T_{[q,p]})$ for all $q, p$.

Let $B \subset GL_n$ denote the Borel group of *lower* triangular matrices, so that $\mathcal{F}\ell_n = B\backslash GL_n$ is the manifold of flags in $\mathbf{k}^n$. Intersecting $\overline{X}_w$ with $GL_n \subset M_n$ yields the variety $\tilde{X}_w$ of all invertible matrices mapping to the Schubert variety $X_w \subseteq \mathcal{F}\ell_n$. Here, we define $X_w$ to be the closure in $\mathcal{F}\ell_n$ of the orbit $Bw^TB^+$, where $B^+$ denotes the upper triangular matrices in $GL_n$.

Heuristically, in the case $\mathbf{k} = \mathbb{C}$, the matrix Schubert variety determines a $B$-equivariant cohomology class $[\overline{X}_w]_B \in H^*_B(M_n)$ that maps to the corresponding class $[\tilde{X}_w]_B \in H^*_B(GL_n)$ under the inclusion $GL_n \hookrightarrow M_n$. Letting $T$ denote the maximal torus in $B$, observe that $H^*_B(M_n) = H^*_T(M_n) = \mathbb{Z}[x_1, \ldots, x_n]$ because $B$ retracts to $T$ and $M_n$ is $T$-equivariantly contractible. Therefore $[\overline{X}_w]_B \in \mathbb{Z}[\mathbf{x}]$ is well-defined as a polynomial in $\mathbf{x} = x_1, \ldots, x_n$. Since the quotient $GL_n \twoheadrightarrow \mathcal{F}\ell_n$ induces a natural isomorphism $H^*_B(GL_n) \cong H^*(\mathcal{F}\ell_n)$, we find that the polynomial $[\overline{X}_w]_B = [\overline{X}_w]_T$ represents the Schubert class $[X_w]$ on $\mathcal{F}\ell_n$.

The previous paragraph accomplishes the goal of singling out unique polynomial representatives for Schubert classes. It can be made completely precise when $\mathbf{k} = \mathbb{C}$ by arguments derived from [Kaz97] and appearing also in [FR01]. These techniques demonstrate why double Schubert polynomials (applied to the Chern roots of flagged vector bundles) are the characteristic classes for degeneracy loci [Ful92]: the mixing space construction of Borel whose cohomology is the equivariant cohomology of $M_n$ agrees with the classifying space for maps between flagged vector bundles.

Instead of relating equivariant cohomology of $M_n$ to ordinary cohomology of $\mathcal{F}\ell_n$, we employ the notion of multidegree—a commutative algebra approach to torus-equivariant cohomology (cf. [Tot99, Bri98, EG98]) on vector spaces. Multidegrees work over arbitrary fields $\mathbf{k}$, and are in any case essential for showing geometrically why the coefficients of $[\overline{X}_w]_T$ are positive.

1.2. **Schubert polynomials as multidegrees.** For the purpose of dealing with multidegrees in complete generality, let $\mathbf{k}[\mathbf{z}]$ be the polynomial ring in $m$ variables $\mathbf{z} = z_1, \ldots, z_m$, with a grading by $\mathbb{Z}^n$ in which each variable $z_i$ has **exponential weight** $\mathrm{wt}(z_i) = \mathbf{x}^{\mathbf{a}_i}$ for some vector $\mathbf{a}_i \in \mathbb{Z}^n$. In the case of interest for later purposes, $m = n^2$ with $\mathbf{z} = (z_{ij})_{i,j=1}^n$ and $\mathrm{wt}(z_{ij}) = x_i$.

Every finitely generated $\mathbb{Z}^n$-graded module $\Gamma = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} \Gamma_{\mathbf{b}}$ over $k[\mathbf{z}]$ has a graded free resolution $\mathcal{E}_{\bullet} : 0 \leftarrow \mathcal{E}_0 \leftarrow \mathcal{E}_1 \leftarrow \cdots \leftarrow \mathcal{E}_m \leftarrow 0$. Suppose $\mathcal{E}_i = \bigoplus_{k=1}^{\beta_i} k[\mathbf{z}](-\mathbf{b}_{ik})$, so that the $k^{\text{th}}$ summand of $\mathcal{E}_i$ is generated in $\mathbb{Z}^n$-graded degree $\mathbf{b}_{ik}$.

**Definition 1.2.1.** The $K$**-polynomial** of $\Gamma$ is $\mathcal{K}(\Gamma; \mathbf{x}) = \sum_i (-1)^i \sum_k \mathbf{x}^{\mathbf{b}_{ik}}$.

Geometrically, the $K$-polynomial of $\Gamma$ represents the class of the sheaf $\tilde{\Gamma}$ on $\mathbf{k}^m$ in equivariant $K$-theory for the action of the $n$-torus whose weight lattice is $\mathbb{Z}^n$. Combinatorially, when the $\mathbb{Z}^n$-grading is **positive**, meaning that the ordinary weights $\mathbf{a}_1, \ldots, \mathbf{a}_n$ lie in a single open half-space in $\mathbb{Z}^n$, the $K$-polynomial of $\Gamma$ is the numerator of the $\mathbb{Z}^n$-graded

Hilbert series $H(\Gamma;\mathbf{x})$:

$$H(\Gamma;\mathbf{x}) \quad := \quad \sum_{\mathbf{b}\in\mathbb{Z}^n} \dim_{\mathbf{k}}(\Gamma_{\mathbf{b}}) \cdot \mathbf{x}^{\mathbf{b}} \quad = \quad \frac{\mathcal{K}(\Gamma;\mathbf{x})}{\prod_{i=1}^m (1-\mathrm{wt}(z_i))}.$$

Positivity occurs when $\mathbf{z} = (z_{ij})$ and $\mathrm{wt}(z_i) = x_i$, where the denominator is $\prod_{i=1}^n (1-x_i)^n$.

Given any Laurent monomial $\mathbf{x}^{\mathbf{a}}$, the rational function $\prod_{j=1}^n (1-x_j)^{a_j}$ can be expanded as a well-defined (i.e. convergent in the $\mathbf{x}$-adic topology) formal power series $\prod_{j=1}^n (1 - a_j x_j + \cdots)$ in $\mathbf{x}$. Doing the same for each monomial in an arbitrary Laurent polynomial $\mathcal{K}(\mathbf{x})$ results in a power series denoted by $\mathcal{K}(\mathbf{1}-\mathbf{x})$.

**Definition 1.2.2.** The **multidegree** of a $\mathbb{Z}^n$-graded $\mathbf{k}[\mathbf{z}]$-module $\Gamma$ is the sum $\mathcal{C}(\Gamma;\mathbf{x})$ of the lowest degree terms in $\mathcal{K}(\Gamma;\mathbf{1}-\mathbf{x})$. If $\Gamma = \mathbf{k}[\mathbf{z}]/I$ is the coordinate ring of a subscheme $X \subseteq \mathbf{k}^m$, then write $[X]_{\mathbb{Z}^n} = \mathcal{C}(\Gamma;\mathbf{x})$.

The letters $\mathcal{C}$ and $\mathcal{K}$ stand for 'cohomology' and '$K$-theory', and the relation between them ('take lowest degree terms') reflects the Grothendieck–Riemann–Roch transition from $K$-theory to its associated graded ring. When $\mathbf{k} = \mathbb{C}$ is the complex numbers, the (Laurent) polynomials denoted by $\mathcal{C}$ and $\mathcal{K}$ are honest torus-equivariant cohomology and $K$-classes on affine space.

The motivating example is the case $X = \overline{X}_w \subseteq M_n$. Recall that the $i^{\text{th}}$ **divided difference** operator $\partial_i$ acts on polynomials $f \in \mathbb{Z}[\mathbf{x}]$ by $\partial_i(f) = (f - s_i f)/(x_i - x_{i+1})$, where the $i^{\text{th}}$ transposition $s_i \in S_n$ switches the variables $x_i$ and $x_{i+1}$ in the argument of $f$. After calculating explicitly that $[\overline{X}_{w_0}]_{\mathbb{Z}^n} = \prod_{i=1}^n x_i^{n-i}$ for the long permutation $w_0 = n\cdots 321 \in S_n$, we use a direct geometric argument using multidegrees to show:

**Theorem 1.2.3.** *If* $\mathrm{length}(ws_i) < \mathrm{length}(w)$ *then* $[\overline{X}_{ws_i}]_{\mathbb{Z}^n} = \partial_i[\overline{X}_w]_{\mathbb{Z}^n}$. *Therefore* $[\overline{X}_w]_{\mathbb{Z}^n}$ *equals the Schubert polynomial* $\mathfrak{S}_w(\mathbf{x})$.

More generally, for the $\mathbb{Z}^{2n}$-grading in which $\mathrm{wt}(z_i) = x_i/y_j$, the proof actually shows that $[\overline{X}_w]_{\mathbb{Z}^{2n}}$ is the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$. Although all of the results below hold in this 'double' setting, we stick to the $\mathbb{Z}^n$-grading for clarity.

1.3. **Multidegrees as positive sums.** Multidegrees, like ordinary degrees, are additive on unions of schemes with disjoint support and equal dimension. To make a precise statement, let $\mathrm{mult}_X(\Gamma)$ denote the multiplicity of $\Gamma$ along the variety $X$.

**Theorem 1.3.1.** *Fix a* $\mathbb{Z}^n$*-graded module* $\Gamma$*, and let* $X_1,\ldots,X_r$ *be the maximal-dimensional irreducible components of the support variety of* $\Gamma$*. Then*

$$\mathcal{C}(\Gamma;\mathbf{x}) \quad = \quad \sum_{\ell=1}^r \mathrm{mult}_{X_\ell}(\Gamma)\cdot[X_\ell]_{\mathbb{Z}^n}.$$

Multidegrees are constant in flat families, including the Gröbner degenerations of the next lemma. For positive gradings this is easy, by the constancy of Hilbert series.

**Lemma 1.3.2.** *Fix a* $\mathbb{Z}^n$*-graded module* $\Gamma$*, and let* $\Gamma \cong F/K$ *be an expression of* $\Gamma$ *as the quotient of a free module* $F$ *with kernel* $K$*. The multidegree* $\mathcal{C}(\Gamma;\mathbf{x})$ *equals the multidegree of* $F/\mathsf{in}(K)$ *for the initial submodule* $\mathsf{in}(K)$ *of* $K$ *under any term order.*

When $\mathbf{k}^m = M_n$ and $\mathrm{wt}(z_i) = x_i$, the multidegree $[L]_{\mathbb{Z}^n}$ of a coordinate subspace $L =$ zero set of $\langle z_{i_1 j_1},\ldots,z_{i_r j_r}\rangle$ is just the monomial $\mathbf{x}^L := x_{i_1}\cdots x_{i_r}$. Therefore multidegrees can always be expressed as positive sums of monomials.

**Corollary 1.3.3.** *Suppose $\mathcal{L}$ is the zero scheme of an initial ideal $\mathsf{in}(I(\overline{X}_w))$ of the ideal of $\overline{X}_w$ for some term order. The equality $[\overline{X}_w]_{\mathbb{Z}^n} = \sum_L \mathrm{mult}_L(\mathcal{L}) \cdot \mathbf{x}^L$ writes the multidegree $\mathfrak{S}_w(\mathbf{x})$ of $\overline{X}_w$ as a sum of monomials with positive coefficients, where the sum is over reduced subspaces of $\mathcal{L}$ having maximal dimension.*

1.4. **Gröbner bases, antidiagonals, and Hilbert series.** A positive sum as in Corollary 1.3.3 lends itself to combinatorial analysis only if we understand the initial ideal. Theorem 1.4.2, which determines an explicit initial ideal, will therefore be our central result. Ultimately it connects the geometry of Schubert varieties to the combinatorics of permutations, using the methods of Section 2 (which are themselves based on techniques in the proof of Theorem 1.4.2).

The matrix Schubert variety $\overline{X}_w$ is cut out set-theoretically by determinants in the generic matrix $Z$. These equations in fact define $\overline{X}_w$ scheme-theoretically; see [Ful92] or Corollary 1.5.1, below.

**Definition 1.4.1.** Let $w \in S_n$ be a permutation and $r_{qp} = \mathrm{rank}(w^T_{[q,p]})$ for each $q, p$.

1. The **Schubert determinantal ideal** $I_w \subset \mathbf{k}[\mathbf{z}]$ is generated by all minors in $Z_{[q,p]}$ of size $1 + r_{qp}$ for all $q, p$, where $Z = (z_{ij})$ is the matrix of variables.
2. The **antidiagonal ideal** $J_w$ is generated by the antidiagonals of the minors of $Z = (z_{ij})$ generating $I_w$.

Here, the **antidiagonal** of a square matrix or a minor is the product of the entries on the main antidiagonal.

This broad class of determinantal ideals includes all ideals "cogenerated by a fixed minor", as in [HT92]. More generally, the generators of every ladder determinantal ideal coincide with the generators of the Schubert determinantal ideal $I_w$ for some vexillary (also known as 2143-avoiding, or single-shaped) permutation $w$. Special cases of these vexillary determinantal ideals are the objects of study in [Con95, MS96a, MS96b, CH97, GM00, GL00], for example. Theorem 1.2.3 yields determinantal formulae for $\mathbb{Z}^{2n}$ and $\mathbb{Z}^n$-graded multidegrees in these cases, because vexillary double Schubert polynomials $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ are multiSchur polynomials; see [Mac91, Ful92].

An **antidiagonal term order** is any term order in which the leading monomial of every minor is its antidiagonal; such orders are easy to construct.

**Theorem 1.4.2.** *For any antidiagonal term order, $\mathsf{in}(I_w) = J_w$; in other words, the union over $q, p$ of the $(1 + r_{qp})$-minors in $Z_{[q,p]}$ constitute a Gröbner basis.*

The hardest step in the proof of Theorem 1.4.2 shows that the Hilbert series of $\{\mathbf{k}[\mathbf{z}]/J_w\}_{w\in S_n}$ satisfy the recursion defining the **Grothendieck polynomials** $\mathcal{G}_w(\mathbf{x})$ [LS82b]. To be precise, $\mathcal{G}_w(\mathbf{x})$ can be defined by downward induction on the weak order of $S_n$, in analogy with Schubert polynomials. This time, however, start with $\mathcal{G}_{w_0}(\mathbf{x}) = \prod_{i=1}^n (1 - x_i)^{n-i}$ and use the **Demazure** or **isobaric divided difference** operators $\overline{\partial}_i$, which act by $\overline{\partial}_i(f) = (x_{i+1}f - x_i s_i f)/(x_{i+1} - x_i)$. Demazure operators work just as well on power series, such as $H_w := H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x})$ for $w \in S_n$, as they do on polynomials.

**Theorem 1.4.3.** *If* $\mathrm{length}(ws_i) < \mathrm{length}(w)$ *then* $H_{ws_i} = \overline{\partial}_i H_w$. *Thus*

$$H(\mathbf{k}[\mathbf{z}]/J_w; \mathbf{x}) \quad = \quad \frac{\mathcal{G}_w(\mathbf{x})}{\prod_{i=1}^n (1 - x_i)^n}.$$

This Hilbert series calculation requires substantial tailor-made combinatorics, giving rise to Section 2. Assume Theorem 1.4.3 henceforth in this exposition.

*Proof sketch for Theorem 1.4.2.* Directly from the definitions of $\partial_i$ and $\overline{\partial}_i$, taking the lowest degree terms in $\mathcal{G}_w(\mathbf{1}-\mathbf{x})$ yields $\mathfrak{S}_w(\mathbf{x})$. Therefore the multidegree of $\mathbf{k}[\mathbf{z}]/J_w$ equals $\mathfrak{S}_w(\mathbf{x})$ and agrees with $[\overline{X}_w]_{\mathbb{Z}^n}$ by Theorem 1.2.3.

After (nontrivially) showing that $J_w$ is pure of dimension $\dim \overline{X}_w$, an easy lemma concerning squarefree monomial ideals reveals why the equality $[\overline{X}_w]_{\mathbb{Z}^n} = [\mathcal{L}_w]_{\mathbb{Z}^n}$ and the containment $\mathsf{in}(I(\overline{X}_w)) \supseteq J_w$ together imply $\mathsf{in}(I(\overline{X}_w)) = J_w$. Since the containments $\mathsf{in}(I(\overline{X}_w)) \supseteq \mathsf{in}(I_w) \supseteq J_w$ are obvious, we conclude that $\mathsf{in}(I(\overline{X}_w)) = \mathsf{in}(I_w) = J_w$. □

1.5. **Applications.** The theorems in the previous section provide formulae for the $K$-polynomials (i.e. Hilbert series) of Schubert determinantal varieties.

**Corollary 1.5.1.** *The ideal of $\overline{X}_w$ is $I_w$, and $\mathcal{K}(\mathbf{k}[\mathbf{z}]/I_w;\mathbf{x}) = \mathcal{G}_w(\mathbf{x})$.*

*Proof.* That $I_w$ is reduced follows from Theorem 1.4.2 and the fact that $J_w$ is reduced. The advertised $K$-polynomial comes from Theorem 1.4.2 and Theorem 1.4.3. □

Being in $K$-theory rather than cohomology, Corollary 1.5.1 is substantially stronger than Theorem 1.2.3. Although the former also follows directly from known results [LS82b] with only a little work (the class of the structure sheaf of $\overline{X}_w$ in $K^\circ_T(M_n)$ maps to that of $X_w$ in $K^\circ(\mathcal{F}\ell_n)$), the Hilbert series in Theorem 1.4.2 must in any case be computed during the course of proving Theorem 1.4.2. Therefore we recover the appropriate results from [LS82b] as consequences.

The antidiagonal ideal $J_w$ is the Stanley–Reisner ideal for a simplicial complex $\mathcal{L}_w$ (thought of as a set of subspaces in $M_n$). The combinatorial implications of Theorem 1.4.3 and the following corollary will be clarified after our analysis of the simplicial complexes $\mathcal{L}_w$ for $w \in S_n$ and their generalizations to arbitrary Coxeter groups, which occupies all of Section 2.

**Corollary 1.5.2.** $\mathfrak{S}_w(\mathbf{x}) = [\overline{X}_w]_{\mathbb{Z}^n} = [\mathcal{L}_w]_{\mathbb{Z}^n} = \sum_{\text{facets } L\in\mathcal{L}_w} \mathbf{x}^L$.

The set of minors in $I_w$ forming a minimal (but not reduced) Gröbner basis can be characterized in terms of essential sets [Ful92]. As a consequence, we crystallize the relation between determinantal ideals and open subsets (“opposite cells”) in Schubert varieties of $\mathcal{F}\ell_n$. In particular, let $\mathfrak{C}$ be a **local condition**, meaning that $\mathfrak{C}$ holds for a variety whenever it holds on each subvariety in some open cover. Such local conditions include normality, Cohen–Macaulayness, and rational singularities.

**Theorem 1.5.3.** *Assume that the local condition $\mathfrak{C}$ holds for a variety $X$ whenever it holds for the product of $X$ with any vector space. Then $\mathfrak{C}$ holds for every Schubert variety in every flag variety if and only if $\mathfrak{C}$ holds for all matrix Schubert varieties.*

Theorem 1.5.3 serves to systematize and even reverse the flow of results from the algebraic geometry of flag manifolds to the literature on determinantal ideals. However, we know of no new consequences that can be derived from it.

1.6. **An example.** The following example illustrates the results thus far.

**Example 1.6.1.** Let $w = 2143 \in S_4$. The matrix Schubert variety $\overline{X}_w$ is then the set of $4 \times 4$ matrices $Z = (z_{ij})$ whose upper left entry is zero and whose upper left $3 \times 3$ block has rank at most one. The equations defining $\overline{X}_{2143}$ are the determinants

$$I_{2143} \quad = \quad \left\langle z_{11}, \quad \det \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = -z_{13}z_{22}z_{31} + \dots \right\rangle.$$

Note that this is *not* a Gröbner basis with respect to term orders that pick out the diagonal term $z_{11}z_{22}z_{33}$ of the second generator, since $z_{11}$ divides that. The term orders that interest us pick out the *anti*diagonal term $-z_{13}z_{22}z_{31}$.

When we Gröbner-degenerate the matrix Schubert variety to the scheme defined by the initial ideal $J_{2143} = \langle z_{11}, -z_{13}z_{22}z_{31}\rangle$, we get a union of three coordinate subspaces

$$L_{11,13}, L_{11,22}, \text{ and } L_{11,31}, \quad \text{with ideals} \quad \langle z_{11}, z_{13}\rangle, \langle z_{11}, z_{22}\rangle, \text{ and } \langle z_{11}, z_{31}\rangle.$$

The resulting equation in $T$-equivariant cohomology, or on multidegrees, reads:

$$\begin{array}{rcccccc} [\overline{X}_{2143}]_{\mathbb{Z}^n} & = & [L_{11,13}]_{\mathbb{Z}^n} & + & [L_{11,22}]_{\mathbb{Z}^n} & + & [L_{11,31}]_{\mathbb{Z}^n} \\ & = & x_1^2 & + & x_1x_2 & + & x_1x_3 \end{array}$$

in $\mathbb{Z}[x_1, x_2, x_3, x_4] \cong H_T^*(M_4)$. The $\mathbb{Z}^n$-graded $K$-polynomial $\mathcal{K}(\mathbf{k}[\mathbf{z}]/I_{2143}; \mathbf{x})$ is equal to $\mathcal{G}_{2143}(\mathbf{x}) = (1 - x_1)(1 - x_1x_2x_3)$. Thus

$$\mathcal{K}(\mathbf{k}[\mathbf{z}]/I_{2143}; \mathbf{1} - \mathbf{x}) \quad = \quad x_1(x_1 + x_2 + x_3 - x_1x_2 - x_1x_3 - x_2x_3 + x_1x_2x_3),$$

whose lowest degree terms agree with $\mathfrak{S}_{2143}(\mathbf{x}) = [\overline{X}_{2143}]_{\mathbb{Z}^n} = x_1(x_1 + x_2 + x_3)$. □

## 2. Mitosis, rc-graphs, and subword complexes

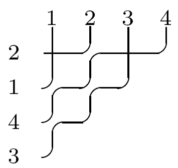
**2.1. Pipe dreams and rc-graphs.** The facets of the initial complex $\mathcal{L}_w$ correspond to certain subsets of the grid $[n] \times [n]$. The obvious question becomes: which subsets? The answer rests on drawing subsets of the grid in the "right" way.

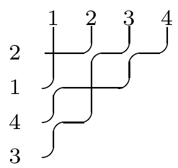
**Definition 2.1.1.** A **pipe dream** is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, identified as the set of crosses in a tiling by **crosses** $+$ and **elbow joints**.

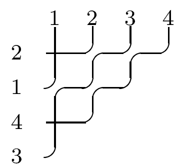
**Example 2.1.2.** Pictorially, each subspace $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$ from Example 1.6.1 represents a subset of the $4 \times 4$ grid: place a '+' in each box containing a generator for its ideal. In other words, given a coordinate subspace $L$, form the diagram $D_L$ by placing a '+' at $(i, j)$ if every matrix in $L$ has $(i, j)$ entry zero:

$$\langle z_{11}, z_{13}\rangle = \begin{array}{|c|c|c|c|}\hline + & & + & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline\end{array}, \quad \langle z_{11}, z_{22}\rangle = \begin{array}{|c|c|c|c|}\hline + & & & \\ \hline & + & & \\ \hline & & & \\ \hline & & & \\ \hline\end{array}, \quad \langle z_{11}, z_{31}\rangle = \begin{array}{|c|c|c|c|}\hline + & & & \\ \hline & & & \\ \hline + & & & \\ \hline & & & \\ \hline\end{array}$$

In the figures below, we draw the zero entries '+' by crossing pipes, and the nonzero entries by elbow joints (imagine that the lower-right triangle is filled with elbows).








These are the three "rc-graphs", or "planar histories", for the permutation 2143. □

The term 'rc-graph' was coined in [BB93], although [FK96] made the definition.

**Definition 2.1.3.** An **rc-graph** is a pipe dream in which each pair of pipes crosses at most once. If $D$ is an rc-graph and $w \in S_n$ is the permutation such that the pipe entering row $i$ exits from column $w(i)$, then $D$ is said to be an **rc-graph for** $w$. The set of rc-graphs for $w$ is denoted by $\mathcal{RC}(w)$.

The next theorem completes the transition from the geometry of $\mathcal{F}\ell_n$ to the combinatorics of $S_n$ through the algebra of determinants. Recall that the itinerary has been: Schubert variety $X_w \rightsquigarrow$ matrix Schubert variety $\overline{X}_w \rightsquigarrow$ determinantal ideal $I_w \rightsquigarrow$ antidiagonal ideal $J_w \rightsquigarrow$ initial complex $\mathcal{L}_w \rightsquigarrow$ rc-graphs $\mathcal{RC}(w)$.

**Theorem 2.1.4.** *$\mathcal{RC}(w) = \{D_L \mid L$ is a facet of $\mathcal{L}_w\}$. In other words, rc-graphs for $w$ are complements of maximal supports of monomials outside $J_w$.*

The proof uses results of [BB93]. Theorem 2.1.4 has the famous 'BJS' formula as a consequence, by Corollary 1.5.2. Previous proofs [BJS93, FS94] were combinatorial.

**Corollary 2.1.5.** *$\mathfrak{S}_w(\mathbf{x}) \;=\; \sum_{D\in\mathcal{RC}(w)} \mathbf{x}^D$, where $\mathbf{x}^D = \prod_{(i,j)\in D} x_i$.*

2.2. **Mitosis algorithm.** The proof in Section 1.4 that the $K$-polynomial of $\mathbf{k}[\mathbf{z}]/J_w$ equals $\mathcal{G}_w(\mathbf{x})$ works by constructing an operator $\varepsilon_i^w$ that takes each monomial outside $J_w$ to a positive sum of monomials outside $J_{ws_i}$. Interpreted in terms of rc-graphs, this procedure lists the coefficients of Schubert polynomials by downward induction on the weak order of $S_n$. Our algorithm serves as a geometrically motivated improvement on the famous conjecture of Kohnert [Mac91, Appendix to Chapter IV, by N. Bergeron], which is similarly inductive but employs Rothe diagrams instead of rc-graphs.

Given a pipe dream in $[n]\times[n]$, define the column index

$$\mathrm{start}_i(D) \;=\; \min(\{j \mid (i,j)\notin D\}\cup\{n+1\})$$

of the leftmost empty box in row $i$. Thus in the region to the left of $\mathrm{start}_i(D)$, the $i^{\text{th}}$ row of $D$ is filled solidly with crosses. Let

$$\mathcal{J}_i(D) \;=\; \{\text{columns } j \text{ strictly to the left of } \mathrm{start}_i(D) \mid (i+1,j) \text{ has no cross in } D\}$$

For $p\in\mathcal{J}_i(D)$, construct the **offspring** $D_p$ as follows. First delete the cross at $(i,p)$ from $D$. Then take all of the crosses in row $i$ of $\mathcal{J}_i(D)$ that are to the left of column $p$, and move each one down to the empty box below it in row $i+1$.

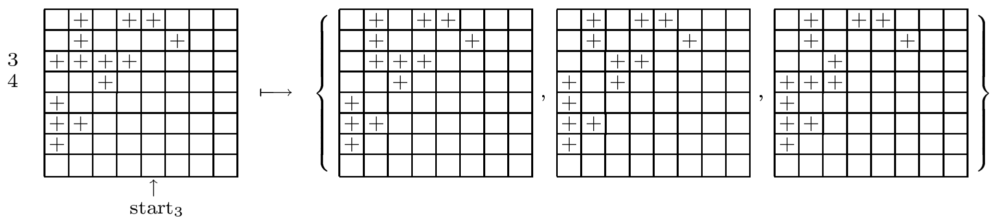
**Definition 2.2.1.** The $i^{\text{th}}$ **mitosis** operator sends a pipe dream $D$ to

$$\mathrm{mitosis}_i(D) \;=\; \{D_p \mid p\in\mathcal{J}_i(D)\}.$$

Write $\mathrm{mitosis}_i(\mathcal{P}) = \bigcup_{D\in\mathcal{P}}\mathrm{mitosis}_i(\mathcal{P})$ whenever $\mathcal{P}$ is a set of pipe dreams.

Observe that all of the action takes place in rows $i$ and $i+1$, and $\mathrm{mitosis}_i(D)$ is an empty set whenever $\mathcal{J}_i(D)$ is empty.

**Example 2.2.2.** The left pipe dream $D$ below is an rc-graph for $w = 13865742$.




The set of three pipe dreams on the right is obtained by applying $\mathrm{mitosis}_3$, since $\mathcal{J}_3(D)$ consists of columns 1, 2, and 4. □

**Theorem 2.2.3.** *$\mathcal{RC}(ws_i)$ is the disjoint union $\dot\bigcup_{D\in\mathcal{RC}(w)}\mathrm{mitosis}_i(D)$. Therefore, if $s_{i_1}\cdots s_{i_k}$ is a reduced expression for $w_0w$ and $D_0$ is the unique rc-graph for $w_0$, then*

$$\mathcal{RC}(w) \;=\; \mathrm{mitosis}_{i_k}\cdots\mathrm{mitosis}_{i_1}(D_0).$$

The unique rc-graph $D_0$ in Theorem 2.2.3 has crosses strictly above the main antidiagonal, and no other crosses. That is, $(i,j)\in D_0$ if and only if $i+j\le n$.

Although combinatorial considerations in the proof of Theorem 1.4.2 were instrumental in figuring out how to define mitosis in the first place, it is possible to give a complete proof of Theorem 2.2.3 based entirely on the BJS formula in Corollary 2.1.5 and the characterization of Schubert polynomials by divided differences, along with elementary combinatorial properties of rc-graphs [Mil02]. This argument exploits an involution on $\mathcal{RC}(w)$ which, for grassmannian (i.e. unique descent) permutations $w$, reduces to a well-known involution on semistandard Young tableaux.

2.3. **Subword complexes.** Given a pipe dream $D$, say that a '+' at $(q, p) \in D$ sits on the $i^{\text{th}}$ **antidiagonal** if $q + p - 1 = i$. Let $Q_D$ be the ordered list of simple reflections $s_i$ corresponding to the antidiagonals on which the crosses sit, starting from the northeast corner of $D$ and reading *right to left* across each row starting from the top and snaking downwards. Induction on the number of crosses in $D$ proves:

**Lemma 2.3.1.** *The list $Q_D$ constitutes a reduced expression for $w$ if and only if the pipe dream $D$ is an rc-graph for $w$.*

For example, the rc-graph $D_0$ in Theorem 2.2.3 corresponds to the reduced expression $w_0 = s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4$ when $n = 5$, while the full $3 \times 3$ grid yields the list $s_3s_2s_1s_4s_3s_2s_5s_4s_3$ of adjacent transpositions in $S_6$.

Here is the generalization of $\mathcal{L}_w$ to arbitrary words in Coxeter systems $(\Pi, \Sigma)$.

**Definition 2.3.2.** A **word** of size $m$ is an ordered list $Q = (\sigma_1, \ldots, \sigma_m)$ of elements of $\Sigma$. An ordered sublist $P$ of $Q$ is called a **subword** of $Q$.

1. $Q$ **represents** $\pi \in \Pi$ if the ordered product of the simple reflections in $Q$ is a reduced decomposition for $\pi$.
2. $Q$ **contains** $\pi \in \Pi$ if some sublist of $Q$ represents $\pi$.

The **subword complex** $\Delta(Q, \pi)$ is the set of subwords $P \subseteq Q$ whose complements $Q \smallsetminus P$ contain $\pi$.

**Example 2.3.3.** Theorem 2.1.4 and Lemma 2.3.1 say that if $\Pi = S_{2n}$ and $Q_{n\times n}$ is the word represented by all of $[n] \times [n]$, then $\mathcal{L}_w = \Delta(Q_{n\times n}, w)$ when $w \in S_n \subset S_{2n}$. Thus the combinatorics and Stanley–Reisner theory of general subword complexes yields information about Schubert and Grothendieck polynomials; see Corollary 2.4.5. □

**Theorem 2.3.4.** *Every subword complex $\Delta(Q, \pi)$ is vertex-decomposable and so Cohen–Macaulay, and even shellable. $\Delta(Q, \pi)$ is homeomorphic to a ball or sphere.*

The vertex decomposition rests on the fact that the link and deletion of the vertex $\sigma_1 \in \Delta(Q, \pi)$ are both subword complexes. The ball or sphere condition is equivalent to $\Delta(Q, \pi)$ being a manifold, given shellability; it reduces to showing that every codimension 1 face lies in at most two facets, which in turn relies on the exchange condition in Coxeter groups [Hum90, Theorem 5.8]. In view of [BW82], Theorem 2.3.4 suggests that the Bruhat and weak orders "feel" somewhat similar.

Fulton proved that $\overline{X}_w$ is Cohen–Macaulay [Ful92], but he used Cohen–Macaulayness of Schubert varieties [Ram85] to do it. Here we provide new proofs of both.

**Corollary 2.3.5.** *Every matrix Schubert variety $\overline{X}_w$, and hence every Schubert variety $X_w \subseteq B\backslash GL_n$, is Cohen–Macaulay.*

*Proof.* The Cohen–Macaulayness of $\mathcal{L}_w$ in Theorem 2.3.4 implies that of $\overline{X}_w$ by Theorem 1.4.2 and the flatness of Gröbner degeneration. Now use Theorem 1.5.3. □

2.4. **Combinatorics of Grothendieck polynomials.** Calculating the $\mathbb{Z}^m$-graded Hilbert series (that is, the $K$-polynomial) of the Stanley–Reisner ring for a simplicial ball or sphere $\Delta$ amounts to identifying the boundary faces of $\Delta$. For subword complexes $\Delta(Q, \pi)$, this identification uses a standard tool from Coxeter group theory.

**Definition 2.4.1.** Let $R$ be a commutative ring, and $\mathcal{D}$ a free $R$-module with basis $\{e_\pi \mid \pi \in \Pi\}$. Defining a multiplication on $\mathcal{D}$ by

$$e_\pi e_\sigma \;=\; \begin{cases} e_{\pi\sigma} & \text{if length}(\pi\sigma) > \text{length}(\pi) \\ e_\pi & \text{if length}(\pi\sigma) < \text{length}(\pi) \end{cases} \tag{1}$$

for $\sigma \in \Sigma$ yields the **Demazure algebra** of $(\Pi, \Sigma)$ over $R$. Define the **Demazure product** $\delta(Q)$ of the word $Q = (\sigma_1, \ldots, \sigma_m)$ by $e_{\sigma_1} \cdots e_{\sigma_m} = e_{\delta(Q)}$.

When $\Pi = S_n$ and $\Sigma$ is the set of simple reflections $s_1, \ldots, s_{n-1}$, the algebra $\mathcal{D}$ is generated over $R$ by the usual Demazure operators $\overline{\partial}_i$ (hence the name "Demazure algebra"). In general, the fact that the equations in (1) define an associative algebra is a special case of [Hum90, Theorem 7.1].

**Proposition 2.4.2.** *A face $Q \smallsetminus P$ is in the boundary of $\Delta(Q, \pi)$ if and only if $P$ has Demazure product $\delta(P) \neq \pi$.*

In our final theorem, the variables $\mathbf{z} = z_1, \ldots, z_m$ are identified with the locations in the list $Q = (\sigma_1, \ldots, \sigma_m)$—that is, with the vertices of the subword complex $\Delta(Q, \pi)$. The $\mathbb{Z}^n$-grading is the finest possible, with $n = m$. So as not to confuse notation in applications with $\mathbf{z} = (z_{ij})_{i,j=1}^n$, where the Hilbert series are $\mathbb{Z}^n$-graded and expressed in variables $\mathbf{x}$, we write $\mathbb{Z}^m$-graded $K$-polynomials in the variables $\mathbf{z}$, with each $z_i$ having tautological exponential weight $z_i$.

**Theorem 2.4.3.** *If* length$(\pi) = \ell$ *and $J$ is the Stanley–Reisner ideal of $\Delta(Q, \pi)$, then*

$$\mathcal{K}(\mathbf{k}[\mathbf{z}]/J; \mathbf{z}) \;=\; \sum_{\delta(P)=\pi} (-1)^{|P|-\ell} (\mathbf{1} - \mathbf{z})^P,$$

*where $(\mathbf{1} - \mathbf{z})^P = \prod_{i \in P}(1 - z_i)$.*

The proof uses Hochster's Betti number formula for the **Alexander dual ideal**

$$J^\star \;=\; \langle z_{i_1} \cdots z_{i_k} \mid \langle z_{i_1}, \ldots, z_{i_k} \rangle \text{ contains } J \rangle,$$

which appears in [ER98] and [BCP99] (for instance). Theorem 2.4.3 then follows from the Alexander inversion formula:

**Proposition 2.4.4.** *If $J \subseteq \mathbf{k}[\mathbf{z}]$ is a squarefree monomial ideal and $J^\star$ is its Alexander dual ideal, then $\mathcal{K}(\mathbf{k}[\mathbf{z}]/J; \mathbf{z}) = \mathcal{K}(J^\star; \mathbf{1} - \mathbf{z})$.*

The Alexander inversion formula can be interpreted as another way to define $J^\star$.

As a special case of Theorem 2.4.3 we recover a formula for double Grothendieck polynomials [FK94]. Although the natural "double" specialization for the exponential weight of $z_{ij}$ is $x_i/y_j$, we substitute $z_{ij} \mapsto x_i y_j$ to agree with the notation in [FK94].

**Corollary 2.4.5.** $\mathcal{G}_w(\mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y}) \;=\; \displaystyle\sum_{\delta(D)=w} \prod_{(i,j) \in D} (-1)^{|D|-\ell} (x_i + y_j - x_i y_j).$

*The single version, in which $z_{ij} \mapsto x_i$ and $\mathbf{x}^D$ equals $\prod_{(i,j) \in D} x_i$, reads:*

$$\mathcal{G}_w(\mathbf{1} - \mathbf{x}) \;=\; \sum_{\delta(D)=w} (-1)^{|D|-\ell} \mathbf{x}^D.$$

Note that Corollary 2.1.5 is the sum of lowest degree terms in the latter formula.

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