As a positive real number, $\xi(\omega_{k,N})$ is not an M^{th} root of -1. □

Substitution of $\omega_{k,N}$ for x in (18), justified by Lemma 1, yields

$$f_1(\omega_{k,N}) = \alpha_{k,N,M} \qquad (k = 0, 1, \ldots, N-1),$$

where

$$\alpha_{k,N,M} = \frac{4\,i\cot\varphi_{k,N}}{1+\xi^M(\omega_{k,N})} \cdot \frac{\xi^M(\omega_{k,N}) - \xi(\omega_{k,N})}{\xi(\omega_{k,N}) - 1} \qquad (i^2 = -1). \tag{20}$$

These N values uniquely determine the unknown polynomial $f_1(x)$. To find its coefficients explicitly we use the inversion formula

$$b_k = \sum_{i=0}^{N-1} a_i\,\omega_{k,N}^i \quad (0 \le k \le N-1) \quad \Longleftrightarrow \quad a_i = \frac{1}{N}\sum_{k=0}^{N-1} b_k\,\omega_{k,N}^{-i} \quad (0 \le i \le N-1)$$

which is verifiable by straightforward computation. Writing $f_1(x) = \sum_{i=0}^{N-1} a_i x^i$ and $b_k = f_1(\omega_{k,N})$, we obtain an explicit expression for the unknown coefficients of $f_1(x)$

$$a_{i,1} = a_i = \frac{1}{N}\sum_{k=0}^{N-1} \frac{\alpha_{k,N,M}}{\omega_{k,N}^i}. \tag{21}$$

In the same way we obtain an explicit expression for the unknown coefficients of $f_2(y)$,

$$a_{1,j} = \frac{1}{M}\sum_{k=0}^{M-1} \frac{\alpha_{k,M,N}}{\omega_{k,M}^i}. \tag{22}$$

$$f_1(x) = \frac{1}{N}\sum_{j=0}^{N-1} x^j \sum_{k=0}^{N-1} \frac{\alpha_{k,N,M}}{\omega_{k,N}^j} \tag{23}$$

where $\alpha_{k,N,M}$ and $\omega_{k,N}$ are given in (20) and (19), respectively.

By interchanging the order of summations in (23), or by using Lagrange Interpolation Formula on the data from (20), we can express $f_1(x)$ with a single summation sign:

$$f_1(x) = \frac{x^N+1}{N}\sum_{k=0}^{N-1} \frac{\alpha_{k,N,M}}{1 - x\omega_{k,N}^{-1}}.$$

In the same way we obtain the other unknown function

$$f_2(y) \;=\; \frac{1}{M}\sum_{j=0}^{M-1} y^j \sum_{k=0}^{M-1} \frac{\alpha_{k,M,N}}{\omega_{k,M}^j} \;=\; \frac{y^M+1}{M}\sum_{k=0}^{M-1} \frac{\alpha_{k,M,N}}{1 - y\omega_{k,M}^{-1}}.$$

Finally we have the following explicit expression for the entire generating function $F(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}x^i y^j$:

$$F(x,y) = \frac{xy\left(4\,\frac{x^N - x}{x-1}\,\frac{y^M - y}{y-1} - (x^N+1)(y^M+1)\left(\frac{1}{N}\sum_{k=0}^{N-1}\frac{\alpha_{k,N,M}}{1-x\omega_{k,N}^{-1}} + \frac{1}{M}\sum_{k=0}^{M-1}\frac{\alpha_{k,M,N}}{1-y\omega_{k,M}^{-1}}\right)\right)}{4\,xy - (x+y)(1+xy)}.$$

In closing we note that the values $a_{i,j}$ can be given explicitly as double trigonometric sums either using the Discrete Fourier Transform, or by direct diagonalization of the linear system (14) (cf. [4]).

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Department of Mathematics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: Marko.Petkovsek@fmf.uni-lj.si

A BIJECTION FOR LOOPLESS TRIANGULATIONS OF A POLYGON WITH INTERIOR POINTS

DOMINIQUE POULALHON AND GILLES SCHAEFFER

ABSTRACT. Loopless triangulations of a polygon with k vertices in $k+2n$ triangles (with interior points and possibly multiple edges) were enumerated by Mullin in 1965, using generating functions and calculations with the quadratic method.

In this article we propose a simple bijective construction of Mullin's formula. The argument rests on *conjugation of trees*, a variation of the cycle lemma designed for planar maps. In the much easier case of loopless triangulations of the sphere ($k=3$), we recover and prove correct an unpublished construction of the second author.

RÉSUMÉ. Les triangulations sans boucles d'un polygone à k côtés en $k+2n$ triangles (avec des points intérieurs et éventuellement des arêtes multiples) ont été énumérées par Mullin en 1965, à l'aide de séries génératrices et de la méthode quadratique.

Dans cet article, nous proposons une construction bijective simple de la formule de Mullin. L'argument repose sur la *conjugaison d'arbres*, une variation sur le lemme cyclique adaptée à l'énumération des cartes planaires. Dans le cas beaucoup plus facile des triangulations ($k=3$), nous retrouvons et démontrons une construction esquissée par le second auteur.

1. INTRODUCTION

In 1965, R.C. Mullin published the following formula for the number of planar loopless triangulations of a rooted k-gon into $k+2n$ triangles (see below for precise definitions):

$$T^*_{k,n} = |\mathcal{T}^*_{k,n}| = \frac{2^{n+2}(2k+3n-1)!(2k-3)!}{(n+1)!(2k+2n)!(k-2)!^2} \tag{1}$$

for all $k \geqslant 2$ and $n \geqslant 0$ (see [Mul65] or [GJ83, p145]), which extends the well-known formula for triangulations of a k-gon without interior points:

$$T^*_{k,-1} = |\mathcal{T}^*_{k,-1}| = \frac{(2k-4)!}{(k-1)!(k-2)!} \tag{2}$$

for all $k \geqslant 3$. By duality this formula also accounts for the number of rooted non-separable planar maps with a root vertex of degree k and $k+2n$ vertices all of degree three.

In his work, R.C. Mullin was closely following the seminal steps of W.T. Tutte in his *census* papers [Tut62a, Tut62b, Tut63]. In particular Formula (1) extends Tutte's formula

$$T_n = T^*_{3,n-2} = \frac{2^{n+1}(3n)!}{n!(2n+2)!} \tag{3}$$

for rooted loopless triangulations of the sphere with $2n$ triangles (or non-separable cubic maps with $2n$ vertices). The proof itself relies, following Tutte, on a recursive decomposition of triangulations that yields a recurrence for their number. Encoding the latter into generating functions then allows for a solution through the quadratic method and a few pages of calculus.

Ever since their discovery, efforts have been made to find derivations reflecting the elegant and simple product form of this and other formulas of Tutte for planar maps. In particular a construction based on the *conjugation of trees* principle was proposed in the second author's

PhD thesis [Sch98] for Formula (3) and a few other formulas of Tutte (all, bipartite, non-separable maps). A new generalization of both Tutte's formula and a formula of Hurwitz was also proved along these lines to enumerate planar constellations [BMS00].

However two parameter formulas for triangulations like (1) seem to resist conjugation of trees. In this article we introduce a slight variation of the family under consideration, which cardinality can be easily deduced from $T^*_{k,n}$, and that appears more suitable for bijective constructions.

In view of this family $\mathcal{T}_{k,n}$, Mullin's formula reads

$$T_{k,n} \;=\; |\mathcal{T}_{k,n}| \;=\; \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}. \tag{4}$$

The purpose of the present article is to provide a bijective construction of the latter formula. A main ingredient of our construction is again the *conjugation of trees* principle, and this confirms the adequacy of this approach to the bijective enumeration of planar maps.

However the bijection involves two new ingredients with respect to the treatment of Tutte's formulas. On the one hand, a *special* vertex is introduced in the construction, that allows to account for parameter k of Mullin's formulas. On the other hand, as opposed to the case of constellations [BMS00], the inverse construction does not rely on breadth-first search. Instead, in order to deal with non-separability, one has to resort on more difficult recursive arguments.

The rest of the article is organized as follows: after Formula (4) for the cardinality of $\mathcal{T}_{k,n}$ has been proved equivalent to Formula (1) for $\mathcal{T}^*_{k,n}$, we exhibit a simple family $\mathcal{E}_{k,n}$ of trees (balanced blossom trees) that are clearly enumerated by the same formula, and we define in a few lines an application φ from $\mathcal{E}_{k,n}$ that we claim onto $\mathcal{T}_{k,n}$ (Section 2, 3). Then comes the harder part, as often with bijections, namely the proof for the unbeliever that the image of the application φ is indeed $\mathcal{T}_{k,n}$ and that it is one-to-one (Section 4 and 5).

2. The enumerative formula for rooted loopless triangulations

2.1. **Definitions around planar maps.** Let us make more precise the definitions of the objects under consideration. A (planar) map is a two-cell embedding of a connected planar graph into the oriented sphere considered up to orientation preserving homeomorphisms of the sphere. Multiple edges are allowed. The degree of a vertex or a face is the number of (sides of) edges incident to that vertex or face. A face is a k-gon if it has degree k and it is incident to k distinct vertices.

A planar map is non-separable if it contains no cut-vertex, that is to say no vertex that can be cut into two vertices (each taking part of the edges) in a way that the resulting graph would not be connected anymore.

A map is rooted if one edge is chosen and oriented. This ensures that the considered object has a trivial automorphism group. The startpoint of the root (edge) and the face on its right hand side (which is well defined since the sphere was taken oriented) are called respectively root vertex and root face. Unless explicitely mentioned, the root face is taken as infinite face when representing maps in the plane.

The dual M^* of a map M is obtained from M by putting a vertex in each face of M and an edge of M^* across each edge of M. If M is rooted, the root edge of M^* is the dual of the root edge of M, oriented in such a way that the root vertex of M^* is the dual of the root face of M. This construction is clearly involutive on unrooted maps (see Figure 1).

2.2. **Rooted loopless triangulations.** A triangulation is a planar map such that each face has degree three. We will only consider loopless triangulations, hence faces are "real"

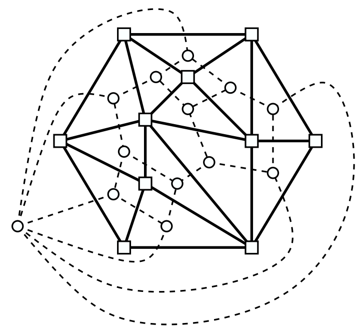


FIGURE 1. A triangulation of an hexagon and its dual

triangles, in the sense that they are 3-gons. However they are only "topological" triangles, in the sense that multiple edges are allowed so that these triangulations do not necessarily admit a representation with straight edges.

A loopless triangulation of a rooted k-gon is a planar map such that the root face is a k-gon while all other faces have degree three. A rooted triangulation of a k-gon is the same thing except that the distinguished k-gon need not be the root face. The terminology refers to the possibility, in order to draw the map in the plane, to take the k-gon as infinite face.

A loopless triangulation of a k-gon has $k + 2n$ triangles for some integer $n \geqslant -1$, and hence $2k + 3n$ edges and $k + n + 1$ vertices (k exterior and $n + 1$ interior ones). Let $\mathcal{T}_{k,n}$ be the set of rooted loopless triangulations of a k-gon into $k + 2n$ triangles. Then

$$k\,T_{k,n} \;=\; 2(2k+3n)\,T^{*}_{k,n},$$

as immediately follows upon considering doubly rooted triangulations with one root on the polygon and the other anywhere: these can be regarded either as rooted loopless triangulations of a k-gon in which an edge of the k-gon is distinguished (and oriented so that the k-gon is on its right hand side), or as loopless triangulations of a rooted k-gon in which an edge is distinguished and oriented.

Hence Mullin's formula becomes

$$T_{k,n} \;=\; 2^{n+3}\frac{(2k+3n)!(2k-3)!}{k\,(n+1)!(2k+2n)!(k-2)!^2},$$

and can be rewritten as previously claimed:

$$T_{k,n} \;=\; \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}.$$

This formula holds for any $k \geqslant 2$ and any $n \geqslant -1$: it specializes correctly for $k \geqslant 3$, $n = -1$, according to Formula (2); as for the degenerate case $k = 2$, $n = -1$, which can only be interpreted as the case of a loop at the special vertex, it boils down to 1.

Observe that $2n\,T_n = T_{3,n-2}$: this corresponds to the fact that a map in $\mathcal{T}_{3,n-2}$ can be viewed as a rooted loopless triangulation with $2n$ triangles among which one is distinguished (the 3-gone).

2.3. **Dual family.** A cubic map is a map with all vertices of degree three, and a near-cubic map is a map with all vertices of degree three, except maybe one. Let $\mathcal{C}_n$ and $\mathcal{C}_{k,n}$ be respectively the set of non-separable cubic maps with $2n$ vertices and the set of non-separable near-cubic maps with a special vertex of degree k and $k + 2n$ vertices of degree three. They are respectively the dual sets of $\mathcal{T}_n$ and $\mathcal{T}_{k,n}$.

3. The constructive census of triangulations

In this section we construct a set of simple objects counted by $T_{k,n}$ and a transformation of these objects that we claim is a bijection onto $\mathcal{T}_{k,n}$.

Terminology for trees. All the trees we are interested in are planted plane trees. In the context of planar maps, it is convenient to define a plane tree as a planar map with only one face, although this is equivalent to classical recursive definitions. Planted means that one vertex of degree one is distinguished and called the root.

We shall consider an enriched terminology for trees, with two kinds of vertices of degree one, *buds* and *leaves*, three kinds of edges, *links*, *inner edges* and *stems*, and three kinds of vertices of larger degrees, *generic*, *pathological* and *special*. Buds and leaves are always incident to stems (as opposed to links or edges) and in pictures, buds are represented by arrows. The root of a planted tree shall always be a leaf (and not a bud). This terminology reflects the very different roles played by otherwise similar items and hopefully makes things clearer once accepted...

3.1. **Planted plane trees.** The first remark is that the following binomial coefficient, taken from Formula (4),

$$A_{k,n} = \binom{2k+3n}{n+1} = \frac{1}{2k+3n+1}\binom{2k+3n+1}{1,n+1,2k+2n-1}$$

is the number of planted plane trees with (see also Figure 4.a)

- one special vertex of degree $2k-2$,
- $n+1$ generic vertices, of degree four,
- $2k+2n$ leaves (including the root) and their $2k+2n$ stems,
- and $n+1$ inner edges connecting the generic and special vertices.

This is nothing but the classical formula for planted plane trees with given numbers of vertices of each degree ([GJ83, p113]). Let us call $\mathcal{A}_{k,n}$ the family of these trees.

Formula (4) now reads

$$T_{k,n} \;=\; \frac{2}{2k+2n}\, 2^{n+1}\binom{2k-2}{k} A_{k,n}, \tag{5}$$

and one can recognize in this formula, the appearance of the numbers of leaves, generic vertices and edges incident to the special vertex.

3.2. **Blossom trees.** Let proceed with the interpretation of the formula by considering the factor

$$B_{k,n} \;=\; 2^{n+1}\binom{2k-2}{k} A_{k,n}.$$

Since a tree A of $\mathcal{A}_{k,n}$ has $n+1$ generic vertices of degree four, the factor 2^{n+1} can be interpreted as the number of ways to select two opposite corners on each generic vertex, while the binomial factor appears as the number of ways to select $k-2$ of the $2k-2$ edges incident to the special vertex.

Given such a selection, let us apply the transformation of Figure 2.a to generic vertices and, that of Figure 2.b to the special vertex. Each generic vertex is expanded into two vertices of degree four joined by a generic link, each carrying a bud. The selected edges on the special vertex are transformed to make room for a special link and two buds attached to a pathological vertex of degree four. Observe that in these constructions buds always immediately precede links in counterclockwise direction around created vertices.

(a) Generic vertices (b) Special vertex

FIGURE 2. From trees to blossom trees

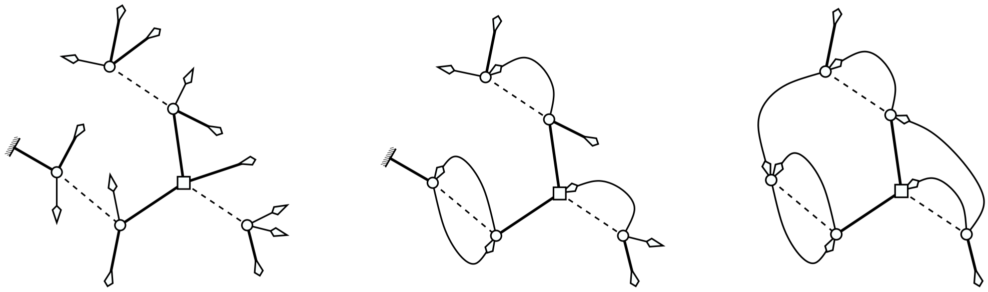


FIGURE 3. The partial closure of an unbalanced blossom tree

The set $\mathcal{B}_{k,n}$ of trees that are constructed in this manner from trees of $\mathcal{A}_{k,n}$ is of course of cardinality $B_{k,n}$. We call them *blossom trees*. By construction blossom trees are exactly the planted plane trees with (see also Figure 3 left, or Figure 4.b)

- one special vertex incident to $k-2$ special links and k edges;
- $k-2$ pathological vertices of degree four, incident to the $k-2$ previous links, each carrying two buds right before the link in counterclockwise order;
- $2n+2$ generic vertices of degree four, organized in $n+1$ pairs connected by generic links, each vertex carrying one bud right before the link in counterclockwise order;
- $2k+2n$ leaves, $2k+2n-2$ buds, and their $4k+4n-2$ stems,
- $n+1$ inner edges connecting some generic, pathological or special vertices.

Formula (4) now reads

$$T_{k,n} = \frac{2}{2k+2n} B_{k,n}, \tag{6}$$

making it inviting to distinguish two leaves among the $2k+2n$.

3.3. **Balanced blossom trees.** The *partial closure* of a blossom tree B consists in the following greedy procedure (see Figure 3 or Figure 4.d). Start with $\ddot{B}^{(0)} = B$, $i=1$.

(1) Find a bud b_i and a leaf ℓ_i such that, walking from b_i to ℓ_i around the infinite face of $\ddot{B}^{(i-1)}$ in counterclockwise direction, no other bud or leaf is met.
(2) Fuse b_i, ℓ_i and their stems into an edge m_i so as to create a bounded face around the previous walk. In particular this new bounded face contains no bud or leaf.
(3) Call $\ddot{B}^{(i)}$ the resulting map and, if it still contains buds, increment i and return to Step (1).

Observe that the latter loop continues until there is no more free bud. The operation in Step (2) is called the *matching* of b and ℓ, and the resulting edge is called a *matching edge*.

The result of this partial closure is a planar map $\ddot{B} = \ddot{B}^{(2k+2n-2)}$ with $k+2n$ vertices of degree four, one special vertex of degree $2k-2$, and two remaining leaves that we call *free* in the infinite face. This map $\ddot{B}$ is independent of the exact order in which buds and leaves have been matched, (exactly like in a balanced parenthesis word, there is only a partial order of inclusion of pairs, and a greedy algorithm performing the matching has a freedom in the order it deals with incomparable pairs).

A blossom tree is called *balanced* if its root is one of the two leaves that remain free throughout partial closure. Let $\mathcal{E}_{k,n}$ be the balanced subset of $\mathcal{B}_{k,n}$. Two blossom trees are called *conjugated* if they can be obtained one from another simply by changing the root leaf. The resulting conjugacy classes of $\mathcal{B}_{k,n}$ are naturally associated with unplanted trees. Matchings between buds and leaves only depend on the conjugacy class of the blossom tree, hence we can also consider the partial closure of an unplanted tree.

Now consider a blossom tree B with root leaf r and let ℓ be one of the two free leaves of B. Taking now ℓ as root of B, a balanced blossom tree with a secondary distinguished leaf r is obtained. This yields[1]:

$$2\, B_{k,n} \;=\; (2k+2n)\, E_{k,n}$$

where $E_{k,n}$ denote the number of balanced blossom trees.

As a consequence, Formula (4) finally reads

$$T_{k,n} \;=\; E_{k,n},$$

and we are lead to seek a bijection between triangulations and balanced blossom trees.

3.4. **Case of $\mathcal{T}_n$.** A similar (but much simpler) construction provides an interpretation of Tutte's enumerative formula for the set $\mathcal{T}_n$ of loopless triangulations with $2n$ triangles, that can be rewritten in the following way:

$$T_n \;=\; \frac{2}{2n+2}\, 2^n\, \frac{1}{2n+1}\binom{3n}{n}. \tag{7}$$

The coefficient $\frac{1}{2n+1}\binom{3n}{n}$ is the number of planted plane ternary trees with n internal nodes, that is trees with n generic vertices of degree four, $n-1$ inner edges and $2n+2$ stems and leaves (including the root). The blossom trees obtained from these trees by the transformation of Figure 2.a have $2n$ generic vertices with their n links and $2n$ buds, $2n+2$ leaves, $4n+2$ stems and $n-1$ inner edges. Let $\mathcal{B}_n$ be the set of these blossom trees without special vertex. After the partial closure of any of these trees, two leaves remain unmatched, so the ratio of balanced blossom trees in $\mathcal{B}_n$ is $\frac{2}{2n+2}$. Hence the corresponding subset $\mathcal{E}_n$ has cardinality

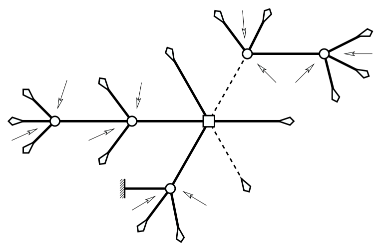
$$E_n \;=\; \frac{2}{2n+2}\cdot 2^n \cdot \frac{1}{2n+1}\binom{3n}{n} \;=\; T_n.$$

Notations. In the rest of the paper, $\mathcal{E}$ denotes the set of all balanced blossom trees (with or without special vertex), and $\mathcal{U}$ the set of all *unplanted* blossom trees. Any tree in $\mathcal{U}$ corresponds to one or two trees in $\mathcal{E}$, depending on its automorphism group.

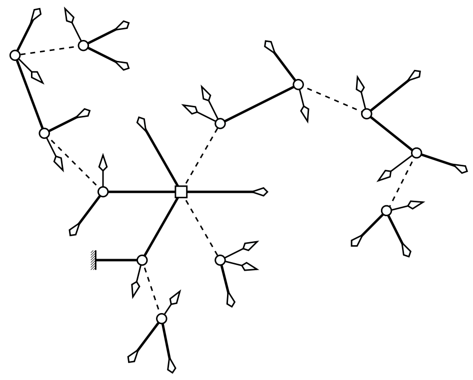
3.5. **The complete closure.** In fact the bijection was almost already completely described. Let us define the *complete closure* φ as an application defined on the set $\mathcal{E}$. Given B a tree of $\mathcal{E}$,

(1) construct the partial closure $\ddot{B}$ of B,
(2) remove all the links and call $\overline{B}$ the result,
(3) fuse the two remaining stems of $\overline{B}$ into a root edge oriented away from the root of B,

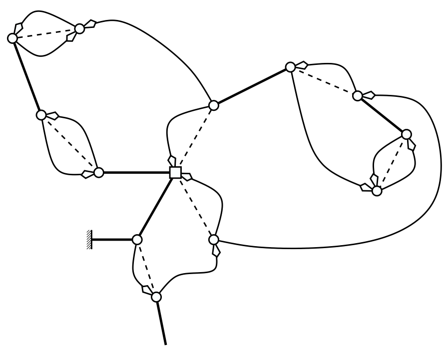
[1] Observe that this relation is the translation for conjugacy classes of trees of the *cycle lemma* for conjugacy classes of Łukasiewicz words. This lemma, initially due to Dworetzki and Motzkin, underlies Raney's combinatorial proof of the Lagrange inversion formula [Lot97, Chap. 11]. This analogy motivates our choice of terminology.



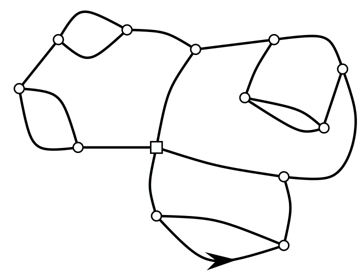
(a) A tree in $\mathcal{A}_{4,4}$, in which some corners and edges are distinguished;



(b) the corresponding (balanced) blossom tree;



(c) its partial closure, revealing the two free leaves;



(d) and the corresponding rooted near-cubic map.

FIGURE 4. An example of complete closure

the resulting rooted planar map is $\varphi(B)$. (See Figure 4 for a complete example.)

Our main result, to be proved in the rest of the paper, is the following theorem.

Theorem 1. *The complete closure φ is a bijection from the set $\mathcal{E}_{k,n}$ (resp. $\mathcal{E}_n$) of balanced blossom trees onto the set $\mathcal{C}_{k,n}$ (resp. $\mathcal{C}_n$) of (near-)cubic maps and by duality onto the set $\mathcal{T}_{k,n}$ (resp. $\mathcal{T}_n$) of triangulations.*

The proof is twofold. First we prove that the complete closure of a tree is indeed a non-separable (near-)cubic map. Then we prove that the application is one-to-one.

4. The closure of a balanced blossom tree is non-separable

Let B be a balanced blossom tree of $\mathcal{E}_{k,n}$. In this section we prove that the complete closure $\varphi(B)$ is indeed a non-separable near-cubic map with the expected number of vertices of each kind.

The vertices of degree four of B, either generic or pathological, are incident to exactly one link. After Step (2) of the complete closure, they result in vertices of degree three. As for the special vertex, it is incident to $k-2$ links and k edges so that it yields a vertex of degree k in $\varphi(B)$. The rooted planar map $\varphi(B)$ hence contains a vertex of degree k and $k+2n$ vertices of degree three. As a consequence, $\varphi(B)$ belongs to $\mathcal{C}_{k,n}$ if and only if it is non-separable, a fact we shall now prove. Similarly, if B belongs to $\mathcal{E}_n$, $\varphi(B)$ belongs to $\mathcal{C}_n$ if and only if it is non-separable.

Observe that, since the matching of buds and leaves only depends on the conjugacy class of a blossom tree, the non-separability of the complete closure is indeed a property of the underlaying unplanted tree, and not of the balanced rooting. From now on, for the sake of convenience, we consider an unplanted blossom tree U in $\mathcal{U}$.

A preliminary observation is that any separating vertex of degree three is incident to a separating edge. It is thus sufficient to prove on the one hand that $\varphi(U)$ has no separating edge (Section 4.1 to 4.4) and on the other hand that the possible special vertex is not separating (Section 4.5).

4.1. **A preliminary lemma on the structure of blossom trees.** Consider U a blossom tree of $\mathcal{U}$ and $e = (v_1, v_2)$ an inner edge or a link of A. The *decomposition* of U at e consists in cutting e in its middle, so as to create two new leaves ℓ_1 and ℓ_2, attached by two stems e_1 and e_2 respectively to v_1 and v_2. As a result, the tree U yields two subtrees $U_1(e)$ and $U_2(e)$, respectively containing v_1 and v_2. A leaf ℓ of U is said *incoming* with respect to e if, in the partial closure of U, it is free or matched to a bud b that does not belong to the same subtree as ℓ (with respect to e). By extension, the matching edge (b, ℓ) is also called incoming with respect to e.

The following lemma is immediate upon counting leaves and buds in each subtree and considering the cyclic orders around v_1 and v_2.

Lemma 1. *Let U be a blossom tree and e an inner edge or a link of U.*

- *If e is an inner edge, then $U_1(e)$ and $U_2(e)$ are well formed blossom trees, with two more leaves than buds (including ℓ_1 and ℓ_2), and thus at least one incoming leaf each.*
- *If e is a link between two generic vertices, then $U_1(e)$ and $U_2(e)$ contain two more leaves than buds (including ℓ_1 and ℓ_2). Moreover in the partial closure of U, the bud adjacent to v_1 in $U_1(e)$ is matched with an incoming leaf of $U_2(e)$.*
- *If e is a link incident to the special vertex (assumed in $U_1(e)$), then $U_1(e)$ has four more leaves than buds and $U_2(e)$ has as many buds as leaves (including ℓ_1 and ℓ_2). As a consequence, in the closure of U, the two buds adjacent to v_2 in $U_2(e)$ are matched to two incoming leaves of $U_1(e)$, and $U_2(e)$ has at least one incoming leaf.*

4.2. **The incremental complete closure.** Let us now consider an application of the (greedy) partial closure procedure of Section 3.3 to U, resulting into the map $\ddot{U}$ through the sequences b_i, ℓ_i, m_i and $\ddot{U}^{(i)}$, for $i \geqslant 1$. Given a matching edge m, obtained from (b, ℓ), we define $e(m)$ to be the unique link incident to the vertex adjacent to b. By construction, for each link e of A there are exactly two indices $j < i$ such that $e(m_i) = e(m_j) = e$. Let us call these indices, the *dates* of e. Finally define a planar map $\underline{U}^{(i)}$ by deleting from $\ddot{U}^{(i)}$ all *generic* links with largest date less or equal to i. In other terms, $\underline{U}^{(i)}$is constructed from $\underline{U}^{(i-1)}$ by adding m_i and removing $e(m_i)$ if it is generic and the other matching edge m_j such that $e(m_j) = e(m_i)$ satisfies $j < i$.

Let finally $\underline{U} = \underline{U}^{(2k+2n-1)}$ be the resulting map. The following technical lemma precisely describes the evolution of connectedness in $\underline{U}^{(i)}$ for $i = 1, \ldots, 2k + 2n - 2$.

Lemma 2. *For all i the planar map $\underline{U}^{(i)}$ is connected. Moreover for any link or inner edge e of U, the graphs induced in $\underline{U}^{(i)}$ respectively by the vertices of $U_1(e)$ and by those of $U_2(e)$ are connected.*

Proof. The lemma is obviously true for the tree $\underline{U}^{(0)} = U$. Assume now the lemma true until $i - 1$ and consider the construction of $\underline{U}^{(i)}$ from $\underline{U}^{(i-1)}$. Let $e' = e(m_i)$ and j be the other date of e'.

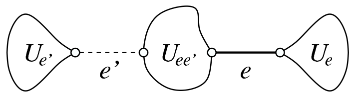


FIGURE 5. Simultaneous decomposition of U at e and e'

There is a deletion only if e' is a generic link and $j < i$. In this case, observe first that, according to Lemma 1, the matching edge m_i connects a vertex of $U_1(e')$ to a vertex of $U_2(e')$ so that by induction hypothesis $\underline{U}^{(i)}$ remains connected upon deleting e'.

Then consider another link or inner edge e and the decompositions of U at e and e': performing both decompositions yields three subtrees, U_e, $U_{ee'}$ and $U_{e'}$ where the indices refer to incidences with e and e' (see also Figure 5). In $\underline{U}^{(i-1)}$, the graphs induced respectively by U_e, $U_{e'}$, and $U_{e'} \cup U_{ee'}$ are connected by induction hypothesis. The deletion of e' does not touch the graph induced by U_e so that we only have to deal with the graph induced by $U_{e'} \cup U_{ee'}$. Since e' is generic, one of m_i or m_j is incident to the endpoint of e' in $U_{ee'}$ and has its other endpoint (the leaf) in $U_{e'}$. Since $U_{e'}$ is connected, the deletion of e' does not disconnect $U_{e'} \cup U_{ee'}$.

□

4.3. Separating edges and generic links.

Lemma 3. *The only separating edges in $\underline{U}$ are inner edges of U that separate the two free leaves.*

Proof. Consider a matching edge m, and let $e = e(m)$ and m' be the second matching edge with $e(m') = e$. Then Lemma 1 asserts that m and m' are incoming with respect to e. In view of Lemma 2, their respective endpoints on both sides of e can be connected to construct a cycle containing m and m'. Moreover, if e is special, the same argument provides a cycle through e and m. Hence neither matching edges nor links can be separating.

Let now e be an inner edge of A that remains separating in $\underline{U}$. Consider the decomposition of U at e. No matching edge connects a vertex of $U_1(e)$ to a vertex of $U_2(e)$, for e would not be separating (Lemma 2). In view of Lemma 1, this implies that there is one free leaf in both subtrees. □

4.4. Matching edges are not separating in $\varphi(U)$.

Lemma 4. *The only separating edges in $\overline{U}$ are inner edges of U that separate the two free leaves.*

Proof. If U has no special vertex, $\overline{U} = \underline{U}$, hence this lemma is equivalent to Lemma 3.

Now suppose that U has a special vertex with degree $2k - 2$. In order to show that the removal of the special links from $\underline{U}$ does not make any matching edge separating, it is sufficient to prove that any two faces that are merged by removing some special links have no common matching edge.

Let us consider the $2k - 2$ subtrees of U at the special vertex v, more precisely defined as the subtrees not containing v in the decomposition of U at any edge or link incident to v. We call such a subtree generic or pathological depending on whether it is attached to the special vertex by an inner edge or by a link. At any step i of the construction of Section 4.2, Lemma 2 ensures that these subtrees induce connected subgraphs of $\underline{U}^{(i)}$, and that, according to Lemma 1, any of them has at least one incoming leaf.

Given a particular ordering of the matchings in the application of the partial closure procedure, let us consider the time j at which, for the first time, an incoming edge of a subtree at v is matched. This is also the first time that a matching edge is created between

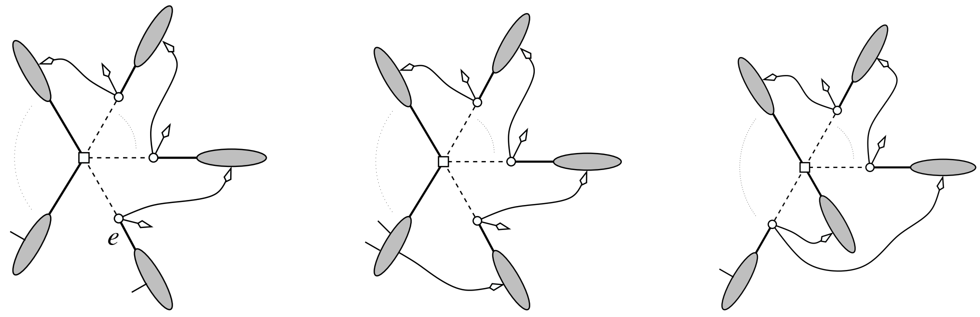


FIGURE 6. Case study for the deletion of special links.

two subtrees. Let us now consider an ordering such that j is as large as possible. In this case, at time j, all the matchings that are internal to each subtree have been performed. More precisely, with the notation of Section 4.2, the ordering is such that, for any $i < j$, b_i and ℓ_i belong to the same generic or pathological subtree at v, and, for any $i \geqslant j$, b_i and ℓ_i belong to different subtrees at v or make a complete turn around v.

Perform then the construction until Step $j - 1$. As already observed, at that moment the subtrees are two by two independent. Moreover, at each pathological vertex, the bud that precedes the link in counterclockwise order is in position to be matched with the first incoming leaf of the next subtree (in counterclockwise order around the special vertex). Every such matching creates a bounded face, which cannot be affected by any further step since it does not contain generic links (Figure 6, left).

Once these $k - 2$ matchings are performed, only two kinds of buds can be matched in such a way that the created bounded face contains a special link: the first unmatched bud of a generic subtree that is to be matched into a pathological one (Figure 6, middle), or the second bud of a pathological vertex that precedes a sequence of subtrees with no more unmatched leaf or bud (Figure 6, right). These matchings also create faces that will not be affected by any further step.

These different cases lead to three different ways for a sequence of faces to be merged into one face by the removal of special links. In each case, we shall argue that, as a whole, these faces do not complete a turn around v.

- In the first case (Figure 6 left), a (non-empty) sequence of bounded faces merge with the infinite face of $\underline{U}$. No bounded faces can thus perform a complete turn around v. Hence two non successive faces in the sequence share no edge, and two successive faces share a special link. In any case they do not have a matching or inner edge in common. As for the infinite face, in view of the disposition of buds, it may only be incident twice to the inner edge marked e in the figure. In this case the shaded subtree below e contains exactly one of the two free leaves so that e separates the two free leaves.
- The second case involves two generic subtrees and a non-empty sequence of pathological ones. Since $k \geqslant 2$, the two generic subtrees are distinct, and there is no complete turn.
- In the third case, there exists a pathological vertex p_1 such that its second bud is matched with a leaf that belongs to a pathological subtree attached on a pathological vertex p_2. This implies that the sequence S of subtrees that follow p_1 and precede p_2 around the special vertex has only one free leaf. In other words, this sequence contains exactly one more generic subtree than pathological ones. Hence the number of involved pathological subtrees is at least the number of involved generic subtrees

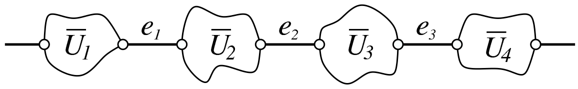


FIGURE 7. Separating inner edges are organised in a sequence in $\overline{U}$.

minus one, and at least one generic subtree at v is not involved. Finally p_1 is different from any involved pathological vertex that follows S and the complete turn is not performed.

We conclude that matching edges are not separating edges and that all separating inner edges still separate the two leaves. Since all links have been removed, the lemma is proved. □

This lemma proves that $\overline{U}$ can be described as an alternating sequence $\overline{U}_1, e_1, \overline{U}_2, \ldots, e_{p-1}, \overline{U}_p$ of submaps $\overline{U}_i$ and edges e_i (with p possibly equal to one), such that $\overline{U}_1$ and $\overline{U}_p$ carry one free leaf each, and no $\overline{U}_i$ contains a separating edge (Figure 7). As a consequence, $\varphi(U)$ has no separating edge.

4.5. **The special vertex.** Suppose that U has a special vertex. The following lemma concludes the proof that $\varphi(U)$ is non-separable.

Lemma 5. *The special vertex v is not a separating vertex of $\varphi(U)$.*

Proof. Assume that the special vertex v is separating in $\overline{U}$ (as given by Step (2) of complete closure) and consider a decomposition of $\overline{U}$ into two components $\overline{U}_1$ and $\overline{U}_2$ connected only at v. This decomposition induces a decomposition of $\ddot{U}$: special links connect v to a vertex of $\overline{U}_1$ or $\overline{U}_2$ and do not interfere; once special links are replaced, generic links appear inside bounded faces and hence inside the two components.

In turn the decomposition of $\ddot{U}$ at v induces a decomposition of the tree U into two sequences of subtrees rooted at v such that there is no matching edge from one to the other. Since Lemma 1 provides in particular an incoming leaf on the first tree of both these sequences, these leaves must be the two free leaves of U.

Returning to $\overline{U}$, we conclude that $\overline{U}_1$ and $\overline{U}_2$ each contain one free leaf. Hence v is not a cut vertex anymore after Step (3) of the complete closure. □

5. THE INVERSE CONSTRUCTION

In this section we define by induction on the number of edges a construction which is inverse to the complete closure.

Let us first consider the minimal cases of non-separable (near-)cubic maps with at most two vertices. The case $k = 2$, $n = -1$ is the degenerate case of the loop at a special vertex and corresponds to the tree with one special vertex of degree two. The case $n = 1$, without special vertex, is the case of a bundle of three edges between two vertices and corresponds to the unique balanced blossom tree with two generic vertices. The case $k = 3$, $n = -1$ is the case of a bundle of three edges between two vertices, one of them being special; the two different rootings of this map correspond to the two balanced rootings of the unique blossom tree with a special vertex of degree three and one single pathological vertex.

Now suppose that C is a rooted non-separable (near-)cubic map with at least three vertices among which, possibly, a special vertex of any degree, and the others of degree three. Let the root edge be oriented from a vertex v_1 to a vertex v_2, and define $\tilde{C}$ by cutting the root edge into two stems with leaves f_1 and f_2. If there exists $B \in \mathcal{E}$ such that $C = \varphi(B)$, then B is necessarily planted on leaf f_1, and reconstructing B is equivalent

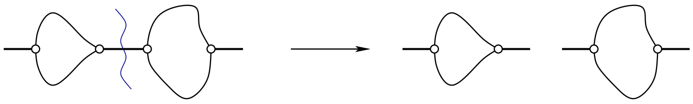


FIGURE 8. Induction with separating edge.

to recovering links between vertices: these links determine which vertices are generic or pathological, and which stems carry leaves or buds.

Given a map $\tilde{C}$, our strategy is to exhibit one (or more) link that exists *necessarily* in any tree U in $\mathcal{U}$ such that $\overline{U} = \tilde{C}$. Moreover we show that the decomposition of U at this link yields subtrees whose images by φ can be uniquely characterized as some strict submaps of $\tilde{C}$. The induction applied to these submaps allows to prove that φ is one-to-one.

The construction depends on whether $\tilde{C}$ is separable.

5.1. **The map $\tilde{C}$ contains a separating vertex other than v_1 or v_2.** Since C is non-separable, the map $\tilde{C}$ is organized as a chain of non-separable components between v_1 and v_2. In this section, *separating vertices* refer to separating vertices distinct from v_1 and v_2. Two cases are distinguished.

– First case: the map $\tilde{C}$ has a separating vertex v that is not the special vertex (see Figure 8). In this case v has degree three and, as already argued, there is a separating edge e. In view of the discussion of the previous section, if there is a tree U in $\mathcal{U}$ such that $\overline{U} = \tilde{C}$ then e is an inner edge of U, and in the decomposition of U at e, the leaves f_1, ℓ_1 and f_2, ℓ_2 are the free leaves of $U_1(e)$ and $U_2(e)$, (so that their partial closure are independent).

Now there is a unique way to recover such a structure. First cut e in $\tilde{C}$ into two stems e_1 and e_2 with leaves ℓ_1 and ℓ_2. The resulting two components of $\tilde{C}$ allow to recover U_1 and U_2 by induction hypothesis and the unique tree U is obtained by fusing back e_1 and e_2 between U_1 and U_2.

– Second case: the special vertex v is the only separating vertex of $\tilde{C}$ (Figure 9). Let C_1 and C_2 be the two non-separable components of $\tilde{C}$ at v. As was already analyzed in Section 4.5, if there is a tree U such that $\overline{U} = \tilde{C}$, then the links or edges incident to v in U are arranged in counterclockwise order into two successive sequences $e_1, \dots, e_p$ with endpoints in C_1 and $e'_1, \dots, e'_q$ with endpoints in C_2, with p and q greater or equal to two in order to avoid separating edges.

Let us prove that the subtree S of U attached to e_1 (resp. to e'_1) is reduced to a special link carrying a pathological vertex. By construction of the two sequences, the incoming leaves of S are free, so that there is at most one such leaf. In view of Lemma 1, there is exactly one. Now if e_1 is an inner edge, Lemma 1 implies that there is no matching edge leaving this subtree, and e_1 is a separating edge of $\tilde{C}$. Therefore e_1 is a special link, which by definition carries a pathological vertex. Finally the subtree cannot be bigger otherwise the pathological vertex would carry an edge and the latter would be separating in $\tilde{C}$.

Hence the tree U is decomposed at v into a special link e_1 that carries a pathological vertex, followed by the tree U_1 formed of $e_2, \dots, e_p$ and their subtrees, by a special link e'_1 that carries a pathological vertex, and by the tree U_2 formed of $e'_2, \dots, e'_p$ and their subtrees. Moreover U_1 and U_2 are well formed blossom trees whose free leaves are respectively matched by the buds of the two pathological vertices.

Now there is a unique way to recover such a structure. First v_1 and v_2 are identified as pathological vertices (since they carry the free leaves). Then, deleting v_1 and v_2 from their respective non-separable component yields two maps C_1 and C_2 from which U_1 and U_2 can

FIGURE 9. Induction with separating special vertex.

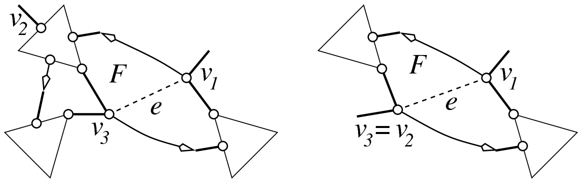


FIGURE 10. Analysis of the possible configurations in the main case of recursion.

be recovered by induction hypothesis. The unique tree U is obtained upon recreating the cyclic order around v.

5.2. **The map $\tilde{C}$ has no other separating vertex than v_1 and v_2.** A first easy case is when the special vertex v carries one of the two free leaves of $\tilde{C}$, say f_1 so that $v = v_1$. Then the analysis is exactly the same as the analysis of the case where v is the only separating vertex (second case of the previous section), with the second sequence reduced to a single leaf: v_2 is found to be pathological and upon deleting v_2 and f_1 the induction hypothesis applies to provide a unique reconstruction.

The main case is when the special vertex v is neither v_1 nor v_2. Assume, without loss of generality, that v is not on the counterclockwise path around the infinite face from v_1 to v_2. (Even if v is incident to the infinite face, it cannot appear in both path from v_1 to v_2 and back from v_2 to v_1.) Let us discuss the constraints on a tree U such that $\overline{U} = \tilde{C}$.

Observe first that v_1 cannot be a pathological vertex: even if the special vertex v is incident to the infinite face, this yields a contradiction in the way its buds are to be matched. Hence v_1 is a generic vertex of U. Let F be the bounded face incident to v_1 in $\tilde{C}$. A generic link e joins v_1 to another vertex v_3 in this face F. Let us consider the subtrees attached to v_1 and v_3 in F (Figure 10).

Since vertex v_1 is adjacent to a free leaf f_1 and a bud, it carries a unique (possibly empty) subtree, which precedes f_1 in counterclockwise order. Call this subtree S_1. According to Lemma 1 and in view of the free leaf f_1, the subtree S_1 has only one incoming leaf. The latter is therefore matched by the bud of v_3 and this matching edge is incident to both F and the infinite face. Moreover there is no other edge incident to F on the path from v_2 to v_3 along the infinite face in counterclockwise direction. Indeed this could only be an inner edge e' (for A to be connected) but then the subtree at e' containing v_2, having two incoming leaf (f_2 and the leaf matched by the bud of v_1) would have, according to Lemma 1 a bud taking part of an incoming matching edge. However this would prevent e' to be on the infinite face.

Consider next the decomposition of U at e and take $U_1(e)$ to contain v_1. In view of its previous definition, the tree S_1 is obtained from $U_1(e)$ upon deleting v_1 and it is a balanced blossom tree. On the other hand, define a tree S_2 from $U_2(e)$ as follows. First delete the bud and the stem inherited from e that are incident to v_3, so that the latter vertex has degree two. Then smooth this vertex out so as to fuse its two incident edges into one single edge e' (which may be a stem). The result is a tree $U_2(e)$ whose closure leaves e' in the infinite face, and whose free leaves are f_2 and the leaf ℓ_1 matched with the bud of v_1 in U.

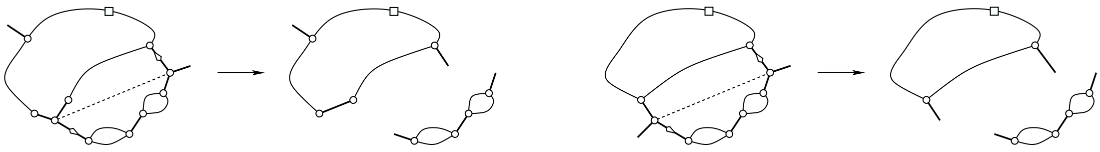


FIGURE 11. Recursive decompositions in the two cases of Figure 10.

Finally there is a unique way to recover the structure (Figure 11). First, taking F to be the bounded face incident to v_1, we dispose of a characterization of vertex v_3 as the first vertex incident to F on the path from v_2 to v_1 around the infinite face in counterclockwise direction. In particular if v_2 is incident to F then $v_3 = v_2$ (as illustrated on the right hand side of Figure 10). Second, the complete closure of the trees S_1 and S_2 are uniquely obtained as follows. Delete f_1 and its stem and cut v_1 so as to create two new leaves ℓ_1 (for the bud of v_1) and ℓ_2 (for the subtree). Detach the edge that follows v_3 along the infinite face from v_2 to v_1: this edge is also incident to F and this operation creates a leaf ℓ_3 in the same component as ℓ_2. Call this component $\tilde{C}_1$. The vertex v_3 remains of degree two and can be smoothed so as to fuse its two incident edges into one single edge e that belongs to a second component, $\tilde{C}_2$, that also contains ℓ_1 and v_2. In view of the previous analysis, the two maps $\tilde{C}_1$ and $\tilde{C}_2$ are the images of S_1 and S_2 by complete closure (upon opening the roots). By induction hypothesis, there exists exactly one couple of such trees. From S_1 and S_2 the tree U is readily recovered and the proof is complete.

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DOMINIQUE POULALHON, LIX, ÉCOLE POLYTECHNIQUE, PALAISEAU
E-mail address: `poulalho@lix.polytechnique.fr`

GILLES SCHAEFFER, CNRS – LORIA, NANCY
E-mail address: `Gilles.Schaeffer@loria.fr`
URL: `http://www.loria.fr/~schaeffe`

ON THE ASYMPTOTIC ANALYSIS OF A CLASS OF LINEAR RECURRENCES

THOMAS PRELLBERG

ABSTRACT. Many problems in enumerative combinatorics can be expressed via linear recurrences whose generating functions satisfy a functional equation. I present the asymptotic analysis for a class of recurrences which lead to a functional equation involving a transformation with a parabolic fixed point. The method used relies heavily on analytic iteration theory. Examples given are Bell numbers, Partition lattice enumerations, and the Takeuchi recursion.

RÉSUMÉ. Beaucoup de problémes en combinatorique eńumérative peuvent être exprimés par des récurrences linéaires dont les fonctions generatrices obeissent des équations fonctionelles. Je vais présenter dans cette note l'analyse asymptotique pour une classe de récurrences qui emmènent à une équation fonctionelle contenant une transformation avec point fixe parabolique. La méthode utilisée est basée sur la théorie de l'itération analytique. Les exemples discutés sont les nombres de Bell, l'énumérations de réseaux partition et la récurrence de Takeuchi.

INSTITUT FÜR THEORETISCHE PHYSIK, TECHNISCHE UNIVERSITÄT CLAUSTHAL, ARNOLD-SOMMERFELD-STRASSE, D-38678 CLAUSTHAL-ZELLERFELD, GERMANY

E-mail address: thomas.prellberg@tu-clausthal.de

PERMUTATIONS BY NUMBERS OF ANTI-EXCEDANCES AND FIXED POINTS

FANJA RAKOTONDRAJAO

Abstract. In this paper we study the bivariated distribution of anti-excedances and fixed points on the symmetric group and give its generating function. We also study the same distribution on the alternating group and its complement, and prove combinatorially a relation between the number of odd permutations and the number of even permutations having a given number of anti-excedances and a given set of fixed points.

Résumé. Nous étudions dans ce papier la distribution de la statistique bivariée des anti-excédances et points fixes sur le groupe symétrique et nous donnons la fonction la fonction génératrice de cette distribution. Nous étudions également cette distribution sur le groupe alterné et son complémentaire, et démontrons combinatoirement une relation entre le nombre de permutations impaires et le nombre de permutations paires ayant un nombre fixé d'anti-excédances et un ensemble donné de points fixes.

1. Introduction

The distribution of fixed point distribution and the so called "Eulerian statistics" over the symmetric group have been studied in depth but always separately ([7], [2], [1]). In this paper we study the bivariated distribution of the Eulerian statistic *anti-excedances* and of the statistic of fixed points on the symmetric group. Recently, the author and Roberto Mantaci [5] have introduced and studied the numbers $a_{n,k}$ that count the derangements over n objects having k anti-excedances, this paper is a generalization of those results. We will study in this paper the numbers $a_{n,k,m}$ that count the permutations over n objects having k anti-excedances and m fixed points. We will give an exponential generating function for these numbers as well as a recursive relation defining them. We will also study the distribution of the bivariate distribution of anti-excedances and fixed points on the alternating group and its complement and prove that the number of odd permutations and the number of even permutations having k anti-excedances and having the same given set F of fixed points differ by 1 for all integers n and k and for all subset F of $[n]$. We will give a combinatorial proof of this result.

Let us denote by $[n]$ the interval $\{1, 2, \cdots, n\}$, by σ a permutation of the symmetric group $\mathcal{S}_n$ and by $\mathcal{A}_n$ the alternating group of rank n, that is, the group of all even permutations.

Definition 1.1. *We will say that $i \in [n]$ is a fixed point for σ if $\sigma(i) = i$.*

Definition 1.2. *We will say that σ presents an anti-excedance (resp. an excedance) in $i \in [n]$ if $\sigma(i) \leq i$ (resp. $\sigma(i) > i$). In this case we will say that i is an anti-exceedant (resp. an exceedant).*

We will denote by $Fix(\sigma)$ the set of the fixed points of σ, by $FIX(\sigma)$ the integer $|Fix(\sigma)|$ and by $AX(\sigma)$ the number of the anti-excedances of σ. We will denote by $\mathcal{F}_{n,m}$ the set of permutations over n objects having m fixed points and by $f_{n,m}$ the cardinality of this set, by $\mathcal{S}_{n,k}$ the set of permutations over n objects having k anti-excedances and by $A_{n,k}$ the cardinality of this set. The numbers $A_{n,k}$ are the classical Eulerian numbers (see [1], [7]). The first values of the numbers $A_{n,k}$ are given in the following table:

$A_{n,k}$						
	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	4	1		
4	0	1	11	11	1	
5	0	1	26	66	26	1

These numbers satisfy the following recursive relation:

$$A_{n,k} = (n-k+1)A_{n-1,k-1} + kA_{n-1,k} \qquad (1 \leq k \leq n).$$

The Eulerian polynomials $A_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{AX(\sigma)}$ have the following generating function (see [1]):

$$A(t,u) = \sum_{n \geq 0} A_n(t)\frac{u^n}{n!} = \frac{(1-t)}{1-t\exp((1-t)u)}.$$

The first values of the numbers $f_{n,m}$ are given in the following table:

$f_{n,m}$						
	m=0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	1	0	1			
3	2	3	0	1		
4	9	8	6	0	1	
5	44	45	20	10	0	1

The numbers $f_{n,m}$ satisfy the following recursive relation (see [6])

$$f_{n,m} = f_{n-1,m-1} + (m+1)f_{n-1,m+1} + (n-1-m)f_{n-1,m} \qquad (0 \leq m \leq n).$$

The generating function of the fixed point distribution on the symmetric group is given by [7]:

$$F(x,u) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{FIX(\sigma)} \frac{u^n}{n!} = \frac{\exp((x-1)u)}{1-u}.$$

2. The anti-excedance and fixed point distribution on the symmetric group

We will give a recursive relation for the distribution of the bivariate statistic (AX, FIX) on the symmetric group $\mathcal{S}_n$, and will give its generating function. We will denote by $\mathcal{F}_{n,k,m}$ the set of permutations in $\mathcal{S}_{n,k}$ having m fixed points (or equivalently, the set of permutations in $\mathcal{F}_{n,m}$ having k anti-excedances) and denote $a_{n,k,m}$ the cardinality of $\mathcal{F}_{n,k,m}$. The first values of these numbers are given in the following tables for some fixed integers

m:

$m=0$				
	k=1	2	3	4
$n=1$	0			
2	1			
3	1	1		
4	1	7	1	
5	1	21	21	1

$m=1$					
	k=3	4	5	6	7
$n=1$	1				
2	0	0			
3	0	3	0		
4	0	4	4	0	
5	0	5	35	5	0

$m=2$					
	k=2	3	4	5	
$n=2$	1				
3	0	0			
4	0	6	0		
5	0	10	10	0	
6	0	15	105	15	0

$m=3$					
	k=3	4	5	6	7
$n=3$	1				
4	0	0			
5	0	10	0		
6	0	20	20	0	
7	0	35	245	35	0

Proposition 2.1. *For all integer n, one has $a_{n,n,n} = 1$.*

Proof. The identity is the only permutation of the symmetric group having n anti-excedances and n fixed points. $\square$

Proposition 2.2. *For a fixed integer m and a fixed integer n, the numbers $a_{n,k,m}$ are symmetric, in the sense that $a_{n,k,m} = a_{n,n-k+m,m}$ for all integers $n \geq 1$ and $m \leq k \leq n$.*

Proof. The bijective map $\sigma \mapsto \sigma^{-1}$ associates a permutation σ in the set $\mathcal{F}_{n,k,m}$ with a permutation in the set $\mathcal{F}_{n,n-(k-m),m}$. $\square$

Remark 2.3. *We have $a_{n,k,m} = 0$ if $k < 0$ or $m > k$.*

Theorem 2.4. *For all integers n, k and m such that $m \geq 0$ and $m \leq k \leq n$, one has:*

$$a_{n,k,m} = a_{n-1,k-1,m-1} + (m+1)a_{n-1,k,m+1} + (n-k)a_{n-1,k-1,m} + (k-m)a_{n-1,k,m}$$

with the initial value $a_{0,0,0} = 1$.

Proof. Notice that all permutation σ' in the symmetric group $\mathcal{S}_n$ is obtained from a permutation σ in $\mathcal{S}_{n-1}$ by multiplying σ on the left by a transposition (i, n) for an integer $i \in [n]$ and we suppose that the integer n is an exceedant for σ. Notice also that if the integer i is an exceedant (resp. an anti-exceedant) for σ, then when we multiply σ by the transposition (i, n) on the left, we create (resp. do not create) a new exceedant for σ'. When it is created, this new exceedant is the integer n itself. Now let us look for the various cases for the integer i:

(1) If $i = n$, then the transposition (i, n) is not a transposition but the 1- cycle (n) and the permutation $\sigma' = (n)\sigma$ has a new anti-exceedant fixed point, which is the integer n itself.
(2) If the integer i is an exceedant for σ, then the permutation $\sigma' = (i, n)\sigma$ has a new anti-excedant, which is the integer n itself, but does not have any new fixed point.
(3) If the integer i is a fixed point of the permutation σ, then the permutation $\sigma' = (i, n)\sigma$ does not have a new anti-exceedant but it has one fewer fixed points than the permutation σ.
(4) If the integer i is an anti-exceedant non fixed point of the permutation σ, then the permutation $\sigma' = (i, n)\sigma$ has neither a new anti-exceedant nor a new fixed point.

It follows straightforwardly that we obtain all the permutations in the set $\mathcal{F}_{n,m}$ having k anti-excedances by considering all the permutation σ indicated in the following four cases and by multiplying them by the appropriate transposition :

(1) $\sigma \in \mathcal{F}_{n-1,m-1}$ having $k-1$ anti-excedances and the only possibility for the choice of "transposition" by which multiply σ is the 1-cycle (n).
(2) $\sigma \in \mathcal{F}_{n-1,m+1}$ having k anti-excedances and there exist $m+1$ possibilities for the choice of the transposition : the transpositions (i, n) where the integer i is a fixed point of the permutation σ.
(3) $\sigma \in \mathcal{F}_{n-1,m}$ having $k-1$ anti-excedances and there exist $n-1-(k-1)$ possibilities for the choice of the transposition : the transpositions (i, n) where the integer i is an exceedant of the permutation σ.
(4) $\sigma \in \mathcal{F}_{n-1,m}$ having k anti-excedances and there exist $k-m$ possibilities for the choice of the transposition : the transpositions (i, n) where the integer i is an anti-exceedant non fixed point of the permutation σ.

□

2.1. **Generating function.** Let us denote by $A(t, x, u)$ the exponential generating function of the distribution of the bivariate statistic $(AX,\ FIX)$ on the symmetric groups, that is, the function $A(t, x, u)$ defined by :

$$A(t, x, u) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} t^{AX(\sigma)} x^{FIX(\sigma)} \frac{u^n}{n!}.$$

Proposition 2.5. *The generating function $A(t, x, u)$ of the distribution of the bivariate statistic $(AX,\ FIX)$ on the symmetric groups satisfies the following differential equation :*

$$xtA = (1 - tu)\frac{\partial A}{\partial u} + t(t-1)\frac{\partial A}{\partial t} + (x-1)\frac{\partial A}{\partial x}$$

with the initial conditions :

$$A(t, 1, u) = \frac{1-t}{1 - t\exp\left((1-t)u\right)}$$

and $A(1, 1, 0) = 0$.

Proof. The differential equation can be easily derived from the recurrence relation given in Theorem 2.4. The initial condition :

$$A(t, 1, u) = \frac{1-t}{1 - t\exp\left((1-t)u\right)}$$

is due to the fact that when we set $x = 1$ the resulting formal series is the well-known generating function of the Eulerian polynomials and when we set $x = 1, t = 1$ and $u = 0$ we obtain the first value of the numbers $a_{n,k,m}$. □

Theorem 2.6. *The generating function*

$$A(t, x, u) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} t^{AX(\sigma)} x^{FIX(\sigma)} \frac{u^n}{n!}$$

of the distribution of the bivariate statistic $(AX,\ FIX)$ on the symmetric group has the following closed form

$$A(t, x, u) = \frac{(1-t)\exp\left((x-1)tu\right)}{1 - t\exp\left((1-t)u\right)}.$$

Proof. The function $\dfrac{(1-t)\exp((x-1)tu)}{1-t\exp((1-t)u)}$ satisfies the differential equation given in the previous proposition, as well as the initial conditions. □

3. Anti-excendances, fixed points and parity

R. Mantaci introduced in [3] the numbers $P_{n,k}$ and $D_{n,k}$ that count respectively the cardinality of the set $\mathcal{A}_{n,k}$ of even permutations having k anti-excedances, and the cardinality of the set $\mathcal{S}_{n,k} \setminus \mathcal{A}_{n,k}$. These numbers satisfy the following relations:

$$P_{n,k} = P_{n-1,k-1} + kD_{n-1,k} + (n-k)D_{n-1,k-1}$$
$$D_{n,k} = D_{n-1,k-1} + kP_{n-1,k} + (n-k)P_{n-1,k-1}$$

for all integers n and $1 \leq k \leq n$ with $P_{0,0} = 1$ and $D_{0,0} = 0$.
Let us denote by:

- $p_{n,m}$ the cardinality of the set of even permutations in $\mathcal{F}_{n,m}$,
- $i_{n,m}$ the cardinality of the set of odd permutations in $\mathcal{F}_{n,m}$.

The following tables report the first values of the numbers $p_{n,m}$ and $i_{n,m}$:

$p_{n,m}$						
	$m=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	0	1			
3	2	0	0	1		
4	3	8	0	0	1	
5	24	15	20	0	0	1

$i_{n,m}$						
	$m=0$	1	2	3	4	5
$n=0$	0					
1	0	0				
2	1	0	0			
3	0	3	0	0		
4	6	0	6	0	0	
5	20	30	0	10	0	0

Proposition 3.1. *The numbers $p_{n,m}$ and $i_{n,m}$ satisfy the following relations:*

$$p_{n,m} = p_{n-1,m-1} + (m+1)i_{n-1,m+1} + (n-m-1)i_{n-1,m}$$
$$i_{n,m} = i_{n-1,m-1} + (m+1)p_{n-1,m+1} + (n-m-1)p_{n-1,m}$$

for all integers n and $0 \leq m \leq n$ with $p_{0,0} = 1$ and $i_{0,0} = 0$.

Proof. We will use the same idea as in Theorem 2.4. Suppose $i \neq n$. If σ is an even (resp. odd) permutation of $\mathcal{S}_{n-1}$, by multiplying $\sigma \in \mathcal{S}_{n-1}$ by the transposition (i,n) we obtain an odd (resp. even) permutation σ' in $\mathcal{S}_n$. When $i = n$, we obtain a permutation having the same parity. □

We denote by

- $P_{n,k,m}$ the cardinality of $\mathcal{A}_{n,k,m}$, the set of even permutations having k anti-excedances and m fixed points,
- $D_{n,k,m}$ the cardinality of $(\mathcal{S}_{n,k} \bigcap \mathcal{F}_{n,m}) \setminus \mathcal{A}_{n,k,m}$, the set of odd permutations having k anti-excedances and m fixed points.

The following tables report the first values of these numbers for some fixed integers m:

$m = 0$

$P_{n,k,0}$				
	$k=1$	2	3	4
$n=1$	0			
2	0			
3	1	1		
4	0	3	0	
5	1	11	11	1

$D_{n,k,0}$					
	$k=1$	2	3	4	5
$n=1$	0				
2	1				
3	0	0			
4	1	4	1		
5	0	10	10	0	

$m = 1$

$P_{n,k,1}$				
	$k=1$	2	3	4
$n=1$	1			
2	0			
3	0	0		
4	0	4	4	
5	0	0	15	0

$D_{n,k,1}$				
	$k=1$	2	3	4
$n=1$	0			
2	0			
3	0	3		
4	0	0	0	
5	0	5	10	5

$m = 2$

$P_{n,k,2}$			
	$k=2$	3	4
$n=2$	1		
3	0		
4	0	0	
5	0	10	10

$D_{n,k,2}$			
	$k=2$	3	4
$n=2$	0		
3	0		
4	0	6	
5	0	0	0

We have the following results:

Proposition 3.2. *For all positive integers n, k and m such that $m \geq 0$ and $m \leq k \leq n$, one has:*

$$P_{n,k,m} = P_{n-1,k-1,m-1} + (m+1)D_{n-1,k,m+1} + (n-k)D_{n-1,k-1,m} + (k-m)D_{n-1,k,m}$$
$$D_{n,k,m} = D_{n-1,k-1,m-1} + (m+1)P_{n-1,k,m+1} + (n-k)P_{n-1,k-1,m} + (k-m)P_{n-1,k,m}$$

with the initial conditions $P_{0,0,0} = 1$ and $D_{0,0,0} = 0$.

Proof. The process described in the Theorem 2.4 to prove the recursive formula for the numbers $a_{n,k,m}$ allows to construct an odd permutation of $\mathcal{S}_n$ starting from an even one of $\mathcal{S}_{n-1}$ and vice-versa, when $i \neq n$. In the case $i = n$, the parity remains the same. □

Let F be a subset of $[n]$ such that $|F| = m$. We denote by

- $\mathcal{F}_{n,F}$ the set of permutations having F as set of fixed points,
- $D_{n,k,F}$ the cardinality of the set $\mathcal{F}_{n,F} \bigcap (\mathcal{S}_{n,k} \setminus \mathcal{A}_{n,k})$
- $P_{n,k,F}$ the cardinality of the set $\mathcal{F}_{n,F} \bigcap \mathcal{A}_{n,k}$,
- $i_{n,F}$ the cardinality of the set $(\mathcal{S}_n \setminus \mathcal{A}_n) \bigcap \mathcal{F}_{n,F}$
- $p_{n,F}$ the cardinality of the set $\mathcal{A}_n) \bigcap \mathcal{F}_{n,F}$

We have the following theorem:

Theorem 3.3. *For all positive integers n, k and m such that $m \geq 0$ and $m \leq k \leq n$ and for all subset F of the interval $[n]$ such that $|F| = m$, one has:*

$$D_{n,k,F} - P_{n,k,F} = (-1)^{n-m}.$$

Proof. A combinatorial proof of this result is the subject of the next separate section of this paper. □

Corollary 3.4. *For all integers n, k and m with $m \geq 0$ and $m \leq k \leq n$ and for all subset $F \subset [n]$ such that $|F| = m$, one has:*

(1) $i_{n,F} - p_{n,F} = (-1)^{n-m}(n-m-1)$

(2) $i_{n,m} - p_{n,m} = (-1)^{n-m}(n-m-1)\binom{n}{m}$

(3) $D_{n,k,m} - P_{n,k,m} = (-1)^{n-m}\binom{n}{m}$

(4) $D_{n,k} - P_{n,k} = (-1)^{n-k+1}\binom{n-1}{k-1}$

Proof. Notice that:

(1)

$$\mathcal{F}_{n,F} = \bigsqcup_{k=m+1}^{n-1} \mathcal{S}_{n,k} \bigcap \mathcal{F}_{n,F}.$$

(2)

$$\mathcal{F}_{n,m} = \bigcup_{\substack{F \subset [n] \\ |F|=m}} \mathcal{F}_{n,F}.$$

(3)

$$\mathcal{S}_{n,k,m} = \bigcup_{\substack{F \subset [n] \\ |F|=m}} \mathcal{F}_{n,F} \bigcap \mathcal{S}_{n,k}.$$

(4)

$$\mathcal{S}_{n,k} = \bigsqcup_{m=0}^{k-1} \mathcal{F}_{n,m} \bigcap \mathcal{S}_{n,k}$$

and

$$\sum_{m=0}^{k-1} (-1)^{n-m}\binom{n}{m} = (-1)^{n-k+1}\binom{n-1}{k-1}.$$

The last identity is a simple combinatorial exercice. □

3.1. Generating functions of the numbers $P_{n,k,m}$ and $D_{n,k,m}$.

Proposition 3.5. *The generating function*

$$P(t,x,u) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_n} t^{AX(\sigma)} x^{FIX(\sigma)} \frac{u^n}{n!}$$

of the distribution of the bivariate statistic (AX, FIX) on the set of even permutations has the closed form:

$$P(t,x,u) = \frac{1}{2}\left\{\frac{\exp\left((x-1)tu\right) - t\exp\left((xt-1)u\right)}{1-t} + \frac{(1-t)\exp\left((x-1)tu\right)}{1-t\exp\left((1-t)u\right)}\right\}.$$

The generating function

$$D(t,x,u) = \sum_{n \geq 0} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \notin \mathcal{A}_n}} t^{AX(\sigma)} x^{FIX(\sigma)} \frac{u^n}{n!}$$

of the distribution of the bivariate statistic (AX, FIX) on the set of odd permutations has the closed form :

$$D(t,x,u) = \frac{1}{2}\left\{\frac{\exp((x-1)tu) - t\exp((xt-1)u)}{t-1} + \frac{(1-t)\exp((x-1)tu)}{1-t\exp((1-t)u)}\right\}.$$

Proof. The series $\sum_{n\geq 0}\sum_{m=0}^{n}\sum_{k=m+1}^{n}(-1)^{n-m}\binom{n}{m}t^k x^m \frac{u^n}{n!}$ has the closed form :

$$\frac{t\exp((xt-1)u) - \exp((x-1)tu)}{1-t}.$$

Therefore, the two functions $P(t,x,u)$ and $D(t,x,u)$ are solutions of the system:

$$\begin{cases} P(t,x,u) + D(t,x,u) = A(t,x,u) = \dfrac{(1-t)\exp((x-1)tu)}{1-t\exp((1-t)u)} \\ D(t,x,u) - P(t,x,u) = \dfrac{t\exp((xt-1)u) - \exp((x-1)tu)}{1-t} \end{cases}$$

□

4. Combinatorial proof of $D_{n,k,F} - P_{n,k,F} = (-1)^{n-m}$

Let Φ be a map of $\mathcal{S}_n$ onto itself defined as follows.
For each permutation σ, let (i,j) be the couple (if it exists) such that :

- (p_0) the integers i and j are not fixed points for the permutation σ,
- (p_1) the integer i is the smallest exceedant of σ such that $\sigma^2(i) \neq j$,
- (p_2) $\sigma(i) < j$,
- (p_3) the integers $\sigma(i)$ and j are two consecutive non fixed points of the permutation σ, that is, if k is any integer such that $\sigma(i) < k < j$, then $\sigma(k) = k$.

The permutation $\sigma' = \Phi(\sigma)$ is obtained by multiplying the permutation σ by $(\sigma(i), j)$ on the right, that is, $\sigma' = \sigma(\sigma(i), j)$. In other terms, the permutation σ' is obtained from σ by exchanging the letters $\sigma(i)$ and j in the word $\sigma(1)\ldots\sigma(n)$.

For example, if we take $\sigma = 3\,2\,4\,1\,5\,7\,8\,6$ then $(i,j) = (3,6)$ and $\sigma' = 3\,2\,6\,1\,5\,7\,8\,4$.

Definition 4.1 (*Critical permutations*)**.** *Let $F = \{f_1, f_2, \cdots, f_m\}$ a subset of $[n]$ and $\{d_1, d_2, \cdots, d_{n-m}\} = [n] \setminus F$ with $d_1 < d_2 < \cdots < d_{n-m}$. Consider the following permutations in the set $\mathcal{F}_{n,F}$:*

$$\Pi_{n,F,i} = (f_1)(f_2)\cdots(f_m)(d_i\, d_{i-1}\cdots d_1\, d_{i+1}\, d_{i+2}\cdots d_{n-m})$$

for all integer $i = 1, \cdots, n-m$. We will call these "critical permutations" and denote by $\mathcal{K}_{n,F}$ the set of the permutations $\Pi_{n,F,i}$ for $i = 1, \cdots, n-m$ and by $\mathcal{K}_{n,m} = \bigcup_{\substack{F\subset[n]\\|F|=m}} \mathcal{K}_{n,F}$

Example 4.2. For $n = 9$ and $F = \{1,3,4,8\}$. We have $m = 4$ and

- $\Pi_{9,F,1} = (1)(3)(4)(8)(2\ 5\ 6\ 7\ 9)$
- $\Pi_{9,F,2} = (1)(3)(4)(8)(5\ 2\ 6\ 7\ 9)$
- $\Pi_{9,F,3} = (1)(3)(4)(8)(6\ 5\ 2\ 7\ 9)$
- $\Pi_{9,F,4} = (1)(3)(4)(8)(7\ 6\ 5\ 2\ 9)$
- $\Pi_{9,F,5} = (1)(3)(4)(8)(9\ 7\ 6\ 5\ 2)$

Remark 4.3. *For all positive integer i and for all subset $F \subset [n]$ such that $|F| = m$, one has $AX(\Pi_{n,F,i}) = m + i$.*

Proposition 4.4. *The map Φ is not defined on the elements of the set $\mathcal{K}_{n,m}$.*

Proof. The critical permutations $\Pi_{n,F,i}$ $(i = 1, \ldots, n-m)$ are the only permutations for which the map Φ is not defined, because there does not exist a couple (i, j) satisfying all the properties. $\square$

Proposition 4.5. *The map Φ preserves the set of fixed points, that is, an integer $\ell \in [n]$ is a fixed point for the permutation σ if and only if it is a fixed point for the permutation $\Phi(\sigma)$.*

Proof. Notice that the two integers $\sigma(i)$ and j that need to be exchanged in σ to compute $\Phi(\sigma)$ are both non fixed points and we have $i < \sigma(i) < j$. After the exchange, we have $\sigma'(i) = \sigma^2(i) = j \neq i$ and $\sigma'(\sigma^{-1}(j)) = \sigma(i) \neq \sigma^{-1}(j)$ because $\sigma^2(i) \neq j$ and $\sigma(i) < j$. $\square$

Proposition 4.6. *The map Φ changes the parity of a given permutation, that is, if σ is an even permutation then $\Phi(\sigma)$ is an odd permutation and vice-versa.*

Proof. The action of Φ consists in multiplying a permutation by a transposition. This operation changes the parity. $\square$

Proposition 4.7. *The map preserves the set of anti-excedances of all permutations, that is, the permutation σ has an anti-excedance in an integer ℓ of the set $[n]$ if and only if $\Phi(\sigma)$ has an anti-excedance in ℓ.*

Proof. The two integers $\sigma(i)$ and j that need to be exchanged in σ to compute $\Phi(\sigma)$ are non fixed points. The exceedant i is both an exceedant for σ and for $\Phi(\sigma)$. Let us look at the possible cases for the integer $\sigma^{-1}(j)$:

(1) if $\sigma^{-1}(j)$ is an exceedant then $\sigma'^{-1}(j)$ is an exceedant as well, since $\sigma^{-1}(j) < \sigma(i) < j$.

(2) if $\sigma^{-1}(j) < j$ (that is, if $\sigma'^{-1}(j)$ is an anti- exceedant), then $\sigma^{-1}(j) < \sigma(i)$.
Suppose that $\sigma^{-1}(j) > \sigma(i)$, then $j > \sigma^{-1}(j) > \sigma(i)$ and $\sigma^{-1}(j)$ would be the smallest non fixed point greater than $\sigma(i)$, which contradicts the hypothesis that j is the smallest non fixed point greater than $\sigma(i)$.

$\square$

Corollary 4.8. *The map Φ preserves the number of fixed points of a given permutation, as well as the number of anti-excedances.*

Theorem 4.9. *The map Φ is a bijection on the set $\mathcal{F}_{n,F} \setminus \mathcal{K}_{n,F}$ onto itself.*

Proof. Notice that if Φ is defined on a permutation σ, then Φ is also defined on $\sigma' = \Phi(\sigma)$, because it is impossible to obtain a critical permutation as image of another permutation. Notice that the two integers that need to be exchanged in σ' to compute $\Phi(\sigma')$ are the same as the two integers that need to be exchanged in σ to compute $\sigma' = \Phi(\sigma)$ from σ. Therefore, if τ is the transposition such that $\Phi(\sigma) = \sigma\tau$ then $\Phi(\Phi(\sigma)) = \Phi(\sigma\tau) = \sigma\tau\tau = \sigma$. Therefore, Φ is an involution and hence is a bijection. $\square$

Theorem 4.10. *For all integers n, k and m such that $m \geq 0$ and $m \leq k \leq n$ and for all subset F of $[n]$ such that $|F| = m$, one has*

$$D_{n,k,F} - P_{n,k,F} = (-1)^{n-m}.$$

Proof. For all integer $k = m+1, \ldots, n$ and for all subset F of $[n]$ such that $|F| = m$, there exists a unique permutation $\Pi_{n,F,k-m}$ of the set $\mathcal{S}_{n,k}$ which is an element of the set $\mathcal{K}_{n,F}$. Furthermore, this permutation $\Pi_{n,F,k-m}$ is always an even permutation if and only if the integer $n-m$ is odd and vice-versa. $\square$

Corollary 4.11. *The map Φ is a bijection on the set $\mathcal{F}_{n,m} \setminus \mathcal{K}_{n,m}$ onto itself.*

Corollary 4.12. *For all integers n, k and m such that $0 \leq m \leq k \leq n$, one has*

$$D_{n,k,m} - P_{n,k,m} = (-1)^{n-m} \binom{n}{m}.$$

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Département de Mathématiques et Informatique, Université d' Antananarivo, 101 Antananarivo, Madagascar

E-mail address: `frakoton@syfed.refer.mg`