ALTERNATE TRANSITION MATRICES FOR BRENTI'S q-SYMMETRIC FUNCTIONS AND A CLASS OF (q,t)-SYMMETRIC FUNCTIONS ARISING FROM PLETHYSM (EXTENDED ABSTRACT)

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ABSTRACT. Brenti [4] introduces a class of q-symmetric functions based on a simple plethysm with the power-sum symmetric functions. Brenti shows that with certain specializations of q, these functions appear in connection with Jack symmetric functions, parking functions, and lattices of non-crossing partitions. Brenti develops combinatorial interpretations for the transition matrices between these new symmetric functions and the standard symmetric function bases. We provide simplified versions of many of these that are sums over significantly smaller classes of combinatorial objects. We then extend Brenti's definitions to symmetric functions on the hyperoctahedral group and give combinatorial interpretations of the analogous transition matrices. We also discuss new generating functions on permutation statistics that arise from Brenti's symmetric functions and our extensions.

RÉSUMÉ. Brenti [4] présente une classe des fonctions symétriques avec un paramètre simple basé sur un plethysm simple avec les fonctions symétriques de puissance-somme. Brenti prouve qu'avec certaines spécialisations du paramètre, ces fonctions apparaissent en liaison avec des fonctions symétriques de Jack, des fonctions se garantes, et des trellis des cloisons de non-croisement. Brenti développe des traductions combinatoires pour les matrices de transition entre ces nouvelles fonctions symétriques et les bases symétriques standard de fonction. Nous fournissons des versions simplifiées de beaucoup de ces derniers qui sont des sommes au-dessus des classes sensiblement plus petites des objets combinatoires. Nous alors étendons des définitions de Brenti aux fonctions symétriques sur le groupe hyperoctahedral et donnons des traductions combinatoires des matrices analogues de transition. Nous discutons également de nouvelles fonctions se produisantes sur les statistiques de permutation qui résultent des fonctions symétriques de Brenti et de nos extensions.

1. Introduction

This work is based on a class of symmetric functions with a parameter q introduced by Brenti in [4] which arise from a simple plethysm with the power-sum symmetric functions. Brenti develops combinatorial interpretations for the transition matrices between these new symmetric functions and the standard symmetric function bases. We provide simplified versions of many of these that have three advantages. First, most involve summing over significantly smaller classes of objects and are therefore much easier to compute. Second, our expressions are given in terms of objects that appear in the standard transition matrices summarized by Beck, Remmel, and Whitehead in [2] and are therefore more recognizable as q-analogues of those matrices. Finally, part of our work involves extending Brenti's symmetric functions to the hyperoctahedral group and our expressions have natural analogues in that setting. In Section 2 below we give examples of two of our results, one of which has a nice corollary involving the expansion of a product of binomial coefficients as a sum of binomial coefficients. In Section 3 we will discuss our generalization of Brenti's results to the hyperoctahedral group B_n. Specifically, we have defined a class of symmetric functions on B_n in two parameters q and t and have developed the combinatorics of the transition matrices that expand the resulting (q,t)-analogues of the standard bases in terms of the standard bases themselves. We give the basic definitions and a sampling of the results here. Finally, in Section 4 we discuss several new generating functions on permutation statistics that arise from Brenti's symmetric functions and our B_n analogues.

2. Transition matrices arising from Brenti's symmetric functions.

We start with some notation. Let Λ_n be the space of homogeneous symmetric functions of degree n and set $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$. Given two bases of Λ_n, $\{a_\lambda\}_{\lambda\vdash n}$ and $\{b_\lambda\}_{\lambda\vdash n}$, we fix some standard order of the partitions of n and then think of the bases as row vectors $\langle a_\lambda\rangle_{\lambda\vdash n}$ and $\langle b_\lambda\rangle_{\lambda\vdash n}$. We define the transition matrix $M(a,b)$ by the equation

$$\langle b_\lambda\rangle_{\lambda\vdash n} = \langle a_\lambda\rangle_{\lambda\vdash n} M(a,b)$$

So $M(a,b)$ is the matrix that transforms the basis $\langle a_\lambda\rangle_{\lambda\vdash n}$ into the basis $\langle b_\lambda\rangle_{\lambda\vdash n}$ and the (λ,μ) entry of $M(a,b)$ is given by

$$b_\mu = \sum_{\lambda\vdash n} a_\lambda M(a,b)_{\lambda\mu}$$

For λ a partition of n, let h_λ, e_λ, p_λ, s_λ, m_λ, and f_λ denote the complete homogenous, elementary, power-sum, Schur, monomial, and forgotten symmetric functions associated with λ, respectively. Brenti's symmetric functions [4] are defined on $\Lambda \otimes \mathbf{Q}(q)$ as follows. For a nonnegative integer r and an alphabet $X = x_1 + x_2 + \cdots$, define

$$p_r^q[X] = p_r[qX] = qp_r[X]$$

Then for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, set

$$p_\lambda^q[X] = p_{\lambda_1}^q[X] p_{\lambda_2}^q[X] \cdots p_{\lambda_l}^q[X] = q^{l(\lambda)} p_\lambda[X]$$

For any symmetric function g, g^q is defined by expanding g in terms of the power basis so that if $g = \sum_\nu a_\nu p_\nu$, then

$$g^q[X] = \sum_\nu a_\nu p_\nu^q[X]$$

Brenti shows that h_λ^q, e_n^{n+1}, and e_n^n have appeared before in connection with Jack symmetric functions, parking functions, and lattices of non-crossing partitions, respectively.

Brenti derives transition matrices that give the expansions of m_λ^q, e_λ^q, h_λ^q and $s_{\lambda/\mu}^q$ in terms of the m's, e's, h's and s's. As mentioned above, we have simplified many of these by summing over significantly smaller classes of objects. We give two examples of these here.

We start with Brenti's expression for $M(m, e^q)$. First, for a 3-dimensional $m \times n \times p$ matrix $A = (A_{i,j,k})_{1\leq i\leq m, 1\leq j\leq n, 1\leq k\leq p}$, define

$$S_{1,2}(A) = \left(\sum_{i=1}^m \sum_{j=1}^n A_{i,j,1}, \ldots, \sum_{i=1}^m \sum_{j=1}^n A_{i,j,p} \right)$$

and analogously for $S_{1,3}(A)$ and $S_{2,3}(A)$.

Also, for a nonnegative integer i, define

$$\binom{q}{i} = \frac{q(q-1)\cdots(q-i+1)}{i!}$$

Then Brenti shows that

$$M(m,e^q)_{\lambda\mu} = \sum_{i=1}^{|\mu|} \mathcal{M}_{\mu,\lambda}(i)\binom{q}{i}$$

where $\mathcal{M}_{\mu,\lambda}(i)$ is the number of $l(\mu)\times l(\lambda)\times i$ (0,1)-matrices A such that $S_{2,3}(A) = \mu, S_{1,3}(A) = \lambda$ and $S_{1,2}(A) > 0$.

Our expression for $M(m,e^q)$ is a sum over two-dimensional matrices.

Theorem 1. *For λ and μ partitions of n,*

$$M(m,e^q)_{\lambda\mu} = \sum_{A\in M(\mu,\lambda)} \prod_{i,j} \binom{q}{A_{i,j}}$$

where $M(\mu,\lambda)$ is the set of $l(\mu)\times l(\lambda)$ matrices with nonnegative integer entries with row sums equal to μ and column sums equal to λ.

We need the following definitions for our next result.

- *μ-brick tabloids of shape λ.* For $\mu \vdash n$, create a set of bricks that have lengths equal to the parts of μ. Then place these bricks in the Ferrers diagram of λ in such a way that each brick lies in a single row and no two bricks overlap. We call each such filling a μ-brick tabloid of shape λ. For example, Figure 1 shows the four $(1,2,3)$-brick tabloids of shape $(3,3)$.
- *Strict n-brick tabloids of shape λ and type μ.* These are μ-brick tabloids of shape λ that have positive integer labels on the bricks. The labels are from the set $\{1,2,\ldots,n\}$ with repetitions allowed, and the labels must strictly increase in rows from left to right. For example, the first tabloid in Figure 1 gives rise to two 2-brick tabloids of shape $(3,3)$ and type $(1,2,3)$ as shown in Figure 2.

Also, for a nonnegative integer i, define

$$\overline{D}_{\lambda,\mu}(q) = \sum_{i=1}^{l(\mu)} \overline{d}_{\lambda,\mu}(i)\binom{q}{i} \tag{1}$$

where $\overline{d}_{\lambda,\mu}(i)$ is the number of strict i-brick tabloids T of shape λ and type μ such that all of the integers 1,2,…,i appear in T.

Brenti shows that

$$M(m,m^q)_{\lambda,\mu} = \overline{D}_{\lambda,\mu}(q)$$

Our expression is:

Theorem 2. *For λ and μ partitions of n,*

$$M(m,m^q)_{\lambda,\mu} = \sum_{T\in\mathcal{B}_{\mu,\lambda}} \prod_{i=1}^{l(\lambda)} \binom{q}{n_i(T)}$$

where $\mathcal{B}_{\mu,\lambda}$ is the set of μ-brick tabloids of shape λ and for $T\in\mathcal{B}_{\mu,\lambda}, n_i(T)$ is the number of bricks in the i^{th} row of T.

We consider a simple example to demonstrate the number of objects involved in the two expressions for $M(m,m^q)$. Let $\lambda = (3,3)$ and $\mu = (1,2,3)$. Then there are four elements of $\mathcal{B}_{\mu,\lambda}$ as shown in Figure 1. Now consider the corresponding $\overline{d}_{\lambda,\mu}(i)$ in Brenti's expression

$$M(m,m^q)_{\lambda,\mu} = \overline{D}_{(\lambda,\mu)}(q) = \sum_{i=1}^{l(\mu)} \overline{d}_{\lambda,\mu}(i)\binom{q}{i}$$

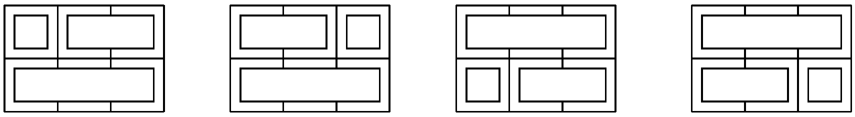

FIGURE 1. The four $(1, 2, 3)$-brick tabloids of shape $(3, 3)$.

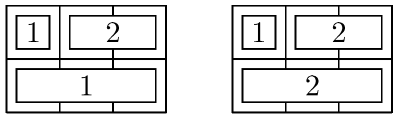

FIGURE 2. The two strict 2-brick tabloids of shape $(3, 3)$ and type $(1, 2, 3)$ corresponding to the first tabloid in Figure 1.

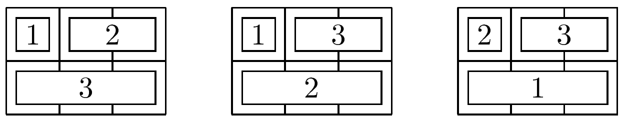

FIGURE 3. The three strict 3-brick tabloids of shape $(3, 3)$ and type $(1, 2, 3)$ that contain all three of the labels 1, 2, and 3, corresponding to the first tabloid in Figure 1.

For each $i = 1$ to 3, to calculate $\overline{d}_{\lambda,\mu}(i)$ we need to label the bricks in each tabloid in Figure 1 with the integers 1 through i such that each label is used at least once and the labels strictly increase in each row. Since each tabloid has at least one row with two bricks, $\overline{d}_{\lambda,\mu}(1) = 0$. Each tabloid in Figure 1 gives rise to two 2-brick tabloids, shown in Figure 2. So $\overline{d}_{\lambda,\mu}(2) = 8$. Finally, there are three 3-brick tabloids labeled with 1, 2, and 3 corresponding to each tabloid in Figure 1, as shown in Figure 3. So $\overline{d}_{\lambda,\mu}(3) = 12$. Hence Brenti's expression involves counting twenty tabloids instead of four.

We note that our expressions for $M(m, e^q)$ and $M(m, m^q)$ follow from Brenti's by letting q be a positive integer and giving a short combinatorial argument. We can also prove our results directly, and it is these proofs that generalize nicely when we extend Brenti's results to the hyperoctahedral group.

It is interesting to note that as a corollary to Theorem 2 we obtain a combinatorial interpretation of the coefficients used to expand an arbitrary product of binomial coefficients as a sum of binomial coefficients:

Corollary 3. *For* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$,

$$\prod_{i=1}^{l} \binom{q}{\lambda_i} = \sum_{i=1}^{n} \overline{d}_{\lambda,1^n}(i) \binom{q}{i}$$

where $\overline{d}_{\lambda,1^n}(i)$ *is the number of ways to fill the Ferrers diagram of* λ *with the integers* $1, 2, \ldots, i$ *so that the entries in rows strictly increase and every integer is used at least once.*

3. A Class of (q,t)-Symmetric Functions on B_n

In this section we define a class of symmetric functions with two parameters on the hyperoctahedral group B_n analogous to Brenti's symmetric functions on S_n. We have a complete list of transition matrices that expand these new symmetric functions in terms of the standard bases for the symmetric functions on B_n. We will present those analogous to $M(m,e^q)$ and $M(m,m^q)$ here.

We start by reviewing the necessary definitions. Let B_n be the hyperoctahedral group on n elements, that is, the wreath product of Z_2 with S_n. We will think of B_n as the group of *signed permutations.* That is, each element $\sigma \in B_n$ can be written as a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where each σ_i can be positive or negative. We can write elements of B_n in cycle notation with cycles of the form

$$(\epsilon_1 i_1, \epsilon_2 i_2, \epsilon_3 i_3, \ldots, \epsilon_m i_m)$$

where $\epsilon_i \in \{-1,1\}$ and it is understood that i_1 is mapped to $\epsilon_2 i_2$, i_2 is mapped to $\epsilon_3 i_3$ and so on.

As an example, the following is an element of B_8:

$$\sigma = -4, 8, -2, -7, -3, -6, 1, 5$$

In cycle notation we have

$$\sigma = (1,-4,-7)(-2,8,5,-3)(-6)$$

We say that a cycle of $\sigma \in B_n$ is a *positive cycle* if the product of the signs in the cycle is positive and a *negative cycle* if the product of the signs is negative. In the example above, the first two cycles are positive cycles and the third cycle is a negative cycle. Let $(\lambda,\mu) \vdash n$ denote an ordered pair of partitions λ and μ such that $|\lambda|+|\mu| = n$. The conjugacy classes of B_n are indexed by such pairs of partitions in the following way. If $B_n(\lambda,\mu)$ denotes the conjugacy class corresponding to (λ,μ), then $B_n(\lambda,\mu)$ is the set of all $\sigma \in B_n$ such that the positive cycles of σ have lengths $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$ and the negative cycles of σ have lengths $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$.

We associate symmetric functions to B_n following Stembridge [5]. Let $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$ be commuting alphabets. The appropriate vector space is defined by

$$\Lambda_{B_n}[X,Y] = \bigoplus_{k=0}^{n} \Lambda_k[X] \otimes \Lambda_{n-k}[Y]$$

The ten standard bases of Λ_{B_n} are products of the standard S_n bases:

$$\begin{array}{ll}
\{p_\lambda[X]p_\mu[Y]\}_{(\lambda,\mu)\vdash n} & \{m_\lambda[X+Y]m_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} \\
\{f_\lambda[X+Y]f_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} & \{m_\lambda[X+Y]f_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} \\
\{f_\lambda[X+Y]m_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} & \{h_\lambda[X+Y]h_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} \\
\{e_\lambda[X+Y]e_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} & \{h_\lambda[X+Y]e_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} \\
\{e_\lambda[X+Y]h_\mu[X-Y]\}_{(\lambda,\mu)\vdash n} & \{s_\lambda[X+Y]s_\mu[X-Y]\}_{(\lambda,\mu)\vdash n}
\end{array}$$

We are now ready to define an analogue of Brenti's symmetric functions on Λ_{B_n}. We define

$$(p_n[X]p_m[Y])^{q,t} = p_n^q[X]p_m^t[Y] = q \cdot p_n[X] \cdot t \cdot p_m[Y]$$

With this definition we have

$$(p_\lambda[X]p_\mu[Y])^{q,t} = p_\lambda^q[X]p_\mu^t[Y] = q^{l(\lambda)}t^{l(\mu)}p_\lambda[X]p_\mu[Y]$$

So in general for $P \in \Lambda_{B_n}$, we expand P in the power basis so that $P = \sum_{(\lambda,\mu)} a_{\mu,\lambda} p_\lambda[X]p_\mu[Y]$ and define $P^{q,t}$ by

$$P^{q,t} = \sum_{(\lambda,\mu)} a_{\mu,\lambda} p^q_\lambda[X]p^t_\mu[Y] = \sum_{(\lambda,\mu)} a_{\mu,\lambda} q^{l(\lambda)} t^{l(\mu)} p_\lambda[X]p_\mu[Y]$$

We will use notation similar to the S_n case to describe transition matrices. For simplicity, we will set

$$\begin{aligned} p_\lambda \tilde{p}_\mu &= p_\lambda[X]p_\mu[Y] \\ h_\lambda \tilde{e}_\mu &= h_\lambda[X+Y]e_\mu[X-Y] \end{aligned}$$

and so on. We define $M(a\tilde{b}, c\tilde{d})$ by the equation

$$\langle c_\lambda \tilde{d}_\mu \rangle_{(\lambda,\mu)\vdash n} = \langle a_\lambda \tilde{b}_\mu \rangle_{(\lambda,\mu)\vdash n} M(a\tilde{b}, c\tilde{d})$$

So $M(a\tilde{b}, c\tilde{d})$ is the matrix that transforms the basis $\langle a_\lambda \tilde{b}_\mu \rangle_{(\lambda,\mu)\vdash n}$ into the basis $\langle c_\lambda \tilde{d}_\mu \rangle_{(\lambda,\mu)\vdash n}$ and the $((\alpha,\beta),(\lambda,\mu))$ entry of $M(a\tilde{b}, c\tilde{d})$ is given by

$$c_\lambda \tilde{d}_\mu = \sum_{(\alpha,\beta)\vdash n} a_\alpha \tilde{b}_\beta M(a\tilde{b}, c\tilde{d})_{(\alpha,\beta),(\lambda,\mu)}$$

We now present results analogous to Theorems 1 and 2.

Theorem 4. *For $(\lambda,\mu) \vdash n$ and $(\rho,\epsilon) \vdash n$,*

$$\begin{aligned} &M(m\tilde{m}, (e\tilde{e})^{q,t})_{(\rho,\epsilon)(\lambda,\mu)} \\ &= \sum_{M \in M((\lambda,\mu)(\rho,\epsilon))} \prod_{\substack{1\le i\le l(\lambda)\\ 1\le j\le l(\rho)}} \binom{q/2+t/2}{M_{i,j}} \prod_{\substack{1\le i\le l(\lambda)\\ l(\rho)+1\le j\le l(\rho)+l(\epsilon)}} \binom{q/2-t/2}{M_{i,j}} \\ &\quad \prod_{\substack{l(\lambda)+1\le i\le l(\lambda)+l(\mu)\\ 1\le j\le l(\rho)}} \binom{q/2-t/2}{M_{i,j}} \prod_{\substack{l(\lambda)+1\le i\le l(\lambda)+l(\mu)\\ l(\rho)+1\le j\le l(\rho)+l(\epsilon)}} \binom{q/2+t/2}{M_{i,j}} \end{aligned}$$

where $M((\lambda,\mu)(\rho,\epsilon))$ is the set of $(l(\lambda)+l(\mu)) \times (l(\rho)+l(\epsilon))$ matrices with nonnegative integer entries such that the row sums form the sequence $\lambda_1,\ldots,\lambda_{l(\lambda)}, \mu_1,\ldots,\mu_{l(\mu)}$ and the column sums form the sequence $\rho_1,\ldots,\rho_{l(\rho)}, \epsilon_1,\ldots,\epsilon_{l(\epsilon)}$ and $M_{i,j}$ denotes the (i,j) entry of the matrix M.

For the next result we need to define B_n versions of brick tabloids. For partitions ρ and ϵ, let $\rho * \epsilon$ be the diagram obtained by attaching the lower right corner of the Ferrers diagram of ρ to the upper left corner of the Ferrers diagram of ϵ. Then for partitions λ and μ, we distinguish λ-bricks from μ-bricks and define the set of (λ,μ)-brick tabloids of shape $\rho * \epsilon$ to be the set of fillings of the diagram $\rho * \epsilon$ with λ-bricks and μ-bricks such that

- each brick lies in a single row
- no two bricks overlap
- λ bricks come before μ bricks in each row.

We will denote this set $\mathcal{B}^{\lambda,\mu}_{\rho*\epsilon}$. For example, Figure 4 shows an element of $\mathcal{B}^{\lambda,\mu}_{\rho*\epsilon}$ for $\lambda = (1,1,2,3)$, $\mu = (1,1,2,3,3)$, $\rho = (2,7)$ and $\epsilon = (2,2,3)$.

We also need an expression analogous to $\overline{D}_{\lambda,\mu}(q)$. For $T \in \mathcal{B}^{\lambda,\mu}_{\rho*\epsilon}$, let $n^\lambda_{i,\rho}(T)$ be the number of λ-bricks in the i-th row of ρ in T and define $n^\mu_{i,\rho}(T)$, $n^\lambda_{i,\epsilon}(T)$, and $n^\mu_{i,\epsilon}(T)$ analogously.

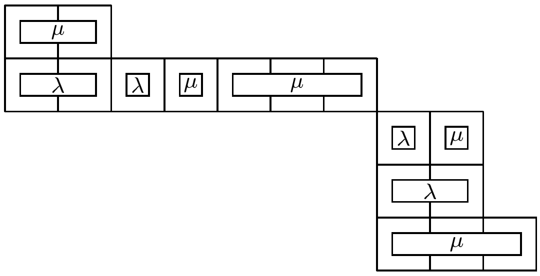

FIGURE 4. An element of $\mathcal{B}^{(1,1,2,2),(1,1,2,3,3)}_{(2,7)*(2,2,3)}$.

Then define

$$\overline{D}_{(\rho,\epsilon),(\lambda,\mu)}(q,t) = \sum_{T\in\mathcal{B}^{\lambda,\mu}_{\rho*\epsilon}} \prod_{i=1}^{l(\rho)} \binom{q/2+t/2}{n^{\lambda}_{i,\rho}(T)}\binom{q/2-t/2}{n^{\mu}_{i,\rho}(T)} \cdot \prod_{i=1}^{l(\epsilon)} \binom{q/2-t/2}{n^{\lambda}_{i,\epsilon}(T)}\binom{q/2+t/2}{n^{\mu}_{i,\epsilon}(T)}$$

The B_n version of Theorem 2 is then

Theorem 5. *For $(\lambda,\mu)\vdash n$ and $(\rho,\epsilon)\vdash n$,*

$$M(m\tilde{m},(m\tilde{m})^{q,t})_{(\rho,\epsilon)(\lambda,\mu)} = \overline{D}_{(\rho,\epsilon)(\lambda,\mu)}(q,t)$$

We note that we can also give expressions analogous to Brenti's version of $\overline{D}_{\lambda,\mu}(q)$.

Theorem 6. *For $(\lambda,\mu)\vdash n$ and $(\rho,\epsilon)\vdash n$,*

$$\overline{D}_{(\rho,\epsilon),(\lambda,\mu)}(q,t) = \sum_{\substack{1\le i+k\le l(\lambda)\\ 1\le j+l\le l(\mu)}} \overline{d}_{\lambda,\mu,\rho,\epsilon}(i,j,k,l)\binom{q/2+t/2}{i}\binom{q/2-t/2}{j}\binom{q/2-t/2}{k}\binom{q/2+t/2}{l}$$

where $\overline{d}_{\lambda,\mu,\rho,\epsilon}(i,j,k,l)$ is the number of (λ,μ)-brick tabloids of shape $\rho*\epsilon$ such that:

- *λ-bricks come before μ bricks in each row*
- *λ bricks in ρ are labelled with $1,2,\ldots,i$ with every value appearing at least once and the labels strictly increasing in rows*
- *μ-bricks in ρ are labelled with $1,2,\ldots,j$, with every value appearing at least once and the labels strictly increasing in rows*
- *λ-bricks in ϵ are labelled with $1,2,\ldots,k$ with every value appearing at least once and the labels strictly increasing in rows*
- *μ-bricks in ϵ are labelled with $1,2,\ldots,l$ with every value appearing at least once and the labels strictly increasing in rows.*

4. Generating Functions

In this section we present some generating functions on permutation statistics that arise from the symmetric functions introduced in the preceding sections. First, Brenti introduces a homomorphism from Λ_n to polynomials in one variable over the rationals which, when applied to the standard bases of Λ_n, gives generating functions on permutation statistics

[3]. Brenti extends this homomorphism to $\Lambda \otimes \mathbf{Q}(q)$ in [4] and derives similar generating functions for Brenti's q-symmetric functions. We will discuss these below and then give extensions to the B_n versions. Next, we present generating functions that arise by substituting Brenti's symmetric functions and extensions into the well-known identity

$$\sum_{n\geq 0} u^n h_n = \frac{1}{\sum_{n\geq 0}(-u)^n e_n}$$

Brenti's homomorphism $\xi : \Lambda_{\mathbf{Q}} \to \mathbf{Q}[x]$ is defined by setting

$$\xi(e_n) = \frac{(1-x)^{n-1}}{n!} \tag{2}$$

Since the e_μ's are a basis of $\Lambda_{\mathbf{Q}}$, this defines ξ on $\Lambda_{\mathbf{Q}}$. The homomorphism is then extended to $\Lambda \otimes \mathbf{Q}(q)$ by letting ξ act on $\mathbf{Q}(q)$ as it does on scalars.

Recall that for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, i is a called an *excedance* of σ if $\sigma_i > i$. We denote the number of excedances in σ by $\mathrm{exc}(\sigma)$. Also let $\mathrm{cyc}(\sigma)$ denote the number of cycles in σ.

Brenti shows the following [4]:

Theorem 7. *For ξ defined in (2) and a positive integer n,*

1. $n!\xi\left(e_n^q\right) = (-1)^n \sum_{\sigma\in S_n} x^{\mathrm{exc}(\sigma)}(-q)^{\mathrm{cyc}(\sigma)}$
2. $n!\xi\left(h_n^q\right) = \sum_{\sigma\in S_n} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}$
3. $\frac{n!}{z_\mu}\xi\left(p_\mu^q\right) = \sum_{\sigma\in S_n(\mu)} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}$
4. $n!\xi\left(s_\mu^q\right) = \sum_{\sigma\in S_n} \chi^\mu(\sigma) x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}$

where $S_n(\mu)$ is the conjugacy class of S_n associated with μ, χ^μ is the irreducible S_n character associated with μ, and if $m_i(\mu)$ denotes the number of parts of μ of size i, then $z_\mu = 1^{m_1(\mu)}\cdots n^{m_n(\mu)} m_1(\mu)!\cdots m_n(\mu)!$.

Beck [1] defines an analogue of Brenti's homomorphism, $\zeta : \Lambda_B \to Q[x]$, by setting

$$\zeta(e_k[X+Y]) = \frac{(1-x)^{k-1} + x(x-1)^{k-1}}{2^k k!} \tag{3}$$

$$\zeta(e_k[X-Y]) = \frac{(1-x)^{k-1} - x(1-x)^{k-1}}{2^k k!} = \frac{(1-x)^k}{2^k k!} \tag{4}$$

As with ξ, we extend ζ to our (q,t)-symmetric functions by treating q and t as scalars.

The statistic that arises here is a B_n version of *decedances.* We first define a linear ordering Θ by setting

$$1 <_\Theta 2 <_\Theta \cdots <_\Theta n <_\Theta \cdots <_\Theta -n <_\Theta \cdots <_\Theta -2 <_\Theta -1$$

Then for $\sigma \in B_n$, write σ in cycle notation as

$$\sigma = (\sigma_{1_1}\sigma_{1_2}\cdots\sigma_{1_{l(1)}})(\sigma_{2_1}\sigma_{2_2}\cdots\sigma_{2_{l(2)}})\cdots(\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_{l(k)}})$$

A B_n decedance occurs at the j^{th} position of the i^{th} cycle if either $1 \leq j < l(i)$ and $\sigma_{i_j} >_\Theta \sigma_{i_{j+1}}$, or $j = l(i)$ and $\sigma_{i_{l(i)}} >_\Theta \sigma_{i_1}$. The number of B_n decedances of σ is denoted $\mathrm{de}_B(\sigma)$.

Beck [1] shows that:

Theorem 8. *For ζ defined in equations (3) and (4) and a positive integer n,*

$$\frac{2^n n!}{z_\lambda z_\mu}\zeta(p_\lambda[X]p_\mu[Y]) = \sum_{\sigma\in B_n(\lambda,\mu)} x^{\mathrm{de}_B(\sigma)}$$

where $B_n(\lambda,\mu)$ is the set of B_n permutations with positive cycles of type λ and negative cycles of type μ.

From this we can derive results analagous to Brenti's S_n results:

Theorem 9. *For ζ defined in equations (3) and (4) and a positive integer n,*

(1) $\frac{2^n n!}{z_\lambda z_\mu}\zeta(p_\lambda^q[X]p_\mu^t[Y]) = \sum_{\sigma\in B_n(\lambda,\mu)} x^{\mathrm{de}_B(\sigma)}q^{\mathrm{poscyc}(\sigma)}t^{\mathrm{negcyc}(\sigma)}$

(2) $2^n n!\zeta(h_n^{q,t}[X+Y]) = \sum_{\sigma\in B_n} x^{\mathrm{de}_B(\sigma)}q^{\mathrm{poscyc}(\sigma)}t^{\mathrm{negcyc}(\sigma)}$

(3) $2^n n!\zeta(h_n^{q,t}[X-Y]) = \sum_{\sigma\in B_n} x^{\mathrm{de}_B(\sigma)}q^{\mathrm{poscyc}(\sigma)}(-t)^{\mathrm{negcyc}(\sigma)}$

(4) $2^n n!\zeta(e_n^{q,t}[X+Y]) = (-1)^n\sum_{\sigma\in B_n} x^{\mathrm{de}_B(\sigma)}(-q)^{\mathrm{poscyc}(\sigma)}(-t)^{\mathrm{negcyc}(\sigma)}$

(5) $2^n n!\zeta(e_n^{q,t}[X-Y]) = (-1)^n\sum_{\sigma\in B_n} x^{\mathrm{de}_B(\sigma)}(-q)^{\mathrm{poscyc}(\sigma)}t^{\mathrm{negcyc}(\sigma)}$

(6) $2^n n!\zeta((s_\lambda[X+Y]s_\mu[X-Y])^{q,t}) = \sum_{\sigma\in B_n} \chi^{\lambda,\mu}(\sigma)x^{\mathrm{de}_B(\sigma)}q^{\mathrm{poscyc}(\sigma)}t^{\mathrm{negcyc}(\sigma)}$

where for $\sigma\in B_n$, $\mathrm{poscyc}(\sigma)$ is the number of positive cycles of σ and $\mathrm{negcyc}(\sigma)$ is the number of negative cycles of σ and $\chi^{\lambda,\mu}$ is the irreducible character of B_n associated with the conjugacy class $B_n(\lambda,\mu)$.

Next we present two results arising from the well-known identity

$$\sum_{n\geq 0} u^n h_n = \frac{1}{\sum_{n\geq 0}(-u)^n e_n} \tag{5}$$

Substituting Brenti's symmetric functions on S_n into (5), we obtain:

Theorem 10. *For a positive integer n,*

$$\sum_{n\geq 0}\frac{u^n}{n!}\sum_{\sigma\in S_n} x^{\mathrm{exc}(\sigma)}q^{\mathrm{cyc}(\sigma)} = \frac{1}{\sum_{n\geq 0}\frac{(u(x-1))^n}{n!}\sum_{k=1}^n S_{n,k}\frac{(q)\downarrow_k}{(1-x)^k}}$$

where $S_{n,k}$ is the Stirling number of the second kind, that is, the number of set partitions of $\{1,2,\ldots,n\}$ into k parts.

We conclude with the analogous result for the B_n symmetric functions.

Theorem 11. *For a positive integer n,*

$$\sum_{n\geq 0}\frac{u^n}{2^n n!}\sum_{\sigma\in B_n} x^{\mathrm{de}_B(\sigma)}q^{\mathrm{poscyc}(\sigma)}t^{\mathrm{negcyc}(\sigma)}$$

$$= \frac{1}{\exp\left\{\frac{u(x-1)(q-t)}{4}\right\}\sum_{n\geq 0}\frac{(u(x-1))^n}{2^n n!}\sum_{k=1}^n S_{n,k}^{\mathrm{odd}}\left(\frac{1+x}{1-x}\right)(q/2+t/2)\downarrow_k}$$

where $S_{n,k}^{\mathrm{odd}}(y) = \sum_{\pi\in P_{n,k}} y^{\#\text{ odd parts}(\pi)}$ and $P_{n,k}$ is the set of partitions of $\{1,2,\ldots,n\}$ into k parts.

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