PLETHYSM OF SCHUR FUNCTIONS AND THE BASIC REPRESENTATION OF $A_2^{(2)}$

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ABSTRACT. Some formulas of plethysms $p_r \circ S_\lambda$ and $p_r \circ Q_\lambda$ are presented. In particular, for the case $p_2 \circ S_\lambda$ with rectangular λ, a relation with the basic representation of the affine Lie algebra of type $A_2^{(2)}$ is discussed.

RÉSUMÉ. Quelques formules de plethysms $p_r \circ S_\lambda$ et $p_r \circ Q_\lambda$ sont présenties. En particulier, pour le cas $p_2 \circ S_\lambda$ avec rectangulaire λ, une relation avec la repréntation basique de l'algèbre de Lie affine de type $A_2^{(2)}$ est discuté.

1. INTRODUCTION

Plethysm is one of fundamental problems in the theory of symmetric functions. Many authors have computed the plethysm $f \circ g$ for particular choices of symmetric functions f and g. This abstract deals with the plethysms $p_r \circ S_\lambda$ and $p_r \circ Q_\lambda$ involving the power sum symmetric functions, where S_λ is the Schur function indexed by the partition λ and Q_λ is Schur's Q-function indexed by the strict partition λ. The case $p_2 \circ S_\lambda$ where λ is the rectangular Young diagram is of particular interest. A nice formula is known for this plethysm which is proved by Carré-Leclerc [1] and Carini-Remmel [2].

One of our main contributions in this abstract is to understand their formula from the viewpoint of representations of affine Lie algebras. By looking at the homogeneous realization of the basic representation of $A_2^{(2)}$ we obtain another formula for $p_2 \circ S_\lambda$ where λ is rectangular.

Our formula should be regarded as an explicit expression of a homogeneous τ-function of a hierarchy of nonlinear differential equations, similar to the case of $A_1^{(1)}$ ([5]). Yet another result in this abstract is a formula for $p_r \circ Q_\lambda$, which is derived from certain factorization theorem of Q-functions.

As for this formula the relation with affine Lie algebras is unclear at this moment. However our proof requires a formula shown by You [17]. Therefore we believe that there is a nice explanation of our formula from the affine Lie algebra point of view.

The abstract is organized as follows. In Section 2 we fix some notations concerning the Schur functions. Section 3 is devoted to a brief review of bar-cores and bar-quotients of strict partitions. In Section 4 we discuss the basic representation of the affine Lie algebra of type $A_2^{(2)}$. The first main result is proved in Section 5. In Section 6 we announce some formulas of plethysms involving Schur's Q-functions. Proofs of these formulas are not given in this abstract.

2. SCHUR FUNCTIONS

We denote by P_n the set of all partitions of n, SP_n the set of all strict partitions of n and OP_n the set of those partitions of n whose parts are odd numbers. Let χ_ρ^λ be the irreducible character of the symmetric group S_n, indexed by $\lambda \in P_n$ and evaluated at the conjugacy class ρ, and ζ_ρ^λ be the irreducible *negative* character of the double cover $\tilde{S}_n$ (cf.

[4]), indexed by $\lambda \in SP_n$ and evaluated at the conjugacy class ρ. Here we recall symmetric functions of variables $\boldsymbol{x} = (x_1, x_2, \cdots)$ which are discussed in this abstract.

Let $p_r(\boldsymbol{x}) = \sum_{i\geq 1} x_i^r$ be the power sum symmetric function for $r \geq 1$. The Schur functions and the "big" Schur functions are defined as follows:

$$S_\lambda(\boldsymbol{x}) = \sum_{\rho\in P_n} z_\rho^{-1}\chi_\rho^\lambda p_\rho(\boldsymbol{x}),$$
$$T_\lambda(\boldsymbol{x}) = \sum_{\rho\in OP_n} z_\rho^{-1}\chi_\rho^\lambda 2^{l(\rho)} p_\rho(\boldsymbol{x}).$$

For $\lambda \in SP_n$ define Schur's Q-function and P-function by

$$Q_\lambda(\boldsymbol{x}) = \sum_{\rho\in OP_n} z_\rho^{-1}\zeta_\rho^\lambda 2^{(l(\lambda)+l(\rho)+\epsilon(\lambda))/2} p_\rho(\boldsymbol{x}),$$
$$P_\lambda(\boldsymbol{x}) = 2^{-l(\lambda)} Q_\lambda(\boldsymbol{x}),$$

where

$$\epsilon(\lambda) = \begin{cases} 0 & \text{if } n - l(\lambda) \text{ is even,} \\ 1 & \text{if } n - l(\lambda) \text{ is odd.} \end{cases}$$

Let $\boldsymbol{x} = (x_1, x_2, \cdots)$ and $\boldsymbol{y} = (y_1, y_2, \cdots)$ be variables. We write

$$\boldsymbol{x}^r = (x_1^r, x_2^r, \cdots),$$
$$\boldsymbol{xy} = (x_i y_j; i \geq 1, j \geq 1).$$

When $\boldsymbol{y}$ is specialized as $\boldsymbol{y} = (1, \omega, \omega^2, \cdots, \omega^{r-1}, 0, 0, \cdots)$ for $\omega = \exp(2\pi\sqrt{-1}/r)$, we write

$$\boldsymbol{x}\omega_r = \boldsymbol{xy}.$$

3. Bar-Cores and Bar-Quotients of Partitions

Fix a positive integer r. An $(r+1)$-tuple of partitions $(\lambda^{c(r)}, \lambda^0, \ldots, \lambda^{r-1})$ is attached to λ ; $\lambda^{c(r)}$ is the r-core of λ and the collection $\lambda^{q(r)} = (\lambda^0, \ldots, \lambda^{r-1})$ is the r-quotient of λ (cf. [15]).

Definition 3.1. Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a strict partition. We define the double of λ by

$$D(\lambda) = (\lambda_1, \cdots, \lambda_l \mid \lambda_1 - 1, \cdots, \lambda_l - 1),$$

in the Frobenius notation.

Example 3.2. Take $\lambda = (4, 2, 1) = \begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{array}$. The double of λ is

$$D(\lambda) = (5, 4, 4, 1) = \begin{array}{ccccc} \circ & \times & \times & \times & \times \\ \circ & \circ & \times & \times & \\ \circ & \circ & \circ & \times & \\ \circ & & & & \end{array}.$$

There is a remarkable property of $D(\lambda)$ as follows.

Proposition 3.3. ([10]) *Let λ be a strict partition and r be a positive odd integer.*
(1) *There exist strict partitions $\lambda^{bc(r)}$ and $\lambda^{b(0)}$ such that*

$$D(\lambda^{bc(r)}) = D(\lambda)^{c(r)},$$
$$D(\lambda^{b(0)}) = D(\lambda)^0.$$

(2)$D(\lambda)^{r-i}$ *is the partition conjugate to* $D(\lambda)^i$ *for* $1 \le i \le (r-1)/2$.

Definition 3.4. (1)The strict partition $\lambda^{bc(r)}$ is called the r-*bar core* of λ.
(2)The collection

$$\lambda^{bq(r)} = (\lambda^{b(0)}, \lambda^{b(1)}, \cdots, \lambda^{b(t)})$$

is called the r-*bar quotient* of λ where $t = (r-1)/2$ and $\lambda^{b(i)} = D(\lambda)^i$ for $1 \le i \le t$.

Example 3.5. We compute the 5-bar quotient of $\lambda = (15, 14, 13, 7, 6, 5, 3, 1)$. Adding 0's in the tail of $D(\lambda)$, if necessary, we always suppose that the size of the vector $D(\lambda)$ is the multiple of r. We see that

$$D(\lambda) + \delta_{15} = (30, 29, 28, 22, 21, 20, 18, 16, 13, 11, 7, 6, 5, 4, 3),$$

where $\delta_{15} = (14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$. To compute the 5-quotient of $D(\lambda)$, we put a set of beads on the positions assigned by $D(\lambda) + \delta_{15}$ as follows.

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 \\ 30 & 31 & 32 & 33 & 34 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \rightarrow ((4,3,1), (1^4), (3,1), (2,1^2), (4)).$$

Read each runner from the bottom and count the number of vacancies above each bead. Thus we have the corresponding partition $D(\lambda)^i$ $(i = 0, \cdots, 4)$. In this case the 5-quotient of $D(\lambda)$ reads $((4,3,1), (1^4), (3,1), (2,1^2), (4))$. This is the 5-bar quotient of λ; $\lambda^{bq(5)} = ((3,1), (1^4), (3,1))$. Moving each bead upwards in the runner successively as far as possible, we get

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \rightarrow (4,3,1) = D(3,1).$$

Thus we see that $D(\lambda)^{c(5)} = (4,3,1)$ and $\lambda^{bc(5)} = (3,1)$. Next we explain the r-*sign* and r-*bar sign* through the example above. Number the beads in the following two ways.
(1)The natural numbering according to the increasing order.
(2)The 5-numbering according to the *layers*.

$$\begin{array}{ccccc} \multicolumn{5}{c}{\text{natural numbering}} \\ 0 & 1 & 2 & 3_1 & 4_2 \\ 5_3 & 6_4 & 7_5 & 8 & 9 \\ 10 & 11_6 & 12 & 13_7 & 14 \\ 15 & 16_8 & 17 & 18_9 & 19 \\ 20_{10} & 21_{11} & 22_{12} & 23 & 24 \\ 25 & 26 & 27 & 28_{13} & 29_{14} \\ 30_{15} & 31 & 32 & 33 & 34 \end{array} \longrightarrow \begin{array}{ccccc} \multicolumn{5}{c}{\text{5-numbering}} \\ 0 & 1 & 2 & 3_4 & 4_5 \\ 5_1 & 6_2 & 7_3 & 8 & 9 \\ 10 & 11_7 & 12 & 13_9 & 14 \\ 15 & 16_{12} & 17 & 18_{13} & 19 \\ 20_6 & 21_{14} & 22_8 & 23 & 24 \\ 25 & 26 & 27 & 28_{15} & 29_{10} \\ 30_{11} & 31 & 32 & 33 & 34 \end{array}$$

When we compare the natural numbering with the 5-numbering we get a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 4 & 5 & 1 & 2 & 3 & 7 & 9 & 12 & 13 & 6 & 14 & 8 & 15 & 10 & 11 \end{pmatrix}.$$

The 5-sign of $D(\lambda)$, which is denoted by $\delta_5(D(\lambda))$, is defined to be the sign of σ.

$$\delta_5(D(\lambda)) = \mathrm{sgn}\sigma = 1.$$

To define the *r-bar sign* $\bar{\delta}_r(\lambda)$ of a strict partition λ, we draw the r-bar abacus of λ. If we take $\lambda = (15, 14, 13, 7, 6, 5, 3, 1)$ as before, then the 5-bar abacus of λ is

$$\begin{array}{ccccc} 0 & ① & 2 & ③ & 4 \\ ⑤ & ⑥ & ⑦ & 8 & 9 \\ 10 & 11 & 12 & ⑬ & ⑭ \\ ⑮ & 16 & 17 & 18 & 19 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}.$$

Number the beads in the following two ways.
(1)The natural numbering according to the increasing order.
(2)The 5-bar numbering according to the *layers*.

$$\begin{array}{ccccc} \multicolumn{5}{c}{\text{natural numbering}} \\ 0 & ①_1 & 2 & ③_2 & 4 \\ ⑤_3 & ⑥_4 & ⑦_5 & 8 & 9 \\ 10 & 11 & 12 & ⑬_6 & ⑭_7 \\ ⑮_8 & 16 & 17 & 18 & 19 \end{array} \longrightarrow \begin{array}{ccccc} \multicolumn{5}{c}{\text{5-bar numbering}} \\ 0 & ①_4 & 2 & ③_6 & 4 \\ ⑤_1 & ⑥_7 & ⑦_5 & 8 & 9 \\ 10 & 11 & 12 & ⑬_8 & ⑭_3 \\ ⑮_2 & 16 & 17 & 18 & 19 \end{array}$$

Since the full description of the r-bar numbering is rather complicated, we refer the reader to [13, pp. 64–65] or [15, pp. 32–33]. When we compare the natural numbering with the 5-bar numbering we get a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 1 & 7 & 5 & 8 & 3 & 2 \end{pmatrix}.$$

The 5-bar sign of λ is defined to be the sign of

$$\bar{\delta}_5(\lambda) = \mathrm{sgn}\sigma = -1.$$

Remark 3.6. (1)If $\lambda \in P_{rn}$ $(r, n \in \mathbb{N})$ has empty r-core, i.e., $\lambda^{c(r)} = \emptyset$, then

$$\delta_r(\lambda) = \chi^{\lambda}_{(r^n)}/|\chi^{\lambda}_{(r^n)}|.$$

(2) If $\lambda \in SP_{rn}$ $(r, n \in \mathbb{N}$, r is odd $)$ has empty r-bar core, i.e., $\lambda^{bc(r)} = \emptyset$, then

$$\bar{\delta}_r(\lambda) = \zeta^{\lambda}_{(r^n)}/|\zeta^{\lambda}_{(r^n)}|.$$

4. Basic Representation of $A_2^{(2)}$

We discuss in this section the basic representation of the affine Lie algebra of type $A_2^{(2)}$ following [7, 8]. Here the Schur functions, Schur's P and Q-functions are described in terms of the so called Sato variables: $u_j = p_j/j$ $(j \geq 1)$ for S_λ, $t_j = 2p_j/j$ $(j \geq 1,\ odd)$ for P_λ and Q_λ. We will denote them by $S_\lambda(u)$, $P_\lambda(t)$, $Q_\lambda(t)$, etc. Put $\Gamma = \mathbb{C}[t_j;\ j \geq 1,\ odd]$, whose basis is chosen as $\{P_\lambda;\ \lambda \in SP_n,\ n \in \mathbb{N}\}$. Associated with the Cartan matrix

$$(a_{ij})_{i,j\in\{0,1\}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix},$$

the Lie algebra $\mathfrak{g}$ of type $A_2^{(2)}$ is generated by $e_i, f_i, \alpha_i^\vee (i = 0,1)$ and d subject to the relations

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \qquad [\alpha_i^\vee, e_j] = a_{ij}e_j, \qquad [\alpha_i^\vee, f_j] = -a_{ij}f_j,$$
$$[e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \qquad (\mathrm{ad}e_i)^{1-a_{ij}}e_j = (\mathrm{ad}f_i)^{1-a_{ij}}f_j = 0 \quad (i \neq j),$$

and

$$[d, \alpha_i^\vee] = 0, \qquad [d, e_j] = \delta_{j,0}e_j, \qquad [d, f_j] = -\delta_{j,0}f_j.$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by $\alpha_0^\vee, \alpha_1^\vee$ and d. Choose the basis $\{\alpha_0, \alpha_1, \Lambda_0\}$ for the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by the pairing

$$< \alpha_i^\vee, \alpha_i > = a_{ij}, \qquad < \alpha_i^\vee, \Lambda_0 >= \delta_{i,0},$$
$$< d, \alpha_j > = \delta_{0,j}, \qquad < d, \Lambda_0 >= 0.$$

The fundamental imaginary root is $\delta = 2\alpha_0 + \alpha_1$.

The basic representation of $\mathfrak{g}$ is by definition the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ_0. The weight system of the basic representation is well known:

$$P(\Lambda_0) = \{\Lambda_0 - p\delta + q\alpha_1 \ ; \ p \leq q^2, p \in \mathbb{N}, \ q \in \frac{1}{2}\mathbb{Z}\}.$$

A weight Λ on the parabola $\Lambda_0 - p\delta + q\alpha_1$ is said to be maximal in the sense that $\Lambda + \delta$ is no longer a weight. For any maximal weight Λ, the multiplicity of $\Lambda - n\delta$ $(n \in \mathbb{N})$ is known to be equal to $p(n)$, the number of partitions of n. A construction of the basic representation in "principal" grading is realized on the space $\Gamma^{(3)} = \mathbb{C}[t_j; \ j \geq 1, \text{ odd}, \ j \not\equiv 0(\text{mod}3)]$ ([8]). A P-function $P_\lambda(t)$ is not necessarily contained in $\Gamma^{(3)}$. However, if the strict partition λ is a 3-bar core, then $P_\lambda(t) \in \Gamma^{(3)}$ and in fact $P_\lambda(t)$ is a maximal weight vector. More generally we "kill" the variables t_{3j} $(j \geq 1, \ odd)$ in the P-function $P_\lambda(t)$ and consider the reduced P-function:

$$P_\lambda^{(3)}(t) := P_\lambda(t)|_{t_3 = t_9 = \cdots = 0} \in \Gamma^{(3)}.$$

It is shown in [14] that $P_\lambda^{(3)}(t)$ is a weight vector for any strict partition λ, and that

$$\{P_\lambda^{(3)}(t); \ \lambda \text{ is a strict partition with no part divisible by } 3\}$$
$$=\{P_\lambda^{(3)}(t); \ \lambda \text{ is a strict partition with } \lambda^{bq(3)} = (\emptyset, \lambda^{b(1)})\}$$

form a weight basis for $\Gamma^{(3)}$. The weight of a reduced P-function with a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with 0 or 1 in such a way that, in each row the sequence (010) repeats from the left as long as possible. If k_0 (resp. k_1) is the number of 0's (resp. 1's) written in the Young diagram, then the weight of the corresponding reduced P-function is $\Lambda_0 - k_0\alpha_0 - k_1\alpha_1$. A removable i-node (i=0,1) is a node $\boxed{i}$ of the boundary of λ which can be removed. An indent i-node (i=0,1) is a concave corner on the rim of λ where a node $\boxed{i}$ can be added. The action of $\mathfrak{g}$ to the reduced P-function $P_\lambda^{(3)}(t)$ is described as follows:

$$e_iP_\lambda^{(3)} = \sum_{\mu \in \mathcal{E}_i^1(\lambda)} P_\mu^{(3)},$$

where $\mathcal{E}_i^1(\lambda)$ is the set of the strict partitions which can be obtained by removing a removable i-node from λ, and

$$f_i P_\lambda^{(3)} = \sum_{\mu\in\mathcal{F}_i^1(\lambda)} P_\mu^{(3)},$$

where $\mathcal{F}_i^1(\lambda)$ is the set of the strict partitions which can be obtained by adding an indent i-node to λ. For instance

$$e_0 P^{(3)}_{(4,3,1)} = P^{(3)}_{(4,2,1)} + P^{(3)}_{(4,3)},$$
$$f_1 P^{(3)}_{(4,3,1)} = P^{(3)}_{(5,2,1)} + P^{(3)}_{(4,3,2)}.$$

Another realization of the basic representation is known, one in the homogeneous grading. The isomorphism between principal and homogeneous realizations is given by Leidwanger [9]. Put

$$\mathcal{B} = \mathbb{C}[u_j,\ s_{2j-1};\ j\geq 1].$$

Define the mapping Φ by

$$\begin{aligned}\Phi\ :\ \Gamma &\xrightarrow{\sim} \mathcal{B}\otimes\mathbb{C}[q,q^{-1}],\\ P_\lambda(t) &\longmapsto 2^{p(\lambda)}\bar{\delta}_3(\lambda)P_{\lambda^{b(0)}}(s)S_{\lambda^{b(1)}}(u)\otimes q^{m(\lambda)},\end{aligned}$$

where

$$p(\lambda) = \sum_{\lambda_i\not\equiv 0 \pmod 3}\left[\frac{\lambda_i-1}{3}\right],$$

and $m(\lambda)$ is determined by

$$\begin{aligned}m(\lambda) = &\ (\text{number of beads on the first runner of } \lambda)\\ &- (\text{number of beads on the second runner of } \lambda).\end{aligned}$$

For example

$$\Phi(P_{(7,5,3,1)}(t)) = 8P_{(1)}(s)S_{(2,1,1)}(u)\otimes q.$$

Leidwanger [9] shows that Φ is indeed an isomorphism and that, if we denote by V the subalgebra of $\mathcal{B}$ generated by u_{2j} and $2^{2j-1}u_{2j-1} - s_{2j-1}$ ($j\geq 1$), then

$$\Phi(\Gamma^{(3)}) = V\otimes\mathbb{C}[q,q^{-1}].$$

The representation of $\mathfrak{g}$ on $V\otimes\mathbb{C}[q,q^{-1}]$, which is induced by Φ, is the basic representation in the homogeneous grading. In fact, if we define the degree in $V\otimes\mathbb{C}[q,q^{-1}]$ by

$$\deg f(u,s)\otimes q^m = 2\deg f(u,s) + m^2,$$

then $\deg\Phi(P_\lambda^{(3)})$ is equal to the number of 0-nodes in λ.

5. Main Result

We first recall a formula of plethysm which is due to Carini and Remmel [2]. For positive integers $l\geq m\geq 0$, let $W(l-m,m)$ denote the set of partitions $\mu = (\mu_1,\cdots,\mu_{2m})$ of $2m(l-m)$ such that $\mu_i + \mu_{2(l-m)+1-i} = 2m$ for $1\leq i\leq m$. It is easily seen that $|W(l-m,m)| = \binom{l}{m}$. Denote by $\square(l-m,m)$ the rectangular Young diagram (m^{l-m}).

Theorem 5.1. ([2])

$$p_2\circ S_{\square(l-m,m)} = \sum_{\mu\in W(l-m,m)} (-1)^{\sum_{i=1}^{l-m}\mu_i} S_\mu.$$

Now fix a strict partition $\Lambda_l = (3l-2, 3l-5, \cdots, 7, 4, 1)$ of length l, which corresponds to a maximal weight vector of the basic representation of $A_2^{(2)}$ in the principal grading. As before, each cell of the Young diagram of Λ_l is supposed to be filled with 0 or 1. Note that each concave corner on the rim of Λ_l is an indent 1-node. Let $\mathcal{F}^m = \mathcal{F}_1^m(\Lambda_l)$ $(0 \le m \le l)$ be the set of the strict partitions which can be obtained by adding m indent 1-nodes to Λ_l. It is obvious that $|\mathcal{F}^m| = \binom{l}{m}$. Our main result in this abstract is the following.

Theorem 5.2.

$$p_2 \circ S_{\square(l-m,m)} = \varepsilon(l,m) \sum_{\mu \in \mathcal{F}_1^m(\Lambda_l)} \bar{\delta}_3(\mu) S_{\mu^{b(1)}},$$

where

$$\varepsilon(l,m) = \begin{cases} (-1)^{\binom{m}{2}} & (0 \le m \le \frac{l}{2}) \\ (-1)^{l(m+1)+\binom{l-m}{2}} & (\frac{l}{2} \le m \le l). \end{cases}$$

Proof. Besides the sign factor $\varepsilon(l,m)$, we only have to show that $\mu^{b(1)} \in W(l-m,m)$ for any $\mu \in \mathcal{F}^m$. Recall the relation between 3-bar quotients and 3-quotients:

$$\mu^{b(1)} = D(\mu)^1.$$

First we see that the 3-abacus of $D(\Lambda_l)$ is

$$\begin{array}{ccc|c}
0 & 1 & 2 & 1 \\
3 & 4 & 5 & \\
 & \vdots & & \\
3l-3 & 3l-2 & 3l-1 & \\
3l & 3l+1 & 3l+2 & l+1 \\
3l+3 & 3l+4 & 3l+5 & \\
 & \vdots & & \\
6l-3 & 6l-2 & 6l-1 & 2l
\end{array}$$

If we add the indent 1-node to Λ_l at the i-th row, then, in the 3-abacus of $D(\Lambda_l)$, the beads at $3i-2$ and $6l-3i+1$ move to $3i-1$ and $6l-3i+2$, respectively:

$$\begin{array}{ccc|c}
0 & 1 & 2 & 1 \\
3 & 4 & 5 & \\
 & \vdots & & \\
3i-3 & 3i-2 & 3i-1 & i \\
 & \vdots & & \\
3l-3 & 3l-2 & 3l-1 & \\
3l & 3l+1 & 3l+2 & l+1 \\
3l+3 & 3l+4 & 3l+5 & \\
 & \vdots & & \\
6l-3i & 6l-3i+1 & 6l-3i+2 & 2l-i+1 \\
 & \vdots & & \\
6l-3 & 6l-2 & 6l-1 & 2l
\end{array}$$

Adding indent 1-nodes successively, we see that, in the 3-abacus of $D(\mu)$ $(\mu \in \mathcal{F}^m)$, the beads at i_1-th, i_2-th, $\cdots$, i_m-th rows in the first runner shift to the second runner as well as the beads at i_1-th, i_2-th, $\cdots$, i_m-th rows from the bottom. Then $D(\mu)_i^1 + D(\mu)_{2(l-m)+1-i}^1$ counts the number of the vacancies of the first runner up to the $2l$-th row; that is $2m$. This proves that $\mu^{b(1)} = D(\mu)^1 \in W(l-m,m)$. By a rather tedious computation, we can fit the sign factor $\varepsilon(l,m)$. $\square$

6. Plethysms Involving Q-functions

In this section we consider the plethysm $p_r \circ Q_\lambda$. For this purpose we need the following *factorization* theorem.

Theorem 6.1. *Let r be a positive odd integer. If $\lambda \in SP_{rn}$ has empty r-bar core, then*

$$2^{-l(\lambda)/2} Q_\lambda(\boldsymbol{x}\omega_r) = \bar{\delta}_r(\lambda) 2^{-l(\lambda^{b(0)})/2} Q_{\lambda^{b(0)}}(\boldsymbol{x}^r) T_{\lambda^{b(1)}}(\boldsymbol{x}^r) \cdots T_{\lambda^{b(t)}}(\boldsymbol{x}^r).$$

Remark 6.2. Since we assume that $\lambda^{bc(r)} = \emptyset$ in Theorem 6.1, we easily verify that

$$l(\lambda) \equiv l(\lambda^{b(0)}) (\mathrm{mod}\ 2).$$

Therefore $2^{(l(\lambda)-l(\lambda^{b(0)}))/2}$ is an integer.

Our proof of Theorem 6.1 relies on a formula shown by You[17]. In the process of the proof we obtain

Corollary 6.3.

$$\delta_r(D(\lambda)) = 1.$$

A special case of Theorem 6.1 is found in [11, 12]. Comparing the spin character table of $\tilde{S}_4$ with that of $\tilde{S}_{12}$, one observes that each character of $\tilde{S}_{12}$ evaluated at the conjugacy classes 3ρ is a linear combination of the characters of $\tilde{S}_4$ with integral coefficients which are simultaneously non-negative or non-positive.

$n=4$	(1^4)	$(3,1)$
$<4>$	2	1
$<3,1>$	4	-1

$n=12$	(3^4)	$(9,3)$	$3-$ *bar core*	
$<12>$	2	1	$\emptyset$	$+1<4>+0<3,1>$
$<11,1>$	-4	-2	$\emptyset$	$-2<4>-0<3,1>$
$<10,2>$	4	2	$\emptyset$	$+2<4>+0<3,1>$
$<9,3>$	4	-1	$\emptyset$	$+0<4>+1<3,1>$
$<8,4>$	-12	0	$\emptyset$	$-2<4>-2<3,1>$
$<7,5>$	12	0	$\emptyset$	$+2<4>+2<3,1>$
$<5,4,2,1>$	8	-2	$\emptyset$	$+0<4>+2<3,1>$
$<5,4,3>$	-16	1	$\emptyset$	$-2<4>-3<3,1>$
$<7,3,2>$	-8	-1	$\emptyset$	$-2<4>-1<3,1>$
$<8,3,1>$	8	1	$\emptyset$	$+2<4>+1<3,1>$
$<6,4,2>$	12	0	$\emptyset$	$+2<4>+2<3,1>$
$<6,5,1>$	-12	0	$\emptyset$	$-2<4>-2<3,1>$
$<9,2,1>$	-8	-1	$\emptyset$	$-2<4>-1<3,1>$
$<6,3,2,1>$	-8	2	$\emptyset$	$-0<4>-2<3,1>$
$<7,4,1>$	0	0	$(7,4,1)$	$+0<4>+0<3,1>$

We give a description of these coefficients. Put

$$t_{\lambda,\mu} = [Q_{\lambda^{b(0)}} T_{\lambda^{b(1)}} \cdots T_{\lambda^{b(t)}}, P_\mu],$$

where $[,]$ is an inner product satisfying $[P_\lambda, Q_\mu] = \delta_{\lambda,\mu}$. It is known that $t_{\lambda,\mu}$ is a non-negative integer [16]. The following theorem is a spin character version of *Littlewood's multiple formula* [11, 12].

Theorem 6.4. *Let r be a positive odd integer. If $\lambda \in SP_{rn}$ has empty r-bar core, then*

$$\zeta^{\lambda}_{r\rho} = \bar{\delta}_r(\lambda) \sum_{\mu \in SP_n} 2^{(l(\mu)-l(\lambda^{b(0)})+\epsilon(\mu)-\epsilon(\lambda))/2} t_{\lambda,\mu} \zeta^{\mu}_{\rho}.$$

for any $\rho \in OP_n$.

Another application of Theorem 6.1 is a formula of plethysm involving P and Q-functions.

Theorem 6.5. *For a positive odd integer r, we have*

$$p_r \circ P_\lambda = \sum_{\mu^{bc(r)}=\emptyset} \bar{\delta}_r(\mu) 2^{(l(\mu)-l(\mu^{b(0)}))/2} t_{\lambda,\mu} P_\mu,$$

$$p_r \circ Q_\lambda = 2^{l(\lambda)} \sum_{\mu^{bc(r)}=\emptyset} \bar{\delta}_r(\mu) 2^{(-l(\mu)-l(\mu^{b(0)}))/2} t_{\lambda,\mu} Q_\mu.$$

For the special case $\lambda = (n)$, the right hand side becomes simpler.

Theorem 6.6. *For a positive odd integer r, we have*

$$(p_r \circ Q_{(n)}) = \sum_{\mu \in H_{r,n}} \bar{\delta}_r(\mu) Q_\mu.$$

Here $H_{r,n} = \{\mu = (\mu_1 \dots \mu_l) \in SP_{rn} |\ \exists i \ s.t.\ \mu_i \equiv k (\mathrm{mod}\ r) \Rightarrow \exists_1 j \ s.t.\ \mu_j \equiv r-k (\mathrm{mod}\ r)\}$.

This formula should be compared with Theorem 5.1.

Example 6.7.

$$\begin{aligned}
p_3 \circ Q_{(1)} &= Q_{(3)} - Q_{(2,1)},\\
p_3 \circ Q_{(2)} &= Q_{(6)} - Q_{(5,1)} + Q_{(4,2)} - Q_{(3,2,1)},\\
p_3 \circ Q_{(3)} &= Q_{(9)} - Q_{(8,1)} + Q_{(7,2)} - Q_{(5,4)} - Q_{(6,2,1)} + Q_{(5,3,1)} - Q_{(4,3,2)},\\
p_5 \circ Q_{(2)} &= Q_{(10)} - Q_{(9,1)} + Q_{(8,2)} - Q_{(7,3)} + Q_{(6,4)} - Q_{(5,4,1)} + Q_{(5,3,2)} - Q_{(4,3,2,1)}.
\end{aligned}$$

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