ASYMPTOTICS OF MULTIVARIATE SEQUENCES

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ABSTRACT. We discuss the problem of coefficient extraction from multivariate generating functions. We concentrate on the authors' ongoing project (see http://www.math.auckland.ac.nz/~wilson/Research/mvGF/). Some representative results are given without proof.

RÉSUMÉ. Nous discutons du problème de l'extraction de coefficients pour les fonctions génératrices en plusieurs variables. Nous nous concentrons sur le projet continu des auteurs (voir http://www.math.auckland.ac.nz/ wilson/Research/mvGF/). Quelques résultats représentatifs sont donnés sans preuve.

1. INTRODUCTION

The generating function $F(z) := \sum_{r=0}^{\infty} a_r z^r$ for the sequence $a_0, a_1, a_2, \dots$ is one of the most useful constructions in combinatorics. If the function F has a simple description, it is usually not too hard to obtain F as a formal power series once one understands a recursive or combinatorial description of the numbers $\{a_r\}$. One may then analyze the analytic properties of F in order to obtain asymptotic information about the sequence $\{a_r\}$. While still part art and part science, this latter analytic step has become quite systematized. [Sta97] in his introduction to enumerative combinatorics gives the example of the function $F(z) = \exp(z + \frac{z^2}{2})$, from which he says "it is routine (for someone sufficiently versed in complex variable theory) to obtain the asymptotic formula $a_r = 2^{-1/2} r^{r/2} e^{-r/2+\sqrt{r}-1/4}$." Routine, in this case, means a single application of the saddle point method. When F has singularities in the complex plane, the analysis is often more direct: the location of the singularities and the behavior of F near these determine almost algorithmically the asymptotic behavior of the sequence $\{a_r\}$. For those not sufficiently versed in complex variable theory, two useful sources are [Hen77] and [Odl95]. The transfer theorems of [FO90] encapsulate much of this knowledge in a very useful way; see also [Wil94] for an elementary introduction.

When the sequence $a_0, a_1, a_2, \dots$ is replaced by a multidimensional array $\{a_{r_1,\dots,r_d}\}$, things become much more hit and miss. Let us use boldface to denote vectors in $\mathbb{C}^d$ or $\mathbb{N}^d$, and use multi-index notation, so that $a_{\mathbf{r}}$ denotes the multi-index $a_{r_1,\dots,r_d}$ and $\mathbf{z}^{\mathbf{r}}$ denotes the product $z_1^{r_1} \cdots z_d^{r_d}$. The generating function $F : \mathbb{C}^d \to \mathbb{C}$ is defined analogously to the one-dimensional generating function by

$$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}.$$

Surprisingly, techniques for extracting asymptotics of $\{a_{\mathbf{r}}\}$ from the analytic properties of F were, until recently, almost entirely missing. In a survey of asymptotic methods, [Ben74] says:

> Practically nothing is known about asymptotics for recursions in two variables even when a generating function is available. Techniques for obtaining asymptotics from bivariate generating functions would be quite useful.

In the intervening 25 years, some results have appeared, addressing chiefly the case where the array $\{a_{\mathbf{r}}\}$ obeys a central limit theorem. Common to all of these is the following method. Treat $\{a_{\mathbf{r}}\}$ as a sequence of $(d-1)$-dimensional arrays indexed by r_d; show that the n^{th} $(d-1)$-dimensional generating function is roughly the n^{th} power of a given function; use this approximation to invert the characteristic function and obtain a Central Limit Theorem. We refer to these methods as GF-sequence methods. The other body of work on multivariate sequences, which we will call the diagonal method, is based on algebraic extraction of the diagonal, as found in [HK71] (see also [Fur67] and later [Lip88] for an algebraic description of the scope of this method; variants are described in [Sta99] and [Pip]).

The most fundamental GF-sequence result is probably [BR83], with extensions appearing in later work of the same authors. [FS] presents a version of the same idea which holds in much greater generality. [GR92] go beyond the central limit case, using the transfer theorems of [FO90] to handle functions that are products of powers with powers of logs. Recent work of Bender and Richmond [BR96, BR99] extends the applicability of the central limit results to many problems of combinatorial interest; see also [Hwa95, Hwa98b], where more precise asymptotics are given, and [Hwa98a], which extends some results to the combinatorial schemes of [FS93]. This does not exhaust the recent work on the problem of multivariable coefficient extraction, but does circumscribe it. It is interesting to note that Odlyzko's survey article [Odl95] devoted only 4% of its space to multivariate problems.

The present authors' recent research on multivariate coefficient extraction has concentrated on functions belonging to a certain class that includes rational functions, although the basic ideas surely extend to a larger class. Desirable features in any theory we develop include: generality of application, complete expansions and not just leading terms, expansions that are as uniform as possible in the direction $\mathbf{r}/|\mathbf{r}|$, and formulae for expansions that lead to effective computation. An ultimate goal is to systematize the extraction of multivariate asymptotics sufficiently that it may be automated in a computer algebra system.

Our methods are analytic, based on complex contour integration, and are outlined below.

For the remainder of this paper, we will assume that the formal power series F converges in a neighborhood of the origin and may be analytically continued everywhere except a set $\mathcal{V}$ of complex dimension $d-1$ which we call the *singular variety*.

A crude preliminary step in approximating $a_{\mathbf{r}}$ is to determine its exponential rate; in other words, to estimate $\log|a_{\mathbf{r}}|$ up to a factor of $1+o(1)$. Let $\mathcal{D}$ denote the (open) domain of convergence of F and let $\log\mathcal{D}$ denote the logarithmic domain in $\mathbb{R}^{d+1}$, that is, the set of $\mathbf{x}\in\mathbb{R}^{d+1}$ such that $e^{\mathbf{x}}\in\mathcal{D}$. If $\mathbf{z}\in\mathcal{D}$ then Cauchy's integral formula

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^{d+1}\int_{T(\mathbf{z})}\frac{F(\mathbf{w})}{\mathbf{w}^{\mathbf{r}+\mathbf{1}}}\,d\mathbf{w} \tag{1}$$

shows that $a_{\mathbf{r}} = O(|\mathbf{z}|^{-\mathbf{r}})$. Letting $\mathbf{z}\to\partial\mathcal{D}$ gives

$$\log|a_{\mathbf{r}}| \le -\mathbf{r}\cdot\log|\mathbf{z}| + o(|\mathbf{r}|),$$

and optimizing in $\mathbf{z}$ gives $\log|a_{\mathbf{r}}|\le\gamma(\mathbf{r})+o(|\mathbf{r}|)$ where

$$\gamma(\mathbf{r}) := -\sup_{\mathbf{x}\in\log\mathcal{D}}\mathbf{r}\cdot\mathbf{x}\,. \tag{2}$$

The cases in which the most is known about $a_{\mathbf{r}}$ are those in which this upper bound is correct, that is, $\log|a_{\mathbf{r}}| = \gamma(\mathbf{r})+o(|\mathbf{r}|)$. To explain this, note first that the supremum in (2) is equal to $\mathbf{r}\cdot\mathbf{x}$ for some $\mathbf{x}\in\partial\log\mathcal{D}$. The torus centred at the origin and containing $\mathbf{x}$ must contain some minimal singularity $\mathbf{z} \in \mathcal{V} \cap \partial\mathcal{D}$. Asking that $\log |a_{\mathbf{r}}| \sim -\mathbf{r} \cdot \mathbf{z}$ is then precisely the same as requiring the Cauchy integral (1) — or the residue integral mentioned above — to be of roughly the same order as its integrand. This is the situation in which it easiest to estimate the integral.

Our approach may now be summarized as follows. Associated to each minimal singularity $\mathbf{z}$ is a cone $\boldsymbol{\kappa}(\mathbf{z}) \subseteq (\mathbb{R}^+)^{d+1}$. Given $\mathbf{r}$, we find one or more $\mathbf{z} = \mathbf{z}(\mathbf{r}) \in \mathcal{V} \cap \partial\mathcal{D}$ where the upper bound is least. We then attempt to compute a residue integral there. This works only if $\mathbf{r} \in \boldsymbol{\kappa}(\mathbf{z})$ and if the residue computation is of a type we can handle.

So far we have performed this residue computation in an explicit way by representing $a_{\mathbf{r}}$ (up to an exponentially smaller term) as a $(d-1)$-dimensional integral of one-variable residues. We have not used the more symmetric Leray residue theory largely because it has not been clear to us how to obtain sufficiently explicit results. Although we have broken the symmetry between coordinates, many of our formulae can be re-symmetrized with moderate effort.

The key analytic tool in extracting asymptotics has been the theory of oscillatory integrals. We reduce the residue computation to the computation of integrals of the form $\int_D \exp(-\lambda f(\mathbf{z}))\psi(\mathbf{z})\, d\mathbf{z}$ where f, ψ are smooth complex and complex-valued with $\operatorname{Re} f \geq 0$, and D is a compact product of simplices and intervals. Several challenges arise because the existing literature apparently does not contain the exact results needed; in particular the boundary terms arising in these integrals cannot always be neglected.

To amplify on this, define a point $\mathbf{z} \in \mathcal{V}$ to be *minimal* if $\mathbf{z} \in \partial\mathcal{D}$ and each z_j is nonzero. Note that $\mathbf{z}$ is minimal if $D(\mathbf{z}) \cap \mathcal{V} \subseteq T(\mathbf{z}$, where $T(\mathbf{z})$ and $D(\mathbf{z})$ are respectively the torus and disk centred at $\mathbf{0}$ and containing $\mathbf{z}$. A minimal point is *strictly minimal* if $D(\mathbf{z}) \cap \mathcal{V} = \{\mathbf{z}\}$. When a minimal point is not strictly minimal, one must add (or integrate) contributions from all points of $\mathcal{V} \cap T(\mathbf{z})$; this step is fairly routine.

There are only three possible types of minimal singularities [PW1, Lemma 6.1]), namely smooth points (where $\mathcal{V}$ is locally a graph of an analytic function); multiple points (where $\mathcal{V}$ is locally a union of graphs of analytic functions), and cone points (all others). It is conjectured that for all three types of points, and any $\mathbf{r} \in \boldsymbol{\kappa}(\mathbf{z})$, we indeed have

$$\log |a_{\mathbf{r}}| = \gamma(\mathbf{r}) + o(|\mathbf{r}|) = -\mathbf{r} \cdot \log |\mathbf{z}| + o(|\mathbf{r}|)\,.$$

This is proved for smooth points in [PW1] via residue integration, and the complete asymptotic series obtained. It is proved in [PW2] for multiple points under various assumptions; the fact that these do not cover all cases seems due more to taxonomical problems rather than the inapplicability of the method. The problem remains open for cone points, along with the problem of computing asymptotics.

The chief purpose of our work is to give a solution to the problem of asymptotic evaluation of coefficients that is as general as possible. An important part of this is re-derivation in a general setting of results obtainable via GF-sequence or *ad hoc* methods. Our results allow us to show that our method successfully finds asymptotics for every function in a certain large class. Familiar examples from this class include: lattice path counting, various known generating functions for polyominos and stacked balls, enumeration of Catalan trees by number of components or surjections by image cardinality (see [FS]), stopping times for certain random walks (see [LL99]), as well as the examples given in the GF-sequence papers of [Ben73] and [BR83]: ordered set partitions enumerated by number of blocks, permutations enumerated by rises, and Tutte polynomials of recursive sequences of graphs.

Nevertheless, our pursuit of this problem was also motivated by some specific applications and challenge problems. These are cases where known methods do not suffice to obtain complete asymptotic information. There is a class of tiling enumeration problems for which

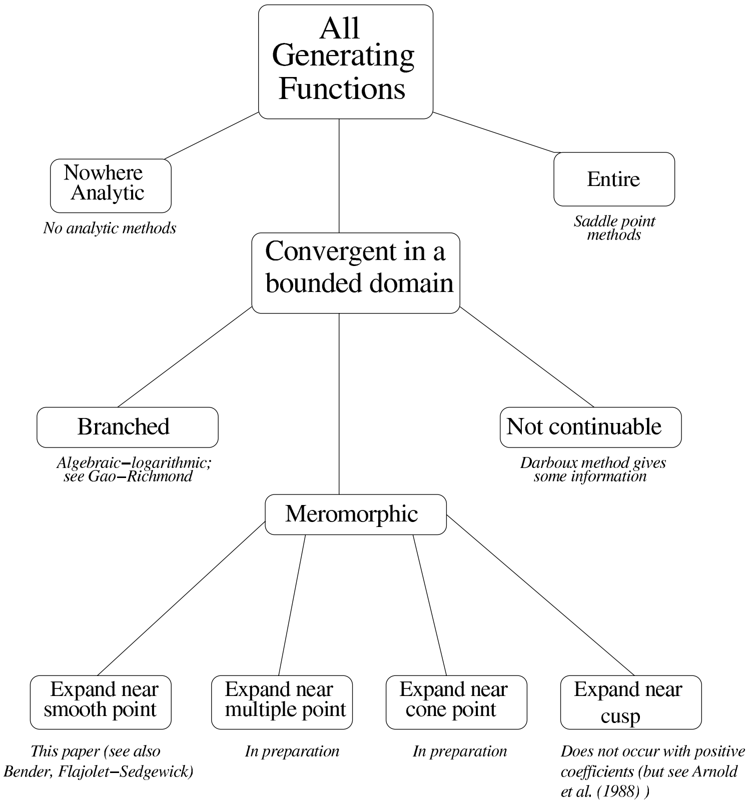

FIGURE 1. Classification of generating functions

an explicit three variable rational generating function may be obtained. This class includes the Aztec Diamond domino tilings of [CEP96]. Asymptotics in the so-called *region of fixation* are obtained from analysis of the smooth points of $\mathcal{V}$, while asymptotics in the region of positive entropy]are derived from analysis of the cone point. [CP] applies a cone point analysis to a tiling enumeration problem for which the only previous results are some pictures via simulation (`http://www.math.ohio-state.edu/ pemantle/Pix/plot.gif`). Another motivation has been to solve the general multivariable linear recursion. Depending on whether one allows forward recursion in some of the variables, one typically obtains either rational or algebraic generating functions (less well-behaved functions can also result — see [BM00] for more details). The general rational function may have any of the types of singularities mentioned above: smooth points, nodes, cones, cusps, branchpoints, etc. Even the simple rational generating function $1/(3 - 3z - w + z^2)$ requires two separate analyses in order to get asymptotics in all directions.

In the rest of this section, we describe the scope of our methods and compare them with previous work. Figure 1 depicts a classification of generating functions and illustrates the remainder of this paragraph. If a formal power series is nowhere convergent, analytic methods are useless. Among those power series converging in some neighborhood of the origin, there are three possibilities: a function may be entire, may have singularities around which analytic continuations exist, or it may be defined only on some bounded subset of $\mathbb{C}^d$. Our methods are tailored to the second class. The third class, although in some sense generic, seldom arises in any problem for which the generating function may be effectively described. Incomplete asymptotic information is available via Darboux' method; details of this method in the univariate case are given in [Hen77] and [Odl95]. The first class can and does arise frequently. Our methods are simply not equipped to handle entire functions, and systematizing the asymptotic analysis of coefficients of entire generating functions remains an important open problem.

Most existing literature on the multivariate coefficient extraction problem deals (in effect, although results are not necessarily stated in this way) with analysis near smooth pole points, whose asymptotics usually exhibit central limit behavior. There are several ways in which our treatment of the smooth case improves upon available analyses.

First, most of the existing results assume that the singular point $\mathbf{z} \in \mathcal{V}$ has positive real coordinates, and that it is strictly minimal in a sense defined in the next section. This assumption often holds when the coefficients $\{a_{\mathbf{r}}\}$ are nonnegative reals, though it will fail if, for example, there is any periodicity. The assumption always fails when the coefficients $\{a_{\mathbf{r}}\}$ have mixed signs, as is the case for example with the generating functions $(1-zw)/(1-2zw+w^2)$ and $1/(1-2zw+w^2)$ for the Chebyshev polynomials of the first and second kinds [Com74, page 50]. GF-sequence methods may be adapted to some of these situations. Indeed, the presentation of these methods by [FS, Theorem 9.7] accomplishes this adaptation in great generality. But certainly there are cases such as the rational generating function $1/(1 - z - w + \beta zw)$, where the points $\mathbf{z}$ with given moduli form a continuum and standard GF-sequence methods are not sufficient.

Second, our methods obtain automatically a full asymptotic expansion of $a_{r_1,\ldots,r_d}$ in decreasing powers of the indices r_j. This is certainly not inherent in the existing results, whose relatively short proofs involve inversion of the characteristic function (see however [Hwa95] and [Hwa96] for something in this direction). The expansion to n terms is completely effective in terms of the first n partial derivatives of $1/F$ at $\mathbf{z}$, as is the error bound.

Third, these results explicitly cover the case where the pole at $\mathbf{z}$ has order greater than 1. The behavior in this case is not according to the central limit theorem. The only existing work addressing this case is [GR92], and they require nonnegativity assumptions, as mentioned above. In the case where $F = G^k$ is an exact power, one could attempt first to solve the problem for G and then to take the k-fold convolution. This is much harder than the present approach, as may be seen by the rather involved computation in [CEP96].

Fourth, the potential for increasing the scope to new applications seems greater for contour methods than for GF-sequence methods. The contour method reduces the asymptotic problem to the problem of an oscillating integral near a singularity, which can almost certainly be done. By contrast, the GF-sequence method requires first an understanding of the sequence of $(d-1)$-dimensional generating functions arising from the given d-dimensional generating function, and then another result in order to transfer this information to asymptotics of the coefficients $a_{\mathbf{r}}$.

Fifth, although our results in the case of smooth pole points are often similar to those obtained by GF-sequence methods, our hypotheses are quite different. Our hypotheses may be universally established for bivariate functions that generate nonnegative values and are meromorphic through their domain of convergence.

Finally, we compare our method to recent results from the diagonal method. It is known [Lip88] that the diagonal sequence $a_{n,n,\ldots,n}$ of a multivariate sequence with rational generating function has a generating function satisfying a linear differential equation over rational functions. Much is known about how to compute this equation (see for example [Chy98]). If one wants asymptotics on the diagonal, or in any direction where the coordinate ratios are rational numbers with small denominators, then these methods give results that are in theory at least as good as ours. The method, unlike ours, is inherently non-uniform in the direction, so there is no hope of extending it to larger sets of directions.

2. RESULTS

To state our results as cleanly as possible, we assume that there are analytic functions G, H of $d+1 \geq 2$ variables with $F = G/H$ in the neighbourhood of the strictly minimal element $\mathbf{z} = \mathbf{1}$ of $\mathcal{V}$ (if $\mathbf{z} \neq \mathbf{1}$, a factor of $\mathbf{z}^{-\mathbf{r}}$ is introduced).

We first discuss our results, in the generic case (that is, with appropriate conditions on transversality of intersection and nonvanishing of G at $\mathbf{z}$), from a qualitative perspective. The following results are valid as $|\mathbf{r}| \to \infty$ for $\mathbf{r}$ in a certain cone $\boldsymbol{\kappa}(\mathbf{z})$ that collapses to a single ray in the smooth case. In fact $\boldsymbol{\kappa}$ can easily be described geometrically as being spanned by the outward normals to the support hyperplanes of $\log\mathcal{D}$ at $\log \mathbf{z}$.

In both smooth and multiple point cases, there is an asymptotic expansion of $a_{\mathbf{r}}$ in descending powers of $|\mathbf{r}|$. If $\mathbf{z}$ is smooth then we have Ornstein-Zernike (central limit) behaviour, $a_{\mathbf{r}} \sim C|\mathbf{r}|^{-d/2}$. By contrast, suppose $\mathbf{z}$ is a multiple point, the intersection of $n+1 \geq 2$ sheets of $\mathcal{V}$.

- If $n = d$ we have asymptotic constancy throughout the cone: $a_{\mathbf{r}} \sim C$ for some C, and the error term is exponentially small (the series has only one term).
- If $n \geq d$, then there are several subcones of $\boldsymbol{\kappa}$ on each of which we get $a_{\mathbf{r}} \sim P(\mathbf{r})$ for some polynomial of degree at most $n - d$. Thus the series is finite.
- If $n \leq d$ then asymptotics start with $|\mathbf{r}|^{n/2-d/2}$.
- Finally, suppose that our transversality conditions are dropped and that all sheets are in fact tangent at $\mathbf{z}$. Then asymptotics start with $|\mathbf{r}|^{n-d/2}$.

We now present a couple of 2-dimensional results and examples contained in [PW1, PW2], chosen because they can be stated with no extra notation and maintain the symmetry between the variables (which is broken in our analysis,, so formulae must be re-symmetrized). The numerous more complex results referred to above require the introduction of considerably more notation in order to give explicit formulae for the leading coefficient of the asymptotic expansion, and are not re-symmetrized. The quantities involved depend on the Weierstrass factorization of H near $\mathbf{z}$ and various derivatives of G and H. To the best of our knowledge, all of these are explicitly computable by symbolic algebra in appropriate commutative rings

Theorem 1. *Let $F = G/H$ be a meromorphic function of two variables, not singular at the origin. Define*

$$Q(z,w) := -w^2 H_w^2 z H_z - w H_w z^2 H_z^2 - w^2 z^2 \left(H_w^2 H_{zz} + H_z^2 H_{ww} - 2 H_z H_w H_{zw} \right).$$

Then

$$a_{r,s} \sim \frac{G(z,w)}{\sqrt{2\pi}} z^{-r} w^{-s} \sqrt{\frac{-wH_w}{sQ(z,w)}}$$

uniformly as (z,w) varies over a compact set of strictly minimal, simple poles of F on which Q and G are nonvanishing, and $(r,s) \in \boldsymbol{\kappa}(z,w)$.

Remark. Usually the expression in the radical will be positive real, as will the coefficients a_{rs}. The result is true in general, though, as long as the square root is taken to be $-(wH_w)^{-1}$ times the principal root of $(-wH_w^3)/(sQ)$. Also note that when $(r,s) \in \mathbf{dir}(z,w)$ then the expression wH_w/s is coordinate-invariant, that is, equal to zH_z/r. Thus the given expression for $a_{r,s}$ has the expected symmetry.

Example 2 (Lattice paths)**.** Let $a_{r,s}$ be the number of nearest-neighbor paths from the origin to (r,s) moving only north, east and northeast; these are sometimes called *Delannoy numbers* [Sta99, page 185]. The generating function is $F(z,w) = 1/(1-z-w-zw)$. The zero set $\mathcal{V}$ of $H = 1-z-w-zw$ is given by $w = (1-z)/(1+z)$, and the minimal points of $\mathcal{V}$ are those where $w \in [0,1]$. With the help of relations that hold when $\mathbf{z} \in \mathcal{V}$ we may compute as follows.

$$\begin{aligned} H_z &= -1 - w \\ -zH_z &= 1 - w \\ Q &= (1-z)(1-w)(1-zw) \\ \frac{zH_z}{wH_w} &= \frac{1-w}{1-z} = \frac{1-w^2}{2w} \end{aligned}$$

with H_w and $-wH_w$ given by reversing z and w. As z varies over $[\varepsilon, 1-\varepsilon]$, the functions Q and $G := 1$ do not vanish. The minimal pair (z,w) that solves $(r,s) \in \mathbf{dir}(z,w)$ is given by $z = (\sqrt{r^2+s^2} - s)/r$ and $w = (\sqrt{r^2+s^2} - r)/s$. Theorem 1 then gives

$$\begin{aligned} a_{rs} &\sim \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{1-z}{s} \frac{1}{1-zw}} \\ &= \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{(r+s-\sqrt{r^2+s^2})^2\sqrt{r^2+s^2}}}, \end{aligned}$$

uniformly when r/s and s/r remain bounded. In particular, when $r = s = n$, this gives the following formula for the n^{th} diagonal coefficient (which may alternatively be obtained by computing the diagonal generating function $(1 - 6s + s^2)^{-1/2}$ according to the method given in [Sta99, Section 6.3]:

$$(\sqrt{2}-1)^{-2n} \sqrt{\frac{1}{2\pi} \frac{2^{-1/4}}{2-\sqrt{2}}}.$$

In the next result, a *boundedly interior* subset of a cone is one that is interior, and bounded away from the walls.

Theorem 3 (2 curves meeting transversally in 2-space)**.** *Let F be a meromorphic function of two variables, not singular at the origin, with $F(z,w) = G(z,w)/H(z,w) = \sum_{r,s} a_{rs} z^r w^s$.*

Suppose that (z,w) is a strictly minimal, double point of $\mathcal{V}$. Let $\mathcal{H}(z,w)$ denote the Hessian of H at (z,w).

Then for each boundedly interior subset K of $\boldsymbol{\kappa}(z,w)$, there is $c > 0$ such that

$$a_{rs} = z^{-r} w^{-s} \left(\frac{G(z,w)}{\sqrt{-z^2 w^2 \det \mathcal{H}(z,w)}} + O(e^{-c|(z,w)|}) \right) \qquad \textit{uniformly for } (r,s) \in K.$$

Example 4 (combinatorial application)**.** An independent sequence of random numbers uniform on $[0,1]$ is used to generate biased coin-flips: if p is the probability of heads then a number $x \le p$ means heads and $x > p$ means tails. The coins will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \le r+s$ will be winning choices?

The probability that n is a winning choice for the player is precisely

$$a_{rs} := \sum_{a+b=n} \binom{n}{a} (2/3)^a (1/3)^b \binom{r+s-n}{r-a} (1/3)^{r-a} (2/3)^{s-b}.$$

Let a_{rs} be this expression summed over n. The array $\{a_{rs}\}_{r,s \ge 0}$ is just the convolution of the arrays $\binom{r+s}{r}(2/3)^r(1/3)^s$ and $\binom{r+s}{r}(1/3)^r(2/3)^s$, so the generating function $F(z,w) :=$

$\sum a_{rs} z^r w^s$ is the product

$$F(z,w) = \frac{1}{(1 - \frac{1}{3}z - \frac{2}{3}w)(1 - \frac{2}{3}z - \frac{1}{3}w)} .$$

Applying Theorem 3 with $G \equiv 1$ and $\det \mathcal{H} = -1/9$, we see that $a_{rs} = 3$ plus a correction which is exponentially small as $r, s \to \infty$ with $r/(r+s)$ staying in any subinterval of $(1/3, 2/3)$. A purely combinatorial analysis of the sum may be carried out to yield the leading term, 3, but says nothing about the correction terms. The diagonal extraction method of [HK71] yields very precise information for $r = s$ but nothing more general in the region $1/3 < r/(r+s) < 2/3$. □

3. Comments and further work

The greatest obstacle to making all these computations completely effective lies in the location of the minimal point $\mathbf{z}$ given $\mathbf{r}$. Assuming the existence of a $\mathbf{z}(\mathbf{r})$ with $\mathbf{r} \in \boldsymbol{\kappa}(\mathbf{z})$, how may we compute $\mathbf{z}(\mathbf{r})$ and test whether it is a minimal point? Since the moduli of the coordinates of $\mathbf{z}$ are involved in the definition of minimality, this is a problem in real rather than complex computational geometry and does not appear easy.

Another natural question is whether there exists such a minimal $\mathbf{z}(\mathbf{r})$. When F generates nonnegative coefficients, the answer is generally yes. Examples show that when the coefficients have mixed signs, the answer is no. We conjecture for every direction there is a (not necessarily minimal) point $\mathbf{z} \in \mathcal{V}$ for which integration near $\mathbf{z}$ yields correct asymptotics. For example, if $G = 1$ and

$$H = (1 - (2/3)w - (1/3)z)(1 + (1/3)w - (2/3)z)$$

then the point $(3/2, 3/4)$ is not minimal but yields asymptotics in the diagonal direction; one sees this by integrating along a deformed torus rather than along $T(3/2, 3/4)$. In fact we conjecture that such a deformation always exists, but the topology seems not transparent enough to yield an easy proof.

The problem of determining asymptotics when $\hat{\mathbf{r}}$ converges to the boundary of $\boldsymbol{\kappa}$ is dual to the problem of letting $\hat{\mathbf{r}}$ converge to $\partial \boldsymbol{\kappa}$ from the outside. Solutions to both of these problems are necessary before we understand asymptotics "in the gaps", that is, in any region asymptotic to and containing a direction in the boundary of $\boldsymbol{\kappa}$. For example, what are the asymptotics for $a_{r, r+\sqrt{r}}$ as $r \to \infty$?

Many of our theorems rule out analysis of a minimal point $\mathbf{z}$ if one of the coordinates z_j is zero. The directions in $\boldsymbol{\kappa}(\mathbf{z})$ will always have $r_j = 0$, in which case the analysis of coefficient asymptotics reduces to a case with one fewer variable. Thus it appears no generality is lost. If we are, however, able to solve the previous problem, wherein $\mathbf{r}$ converges to $\partial \boldsymbol{\kappa}$, then we may choose to let $\mathbf{r}$ converge to something with a zero component. The problem of asymptotics when some $r_j = o(r_k)$ now makes sense and is not reducible to a previous case. Presumably these asymptotics are governed by the minimal point $\mathbf{z}$ still, but it must be sorted out which of our results persist when $z_j = 0$. Certainly the geometry near $\mathbf{z}$ has more possibilities, since it is easier to be a minimal point (it is easier to maintain $|z'_j| \geq |z_j|$ for $\mathbf{z}'$ near $\mathbf{z}$ when $z_j = 0$).

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