GENERALIZED PSEUDO-PERMUTATIONS

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ABSTRACT. We introduce and study combinatorial properties of pseudo-permutations with multiple occurences. We prove that the set of all such generalized pseudo-permutations of a given type have a lattice structure and derive a recursive formula to compute its cardinality, which turns out to be related to Stirling numbers of the second kind. Results about the longest and shortest chains are also obtained.

RÉSUMÉ. Nous introduisons et étudions des propiétés combinatoires des pseudo-permutations avec occurrences multiples. Nous montrons que l'ensemble de toutes ces pseudo-permutations généralisées d'un type donné possède une structure de treillis et dérivons une formule récursive pour calculer sa cardinalité, qui est reliée aux nombres de Stirling du second type. Résultats sur les chaînes les plus longues et les chaînes les plus courtes sont également obtenus.

1. INTRODUCTION

One of the most active and important research area in Computer Science and in particular, Artificial Intelligence is representation of temporal knowledge (see, *e.g.*, [All81, Yu83], where one needs to consider a set of events which happen at certain dates and wants to use this information to solve a problem, take a decision. In this context, it is natural to represent the temporal relations between n events by an ordered sequence of nonempty parts, each part corresponds to events which happen at the same time.

For example, the sequence $(2,4)(1,2)(3)$ means that the events 2 and 4 occur first at the same time, then event 1 and event 2 (for the second time) occur, and event 3 occurs last. If we add the constraint that each event occurred exactly once, then we have a so-called *pseudo-permutation* of order n. This is a new combinatorial object which was introduced in the five author paper [KLNPS] and studied futher in [BHKN01]. They showed that there is a natural partial order on the set of all pseudo-permutation of a given order, and surprisingly, turns out to be a lattice. This new object is not only combinatorially interesting in its own right, but turns out to be closely related to *S-arrangement* in the field of formal languages [DS00, Sch97].

In this paper, we deal with the more general (and more natural) case where each event can occur several times; in other words, pseudo-permutations with multiple occurrences. We show that most of the properties of pseudo-permutations can be generalized to this situation. Our main results are proofs of the lattice property of the set of all generalized (pseudo-)permutation of a given type K, and a recursive formula to compute its size by using Stirling numbers of the second kind. Results about the longest and shortest chains are also obtained. Geometric applications, and applications in representation theory of the symmetric groups will appear in a subsequence paper.

Let us now recall some basic definitions and introduce our cast of characters. Let n be a positive integer number. Let $\mathfrak{S}(n)$ be the group of all permutations of n numbers $1 \le i \le n$. There is a natural partial order on $\mathfrak{S}(n)$ by requiring that σ' is *covered* by σ, and write $\sigma \succ \sigma'$, if σ' is obtained from σ by switching two *consecutive* numbers i and j in σ where $i < j$. The resulting poset is what usually called *permutohedron* of order n, and is denoted by $\mathcal{P}(n)$. A *pseudo-permutation* of order n is a sequence of non-empty parentheses of numbers from 1 to n. Let $\mathcal{PP}(n)$ denote the set of all such pseudo-permutations of order n. Note that the order inside each parentheses is irrelevant (events occur at the same time!), and we make the convention that the numbers in each parentheses will always be written in increasing order.

As in the case of permutations, there is also a partial order on $\mathcal{PP}(n)$. Say that a pseudo-permutation $\mathfrak{s}'$ is *covered* by $\mathfrak{s}$, and write $\mathfrak{s} \succ \mathfrak{s}'$, if and only if it can be obtained from the latter by applying one of the following two operators:

- The merging operators M_i: If each element of the parentheses $(i)^{th}$ is smaller than all elements of the $(i+1)^{th}$, then M can combine these two parentheses into a single one.
- The splitting operators $S_{i,j}$: the $(i)^{th}$ parentheses is split into two, the *second* one is composed of the first j smallest elements.

For example, M_2 ((3567) (1) (2) (4)) = (3567) (12) (4); and $S_{1,3}$ ((3567) (12) (4)) = (7) (356) (12) (4). We also use the notation $\mathcal{PP}(n)$ for the resulting poset of all pseudo-permutations of order n and call it *pseudo-permutohedron of order* n. Figure 1 is an example of such a pseudo-permutohedron for $n = 3$. Of course, the complexity of these posets increase exponentially with n.

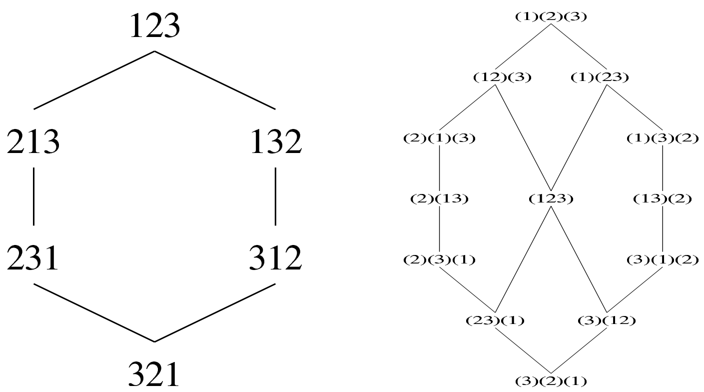

FIGURE 1. The permutohedron $\mathcal{P}(3)$ (left) and the pseudo-permutohedron $\mathcal{PP}(3)$ (right). The order orientation is top-down.

We will now define generalized (pseudo-)permutation. Let m be a positive integer, and $K = (k_1, \ldots, k_n)$ be a composition of m, *i.e.* k_i are positive integers and $\sum k_i = m$. Let I be the sequence of integers $(1, \ldots, 1, \ldots, n, \ldots, n)$ where each integer i appears k_i times. Sometimes, in order to distinguish between numbers of the same symbols, we write $\mathcal{I} = (1_1, \ldots 1_{k_1}, \ldots, n_1 \ldots n_{k_n})$. A *generalized permutation of type* K is, by definition, a permutation of I.

Let $\mathcal{GP}(K)$ be the set of all generalized permutations of type K. The *generalized permutohedron* of type K, also denoted by $\mathcal{GP}(K)$, is the poset over $\mathcal{GP}(K)$ where σ' is *covered* by σ if σ' is obtained from σ by switching two *consecutive* numbers i and j in σ where $i < j$.

In the special case when each letter i appear exactly once, then $\mathcal{GP}(K)$ is just the usual group $\mathfrak{S}(m)$ of all permutations of order m. Given a generalized-permutation, again one can try to put their numbers into non-empty parts by parentheses, such that in every parentheses, each number appears at most once. Such a partition is called a *generalized pseudo-permutation*. Let $\mathcal{GPP}(K)$ be the set of all generalized pseudo-permutations of type K. As in the case of pseudo-permutation, one also has the merging operator M, and the splitting operator S on $\mathcal{GPP}(K)$. The *generalized pseudo-permutohedron* of type K, also denoted by $\mathcal{GPP}(K)$, is the poset over $\mathcal{GPP}(K)$ where $\mathfrak{s}'$ is *covered* by $\mathfrak{s}$ if it can be obtained from $\mathfrak{s}$ by applying one of these two operators.

Figures 2 and 3 are pictures of the generalized-permutohedron $\mathcal{GP}(K)$ and the generalized pseudo-permutohedron $\mathcal{GPP}(K)$ where $K = (2, 1, 1)$.

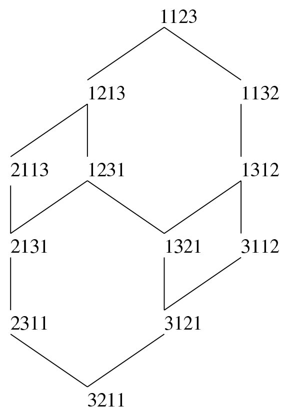

FIGURE 2. The generalized permutohedron of (211).

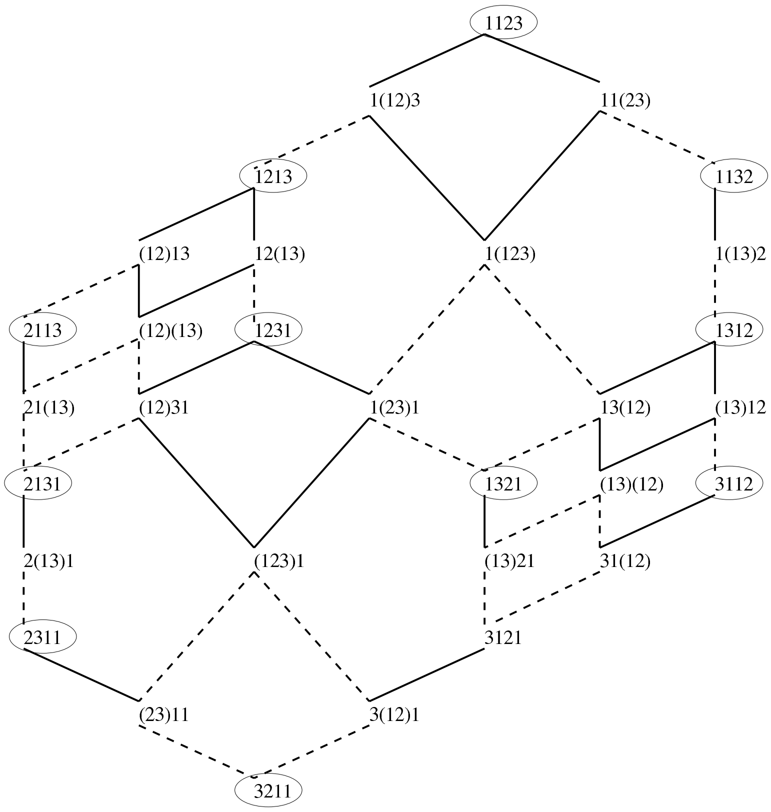

FIGURE 3. The generalized pseudo-permutohedron of (211).

2. Lattice properties of $\mathcal{GP}(K)$ and $\mathcal{GPP}(K)$

Let m be a positive integer and let $K = (k_1, \ldots, k_n)$ be a composition of m. In [KLNPS], it is showed that the poset $\mathcal{P}(m)$ is in fact a lattice. Their proof make use of the inversion table associated with each pseudo-permutation, and complicated construction of the infimum and supremum. We are going to show that the newly introduced posets $\mathcal{GP}(K)$ as well as $\mathcal{GPP}(K)$ enjoy similar property. But in order to prove this, we just have to show that there is a canonical embedding ϕ of $\mathcal{GP}(K)$ into $\mathcal{P}(m)$ whose image is a sub-lattice of $\mathcal{P}(m)$; similar statement holds for $\mathcal{GPP}(K)$. We refer to [DP90] for basic reference about lattice theory.

Let $\mathcal{I}$ be the sequence $(1_1, \ldots, 1_{k_1}, \ldots, n_1, \ldots, n_{k_n})$. We say that the elements i_ts are of the same symbol. We define a natural order between elements of $\mathcal{I}$ as follows: $1_1 < \ldots < 1_{k_1} < \ldots < n_1 < \ldots, n_{k_n}$. The permutohedron $\mathcal{P}(\mathcal{I})$ and the pseudo-permutohedron $\mathcal{PP}(\mathcal{I})$ are defined, similar to $\mathcal{P}(m)$ and $\mathcal{PP}(m)$. In fact, it it evident that $\mathcal{P}(\mathcal{I})$ (resp. $\mathcal{PP}(\mathcal{I})$) is isomorphic as a lattice to $\mathcal{P}(m)$ (resp. $\mathcal{PP}(m)$).

The embedding ϕ from $\mathcal{GP}(K)$ (resp. $\mathcal{GPP}(K)$) to $\mathcal{P}(\mathcal{I})$ (resp. $\mathcal{PP}(\mathcal{I})$) is done by just replacing the sequence $I = (\underbrace{1, \ldots, 1}_{k_1 \text{ times}}, \ldots, \underbrace{n, \ldots, n}_{k_n \text{ times}})$ with $\mathcal{I} = (1_1, \ldots, 1_{k_1}, \ldots, n_1, \ldots, n_{k_n})$. It is easy to see that the image of ϕ of $\mathcal{GP}(K)$ (resp. $\mathcal{GPP}(K)$) is the set of all elements such that for any sub-indices $r < s$, the parentheses containing i_r is on the left of that for i_s. In other words, the image of $\mathcal{GP}(K)$ (resp. $\mathcal{GPP}(K)$) is the interval from the element $(1_1 \ldots 1_{k_1} \ldots n_1 \ldots n_{k_n})$ to the element $(n_1 \ldots n_{k_n} \ldots 1_1 \ldots 1_{k_1})$. It is a standard fact in lattice theory [DP90] that an interval in a lattice is a sub-lattice, hence we have:

Theorem 1. *The generalized permutohedron $\mathcal{GP}(K)$ is isomorphic to a sub-lattice of $\mathcal{P}(m)$, and the generalized pseudo-permutohedron $\mathcal{GPP}(K)$ is isomorphic to a sub-lattice of $\mathcal{PP}(m)$.*

3. Combinatorial properties of $\mathcal{GP}(K)$ and $\mathcal{GPP}(K)$

Our purpose in this section is to compute the cardinalities of $\mathcal{GP}(K)$ and $\mathcal{GPP}(K)$. We also obtain a decomposition of $\mathcal{P}(m)$ (resp. $\mathcal{PP}(m)$) as a disjoint union of $\mathcal{GP}(L)$ (resp. $\mathcal{GPP}(L)$) where L are subsequences of K. As a corollary, we obtain a recursive formula for the cardinality of $\mathcal{GPP}(K)$ in terms of $\mathcal{GPP}(L)$.

Recall that the Young subgroup $\mathfrak{S}(K)$ of $\mathfrak{S}(m)$ is defined as the product:

$$\mathfrak{S}(K) = \mathfrak{S}(k_1) \times \ldots \times \mathfrak{S}(k_n),$$

where the i^{th} component $\mathfrak{S}(k_i)$ acts on the set $\{i_1, \ldots, i_{k_i}\}$. It is clear that $\mathcal{GP}(K)$ is isomorphic to the coset $\mathcal{P}(\mathcal{I})/\mathfrak{S}(K)$, and its cardinality can be computed by the following well-known formula:

$$|\mathcal{GP}(K)| = \binom{m}{k_1, \ldots, k_n} = \frac{m!}{k_1! \ldots k_n!}.$$

For each element σ of $\mathfrak{S}(K)$, we define a map $\phi_\sigma : \mathcal{GP}(K) \to \mathcal{P}(\mathcal{I})$ by letting $\phi_\sigma(\mathfrak{s}) = \sigma(\mathfrak{s})$. For example, if $\mathcal{I} = (1_1, 1_2, 1_3, 2, 3_1, 3_2)$ and if $\sigma = (321) \times id \times (21)$ and $\mathfrak{s} = (23)1(13)1$, then $\phi_\sigma(\mathfrak{s}) = (23_2)1_3(1_2 3_1)1_1$.

It is easy to see that ϕ_σ is an injection which preserves the lattice structure. In other words, its image, denoted by $\mathcal{GP}(\sigma, K)$, is a sub-lattice of $\mathcal{P}(\mathcal{I})$. Note that all $\mathcal{GP}(\sigma, K)$ are isomorphic to $\mathcal{GP}(K)$ (by convention, $\mathcal{GP}(K)$ is $\mathcal{GP}(id, K)$). See Figure 4 for an example of lattice decomposition of $\mathcal{P}(1_1, 1_2, 2)$. We obtain then a decomposition of $\mathcal{P}(m)$:

Proposition 2. *$\mathcal{P}(m)$ is a disjoint union of lattices, each one is isomorphic to $\mathcal{GP}(K)$:*

$$\mathcal{P}(m) \cong \mathcal{P}(\mathcal{I}) = \bigsqcup_{\sigma \in \mathfrak{S}(K)} \mathcal{GP}(\sigma, K).$$

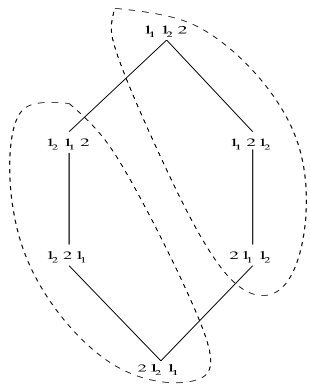

FIGURE 4. The decomposition of $\mathcal{P}(1_1, 1_2, 2)$ for $K = (2, 1)$.

Let us now consider the generalized pseudo-permutohedron $\mathcal{GPP}(K)$. For each σ in $\mathfrak{S}(K)$, we can also construct a lattice-monomorphism

$$\phi_\sigma : \mathcal{GPP}(K) \to \mathcal{GPP}(\sigma, K) \subset \mathcal{PP}(m),$$

which takes $\mathfrak{s}$ to $\sigma(\mathfrak{s})$. For every $\sigma \in \mathfrak{S}(K)$, the image $\mathcal{GPP}(\sigma, K)$ of ϕ_σ is isomorphic to $\mathcal{GPP}(K)$. But unlike the previous situation for $\mathcal{P}(\mathcal{I})$, $\mathcal{PP}(\mathcal{I})$ is not the disjoint union of all $\mathcal{GPP}(\sigma, K)$. Indeed, consider the case when $K = (2, 1)$ (see Figure 5). The three elements

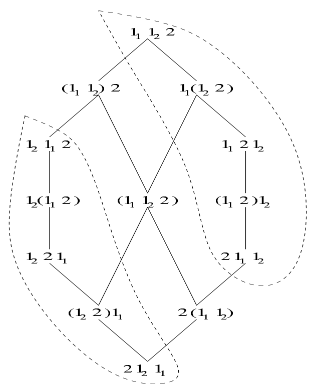

FIGURE 5. The decomposition of $\mathcal{PP}((2, 1))$.

$(1_1 1_2)(2), (1_1 1_2 2)$ and $(2)(1_1 1_2)$ are not in any of $\mathcal{GPP}(\sigma, K)$. The reason for this is that in $\mathcal{PP}(\mathcal{I})$, an element may have a parentheses containing a pair of numbers i_r, i_s of the same symbol, while this is forbidden in $\mathcal{GPP}(\sigma, K)$.

It remains to compute the set of all those extra elements in $\mathcal{PP}(\mathcal{I})$. Let us introduce the set $Par(\mathcal{I})$ of all sequences $\mathcal{J}$ obtained from $\mathcal{I}$ by parenthesizing several elements of the same symbol. For example, if $\mathcal{I} = (1_1, 1_2, 1_3, 2, 3)$, then $\mathcal{J}$ can be $1_3(1_1 1_2)23$ or $(1_1 1_3)1_2 23$ or $1_2 1_1 1_3 23$, etc. We define $\mathcal{GPP}(\mathcal{J})$ as the lattice of all elements $\mathfrak{s}$ obtained from $\mathcal{J}$ by using the two operators M and S, but numbers of the same symbol are not allowed to be merged or split. In other words, for any pair i_r, i_s, the relative position between parentheses containing i_r and that containing i_s does not change.

It is clear that the disjoint union of all lattices $\mathcal{GPP}(\mathcal{J})$ is $\mathcal{PP}(\mathcal{I})$.

$$\mathcal{PP}(\mathcal{I}) = \bigsqcup_{J \in Par(\mathcal{I})} \mathcal{GPP}(J).$$

On the other hand, let $L_J = (l_1, \dots l_n)$ where l_i is the number of parentheses of the symbol i in $\mathcal{J}$, then we see that the lattice $\mathcal{GPP}(\mathcal{J})$ is isomorphic to the lattice $\mathcal{GPP}(L_J)$ (in fact, if we identify a parentheses of the same symbol i in $\mathcal{J}$ to the number i, then $\mathcal{J}$ corresponds to the sequence $(\underbrace{1,\dots,1}_{l_1 \text{ times}}, \dots, \underbrace{n,\dots,n}_{l_n \text{ times}}))$. Write $L \le K$ if $1 \le l_i \le k_i$ for any i, moreover $L < K$ if $L \ne K$. Let $N(K, L)$ be the number of sequences $\mathcal{J}$ such that $L_J = L$, we obtain immediately:

Theorem 3. *The lattice $\mathcal{PP}(m)$ is isomorphic to a disjoint union of lattices $\mathcal{GPP}(L)$ with multiplicity $N(K, L)$.*

$$\mathcal{PP}(m) \cong \mathcal{PP}(\mathcal{I}) = \bigsqcup_{L \le K} N(K, L)\mathcal{GPP}(L). \tag{1}$$

From the above theorem, we need only to compute the multiplicity $N(K, L)$. It is easy to see that $N(K, L)$ is equal to the product $N(k_1, l_1) \times \dots \times N(k_n, l_n)$, where $N(k_i, l_i)$ is the number of partitions of k_i elements into l_i (ordered) parentheses. Hence, $N(k_i, l_i) = l_i! S(k_i, l_i)$, where $S(k_i, l_i)$ is the Stirling number of the second kind [Sta98]:

$$S(p, q) = \frac{1}{q!} \sum_{j=0}^{j=q} (-1)^{(q-j)} \binom{j}{q} j^p,$$

We deduce that:

$$N(k_i, l_i) = \sum_{j=0}^{j=l_i} (-1)^{(l_i - j)} \binom{j}{l_i} j^{k_i}. \tag{2}$$

In particular, $N(K, K) = k_1! \times \dots \times k_n! = |\mathfrak{S}(K)|$, the above relation implies a recursive formula to compute $|\mathcal{GPP}(K)|$:

$$|\mathcal{GPP}(K)| = \frac{|\mathcal{PP}(m)| - \sum_{L<K} N(K, L)|\mathcal{GPP}(L)|}{k_1! \times \dots \times k_n!}. \tag{3}$$

The cardinality of the pseudo-permutohedron $\mathcal{PP}(m)$ has been computed in [KLNPS], which turns out to be related to Eulerian numbers $a_{m,i}$ (See [FS70], [Com70]).

$$|\mathcal{PP}(m)| = \sum_{i=0}^{m-1} 2^i a_{m,i}.$$

Recall that $a_{m,i}$ is the number of permutations in $\mathfrak{S}(m)$ whose descent number is exactly $(i-1)$. By convention, $a_{m,0} = 1$.

Note that from the recursive formula for $|\mathcal{GPP}(K)|$ above, one can show by induction that the cardinality of $\mathcal{GPP}(K)$ is a symmetric function on $(k_1, \dots, k_n)$. That is, if K' is a permutation of K, then $|\mathcal{GPP}(K)| = |\mathcal{GPP}(K')|$. Of course, the same result holds for $|\mathcal{GP}(K)|$. However, $\mathcal{GP}(K')$ (resp. $\mathcal{GPP}(K''))$ is, in general, not isomorphic as a lattice to $\mathcal{GP}(K)$ (resp. $\mathcal{GPP}(K)$). Figure 6 is a picture of $\mathcal{GP}(1, 2, 1)$ which is not isomorphic to $\mathcal{GP}(2, 1, 1)$ (in Figure 6).

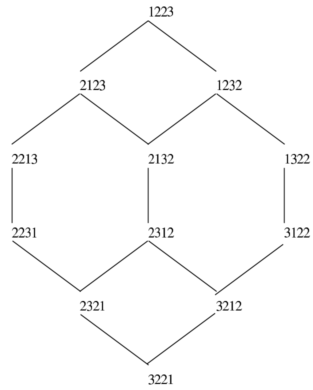

FIGURE 6. The generalized permutohedron of (121)

4. Longest and shortest chains

In this section, we will compute the length of the longest and shortest chains in a pseudo-permutohedron as well as in a generalized pseudo-permutohedron. Here we mean a *chain* in a lattice is a path from the maximal element to the minimal element. In order to prove our results, we need to make use of the notion of inversion of a pair (i, j) and the table of inversions of a pseudo-permutation. This is a crucial ingredient in the proof of the lattice property of the set of pseudo-permutations of order n [KLNPS].

Let $\mathfrak{s}$ be a pseudo-permutation of order n. For every pair of numbers $1 \leq i < j \leq n$, we associate a rational number called inversion, and denoted by *inv(i,j)* as follows:

$$inv(i,j) = \begin{cases} 1/2 & \text{if } i \text{ and } j \text{ are in the same parentheses,} \\ 0 & \text{if the parentheses of } i \text{ is on the left of that of } j, \\ 1 & \text{if the parentheses of } i \text{ is on the right of that of } j. \end{cases}$$

The *table of inversions* of $\mathfrak{s}$ is just the list of all non-zero inv(i,j) in $\mathfrak{s}$. The summation of all such inversion numbers is called the *inversion number* of $\mathfrak{s}$. For example, if $\mathfrak{s} = (4)(13)(2)$, then its table of inversions is:

$$\{\frac{1}{2}(1,3), 1(1,4), 1(2,3), 1(2,4), 1(3,4)\}$$

and its inversion number is $4\frac{1}{2}$. The number on the left of each pair (i, j) is its inversion.

It is easy to see that in the lattice $\mathcal{P}(m)$, each operator increase inversion number by 1. As a result, all chains in $\mathcal{P}(m)$ are of the same length. However, this is different in $\mathcal{PP}(m)$, M and S still increase inversion numbers, the least increase is $\frac{1}{2}$, which correspond to either merging two parentheses, each of a single element, into one; or splitting a parentheses which contains two elements. It follows from this remark that the longest chains should be those which contain only operations described above.

Proposition 4. *There is a natural bijection between the set of chains in $\mathcal{P}(n)$ and the set of longest chains in $\mathcal{PP}(n)$. Moreover, the length of the longest chains in $\mathcal{PP}(n)$ is $n(n-1)$.*

Proof. The bijection is given as follows. Given a chain of $\mathcal{P}(n)$, consider it as a subchain in $\mathcal{PP}(n)$ via the obvious inclusion (parenthesizing every single number); then between any two consecutive pseudo-permutations

$$\ldots(i)(j)\ldots, \ldots(j)(i)\ldots,$$

where $i < j$, insert a pseudo-permutation $\dots(ij)\dots$. The result is a chain of pseudo-permutations of the longest length possible, since it is easy to check that the inversion number increase by $\frac{1}{2}$ in the entire chain. For example, $123, 132, 312, 321$ is a chain in $\mathcal{P}(3)$, the corresponding chain in $\mathcal{PP}(3)$ is:

$$(1)(2)(3), (1)(23), (1)(3)(2), (13)(2), (3)(1)(2), (3)(12), (3)(2)(1).$$

The converse is also clear, using the observation in the previous paragraph. Note that the length of the longest chain in $\mathcal{PP}(n)$ is twice the difference of inversion number between the maximum element $(1)(2)\dots(n)$ and the minimum element $(n)(n-1)\dots(1)$ of $\mathcal{PP}(n)$. Since the inversion number of of $(1)(2)\dots(n)$ is 0, and the inversion number of $(n)(n-1)\dots(1)$ is $\frac{n(n-1)}{2}$, we conclude that the longest length is $n(n-1)$. □

The discussion above also applies to the "generalized" version, *i.e.* all chains in $\mathcal{GP}(K)$ are of the same length and there is a bijection between the set of chains in $\mathcal{GP}(K)$ and the set of longest chains in $\mathcal{GPP}(K)$. The inversion number of the minimal element $(n)\dots(n)\dots(1)\dots(1)$ is $\sum_{i<j} k_i k_j$, so every chain in $\mathcal{GP}(K)$ is of length $\sum_{i<j} k_i k_j$ and the longest length in $\mathcal{GPP}(K)$ is $2\sum_{i<j} k_i k_j$.

We next consider the shortest chains in $\mathcal{PP}(n)$.

Proposition 5. *There are $[(n-1)!]^2$ chains of shortest length in $\mathcal{PP}(n)$, and they each have length $(2n-2)$.*

Proof. We first show that any chain in $\mathcal{PP}(n)$ have length at least $(2n-2)$. Our key observation is that the operations M or S can either eliminate or increase at most one parentheses between any two numbers i and j at a time. Consider a chain from the maximum element $(1)(2)\dots(n)$ to the minimum one $(n)\dots(1)$. At the beginning, there are (n-1) parentheses between 1 and n, so one needs at least (n-1) steps to put 1 and n in the same parentheses. Again, one needs at least $(n-1)$ more moves to move 1 to the position on the far right of n, so that there are $(n-1)$ parentheses between them. Therefore, any chain in $\mathcal{PP}(n)$ has length at least $(2n-2)$. It remains now to provide an explicit chain which has this length $(2n-2)$. Here is an example,

$$(1)(2)\dots(n), (12)(3)\dots(n), \dots, (12\dots n),$$
$$(n)(12\dots n-1), (n)(n-1)(12\dots n-2), \dots, (n)(n-1)\dots(1).$$

Since in a shortest chain, the first $(n-1)$ steps decrease the number of parentheses between 1 and n, the n^{th} elements in the chain must be $(12\dots n)$. It is easy to see that there are $(n-1)!$ different paths from $(1)(2)\dots(n)$ to $(12\dots n)$, all have length $(n-1)$. By symmetry, there are also $(n-1)!$ paths from $(12\dots n)$ to $(n)(n-1)\dots(1)$. A chain of minimum length is obtained by combining these two paths. Conversely, every chain containing $(12\dots n)$ has length $2n-2$. The proposition follows immediately. □

Our method of computing the shortest chains does not work for generalized pseudo-permutations, however. Since there are more constraints over the merging operation M. For example, it is not allowed to merge the two parentheses (12) and (13) because 1 appears in both.

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