ASSOCIATION SCHEMES BASED ON ISOTROPIC SUBSPACES, PART I

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ABSTRACT. The subspaces of a given dimension in a finite classical polar space form the points of an association scheme. When the dimension is zero, this is the scheme of the collinearity graph of the space. At the other extreme, when the dimension is maximal, it is the scheme of the corresponding dual polar graph. These extreme cases have been thoroughly studied. In this article, the general case is examined and a detailed computation of the intersection numbers of these association schemes is initiated.

RÉSUMÉ. Les sous-espaces d'une dimension donnée dans un espace polaire fini classique forment les points d'un schéma d'association. Quand la dimension est zéro, cela constitue le schéma du graphe collinéaire de l'espace. A l'autre extrême, quand la dimension est maximale, c'est le schéma du graphe duel polaire correspondant. On a déjà étudié de façon approfondie ces cas extrêmes. Dans cet article on examinera le cas général, afin d'inaugurer un calcul détaillé des nombres d'intersection de ces schémas d'association.

1. INTRODUCTION

Grassman graphs and dual polar graphs are two well-known classes of distance-regular graphs. The intersection numbers and eigenvalues of these and other distance-regular graphs with "classical parameters" are recorded in [4, Chapter 9]. In the case of the dual polar graphs, they were determined by D. Stanton in [14]. In his Ph.D. dissertation [10], the author of the present article used a number of formulas from [7] to give an alternative derivation of the intersection numbers found in [14]. Also in [10], adjacency in a graph which the author calls a *hyperbolic partner graph* was observed to be part of an association scheme, although this fact is an easy consequence of Witt's Extension Theorem. Actually, there are classes of such graphs and related association schemes. These association schemes include those of the dual polar graphs as well as the collinearity graphs of finite classical polar spaces, and are rather evident generalizations of these. They can also be viewed as a type of analogue of the Grassman graph association schemes, which in turn are "q-analogues" of Johnson graph association schemes.

The association schemes to be considered are defined as follows. Fix an N-dimensional vector space V over a finite field GF(q). In order to simplify the discussion, it will be assumed that q is odd. Equip this space with a non-degenerate form $\langle\,\cdot\,,\,\cdot\,\rangle$ that is symmetric bilinear, alternating bilinear or Hermitian. Let d be the Witt index of this form (*i.e.* the dimension of any maximal isotropic subspace), and assume that $d \geq 1$.

Fix an integer m between 1 and d. The points of the association scheme will be the isotropic (*i.e.* totally singular) m-subspaces of V.

If U and U' are isotropic m-subspaces of V, and if $\dim(U \cap U') = m-k$, and $\dim(U^\perp \cap U')$ $= m-\gamma$, then we will we say that these two subspaces are (k,γ) - *associates.* $U^\perp$ is defined to be $\{\, v \in V \,|\, (\forall u \in U)\, \langle u, v\rangle = 0 \,\}$. Clearly $0 \leq \gamma \leq k \leq m$ here since $U \subseteq U^\perp$. It is not hard to see that $\dim(U \cap U'^\perp)$ will also equal $m-\gamma$ as follows. $(U \cap U'^\perp)^\perp = U^\perp + U'$

$\mu = 0, \nu = 1$ orthogonal D_d $N = 2d$		
$\mu = 0, \nu = \frac{1}{2}$ unitary ${}^2A_{2d-1}$ $N = 2d$	$\mu = \frac{1}{2}, \nu = \frac{1}{2}$ orthogonal B_d $N = 2d + 1$	
$\mu = 0, \nu = 0$ symplectic C_d $N = 2d$	$\mu = \frac{1}{2}, \nu = 0$ unitary ${}^2A_{2d}$ $N = 2d + 1$	$\mu = 1, \nu = 0$ orthogonal ${}^2D_{d+1}$ $N = 2d + 2$

FIGURE 1. Parameters for the six geometry types

has dimension $(N - m) + m - (m - \gamma) = N - (m - \gamma)$, so $U \cap U'^{\perp}$ has dimension $m - \gamma$. The ordered pair (k, γ) will be written using the following unorthodox notation: k_γ. Thus for example, 5_2 really means (5,2). We will speak of k_γ - associates, rather than (k, γ) - associates. Two isotropic subspaces that are k_γ - associates are a distance k apart in the Grassman graph whose vertices are the m-subspaces of V.

Let $\mathcal{N}_m$ denote the collection of all isotropic m-subspaces of V. Let $\mathcal{R}_{m,k_\gamma}$ be the collection of all ordered pairs of isotropic m-subspaces of V that are k_γ-associates. If m is implicit, then $\mathcal{R}_{k_\gamma}$ will be written instead of $\mathcal{R}_{m,k_\gamma}$. The claim of course is that with $\mathcal{N}_m$ as the set of "points", the relations $\mathcal{R}_{k_\gamma}$ $(0 \le \gamma \le k \le m)$ form an association scheme. It is an immediate consequence of Witt's Extension Theorem that when the isometry group of form $\langle \cdot, \cdot \rangle$ acts (in the canonical way) on $\mathcal{N}_m$, this action is generously transitive and the orbitals are in fact the relations $\mathcal{R}_{k_\gamma}$. It follows that these relations do indeed form an association scheme. This is the same reasoning typically used to establish that Grassman graphs and dual polar graphs are distance-transitive (cf. [4, Theorems 9.3.1, 9.3.3, 9.4.3]).

There are actually six distinct types of forms on V ("geometries") to consider. If the form is alternating, then V is said to have a *symplectic geometry* or a geometry of type C_d. Here $N = 2d$. If the form is symmetric, then one speaks of V having an *orthogonal geometry*. Here if n is odd, then n will equal $2d + 1$ and the geometry is said to be of type B_d. But if n is even, then it is possible that either $N = 2d$ or $N = 2d + 2$, and one speaks of geometries of type D_d or ${}^2D_{d+1}$, accordingly. Finally, if the form is Hermitian, then V is said to have a *unitary geometry*. If N is even, then $N = 2d$, and the geometry is said to be of type ${}^2A_{2d-1}$, while if N is odd, then $N = 2d + 1$, and the geometry is said to be of type ${}^2A_{2d}$. Following notation introduced in [10], let $\mu = \frac{1}{2}N - d$ and let ν be such that $\mu + \nu$ equals 0, $\frac{1}{2}$, 1 in the symplectic, unitary, orthogonal cases, respectively. The parameters for the six types of geometries are shown in Figure 1.

We will require some additional terminology and notation. $\mathcal{L}(V)$ will denote the lattice of all subspaces of V. Given $U \in \mathcal{L}(V)$ (*i.e.* a subspace of V), the *radical* of U, denoted $\mathrm{rad}(U)$, is simply the subspace $U \cap U^{\perp}$, which is necessarily isotropic. Note that U is isotropic if and only if $U = \mathrm{rad}(U)$ ($U \subseteq U^{\perp}$). When U is isotropic, the quotient space $U^{\perp}/U$ inherits the form on V by defining $\langle u+U, v+U\rangle = \langle u, v\rangle$, and this inherited form is non-degenerate and produces the same type of geometry on $U^{\perp}/U$ as on V. Also, $W \in \mathcal{L}(V)$ is called *coisotropic* if $W^{\perp} \subseteq W$. Also, for $A, B \in \mathcal{L}(V)$ with $B \subseteq A$, $[A : B]$ will denote $\dim(A/B)$ ($= \dim(A)$ - $\dim(B)$).

In the discussion that follows, standard notation for *q-shifted factorials* and *q-binomial coefficients* will be used. Thus

$$(a;q)_m = \prod_{j=0}^{m-1} (1 - aq^j)\,, \qquad (a,b;q)_m = (a;q)_m (b;q)_m\,,$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1};q)_m}{(q;q)_m} = \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}}.$$

The latter is the number of m-subspaces of an n-dimensional GF(q)-vector space. The number of these that are isotropic, with respect to an implied form, will be denoted

$$\begin{bmatrix} n \\ m \end{bmatrix}_0$$

This value will be computed in the next section.

Define $p_{i_\alpha j_\beta k_\gamma}$ to be the number of ordered triples (U, U', U'') of isotropic m-subspaces such that $(U, U') \in \mathcal{R}_{k_\gamma}$, $(U, U'') \in \mathcal{R}_{j_\beta}$ and $(U', U'') \in \mathcal{R}_{i_\alpha}$. Taking $p^{k_\gamma}_{i_\alpha j_\beta} = p_{i_\alpha j_\beta k_\gamma}/|\mathcal{R}_{k_\gamma}|$, note that this is the number of isotropic m-subspaces U'' such that $(U, U'') \in \mathcal{R}_{j_\beta}$ and $(U', U'') \in \mathcal{R}_{i_\alpha}$, given any $(U, U') \in \mathcal{R}_{k_\gamma}$. These are the intersection numbers of the association scheme. The goal of this paper is to compute $p^{1_0}_{i_\alpha j_\beta}$ as a first step towards computing all of the intersection numbers and eigenvalues for the association scheme $(\mathcal{N}_m\,,\,\{\,\mathcal{R}_{k_\gamma}\,\})$.

A question related to the computation of $p^{k_\gamma}_{i_\alpha j_\beta}$, which will be called the "lattice problem", is the following. Consider the sublattice $\mathcal{M}$ of $\mathcal{L}(V)$ generated by 0 and V, and the isotropic m-subspaces U and U', and the coisotropic subspaces $U^{\perp}$ and $U'^{\perp}$. Since $U \subseteq U^{\perp}$ and $U' \subseteq U'^{\perp}$, when 0 and V are removed from $\mathcal{M}$, the resulting lattice, in the generic case, is the lattice with 18 nodes considered in [3, Exercise 3.7.6]. $\mathcal{M}$ is depicted in Figure 2. We now ask, how many isotropic m-subspaces U'' intersect each of the nodes of $\mathcal{M}$ in some prescribed number (depending on the node) of dimensions? In general this is a very complicated enumeration problem. However, if its solution was known, then $p^{k_\gamma}_{i_\alpha j_\beta}$ could be computed by simply summing such counts. To see this, and still assuming that $(U, U') \in \mathcal{R}_{k_\gamma}$, note that $p^{k_\gamma}_{i_\alpha j_\beta}$ is just the number of isotropic m-subspaces U'' satisfying $\dim(U \cap U'') = m - j$, $\dim(U' \cap U'') = m - i$, $\dim(U^{\perp} \cap U'') = m - \beta$, $\dim(U'^{\perp} \cap U'') = m - \alpha$. So by summing the counts from all of the lattice problem enumerations that impose these four conditions on U'', one would obtain $p^{k_\gamma}_{i_\alpha j_\beta}$.

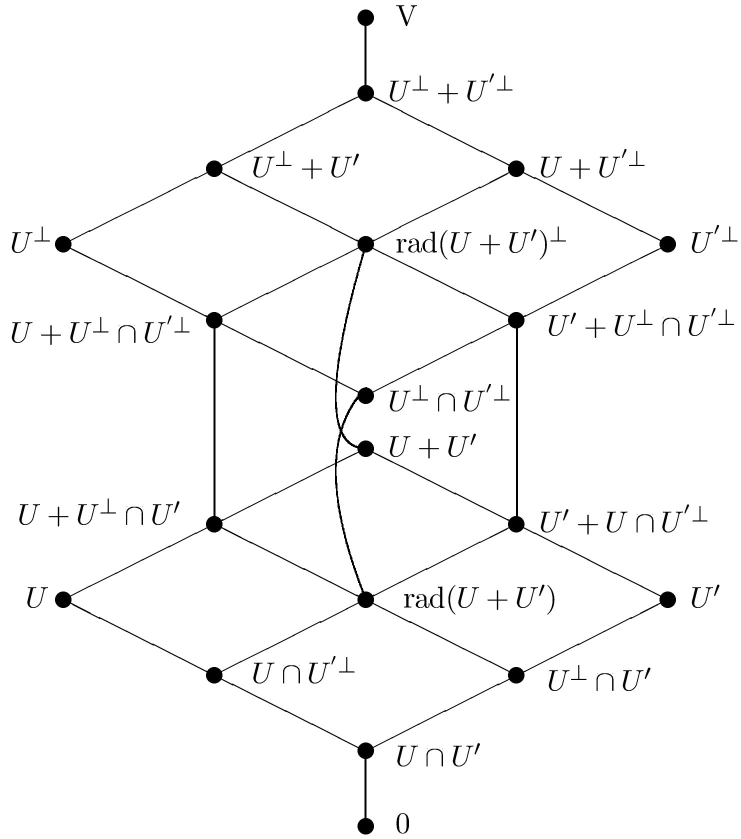


FIGURE 2. The lattice $\mathcal{M}$ generated by $0, V, U, U', U^\perp$ and $U'^\perp$ (generic case)

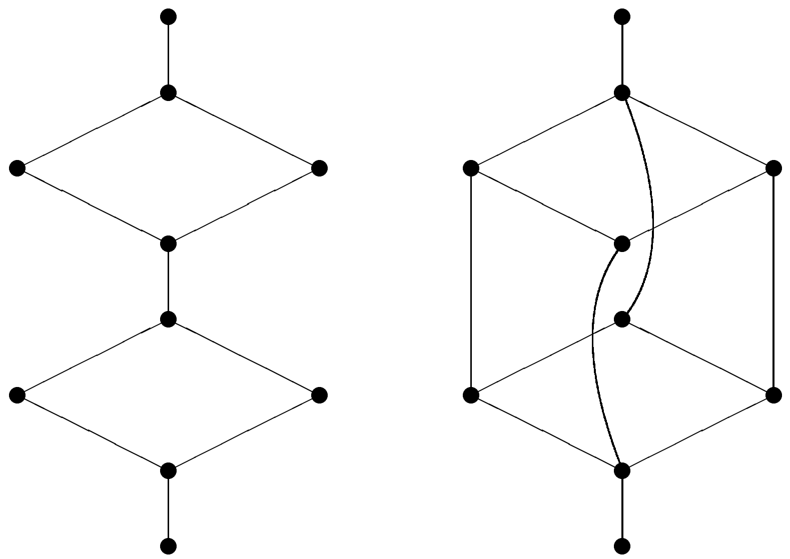
Although the lattice problem seems to be difficult in general, we will consider a couple special cases that are more tractable. The first of these is the case where $U + U' \subseteq U^\perp \cap U'^\perp$, which means that $\gamma = 0$. Here the lattice $\mathcal{M}$ is considerably simpler than the lattice in Figure 2. It is depicted in Figure 3(a). A subcase of this special case is handled in Section 3, and this is used to compute $p^{1_0}_{i_\alpha j_\beta}$.

Another special case is the case where $U \cap U' = \text{rad}(U + U')$. Here $\gamma = k$ ($U^\perp \cap U' = U \cap U'$). This case is depicted in Figure 3(b). A subcase of this special case leads to the computation of $p^{1_1}_{i_\alpha j_\beta}$, which will be computed in a follow-up article.

Remark 1. *By reasoning along the lines used in the theory of distance-regular graphs* (*cf.* [4, Section 4.1]), *it may be seen that together the formulas for* $p^{1_0}_{i_\alpha j_\beta}$ *and* $p^{1_1}_{i_\alpha j_\beta}$ *provide enough information to produce a recursion formula for computing the general intersection numbers* $p^{k_\gamma}_{i_\alpha j_\beta}$.

2. SOME ENUMERATIVE LEMMAS

Most of the lemmas in this section are concerned with counting the number of isotropic subspaces that satisfy certain conditions. They provide the basic tools for handling the enumeration problems discussed in the introduction, at least in special cases. The first two



(a) $U + U' \subseteq U^{\perp} \cap U'^{\perp}$ (b) $U \cap U' = \mathrm{rad}(U + U')$

FIGURE 3. Two special cases of $\mathcal{M}$

lemmas concern a reduction that will be routinely applied in the next section. Both are straightforward to check, and the proof of the first one is omitted.

Lemma 1. *Fix an isotropic subspace Z of V. Let $Z^{\perp}/Z$ inherit the form on V. Let $\varphi : \mathcal{L}(V) \to \mathcal{L}(Z^{\perp}/Z)$ by $\varphi(X) = (Z + (Z^{\perp} \cap X))/Z$. This mapping preserves containment and the $\perp$ operation. That is, $\varphi(X^{\perp}) = \varphi(X)^{\perp}$ for all subspaces X of V. However, it does not in general preserve the lattice operations ($+$ and $\cap$).*

Lemma 2. *Fix an isotropic m-subspace Z of V. Let $Z^{\perp}/Z$ inherit the form on V. Let $\varphi : \mathcal{L}(V) \to \mathcal{L}(Z^{\perp}/Z)$ by $\varphi(X) = (Z+(Z^{\perp}\cap X))/Z$. Fix $A, B_1, ..., B_j \in \mathcal{L}(V)$. Consider the restriction of φ to the collection of isotropic $(m+k)$ - subspaces X, containing Z, contained in $Z + A$, and such that $X \cap B_\alpha \subseteq Z$ for all $\alpha = 1, ..., j$. This restriction is a one-to-one correspondence with the isotropic k-subspaces of $\varphi(A)$ that trivially intersect each $\varphi(B_\alpha)$ $(\alpha = 1, ..., j)$.*

Proof. For a given X (as above), note that $Z \subseteq X \subseteq X^{\perp} \subseteq Z^{\perp}$. So, $\varphi(X) = X/Z$ and $\varphi(X)\cap\varphi(B_\alpha) = [X\cap(Z+(Z^{\perp}\cap B_\alpha))]/Z = (Z+(X\cap B_\alpha))/Z = \varphi(X\cap B_\alpha)$. This is clearly zero since $X \cap B_\alpha \subseteq Z$. Also, $\dim(\varphi(X)) = k$ and $\varphi(X) \subseteq \varphi(Z + A) = \varphi(A)$. So $\varphi(X)$ is as claimed. Conversely, any isotropic k-subspace W of $\varphi(A)$ that trivially intersects each $\varphi(B_\alpha)$ $(\alpha = 1, ..., j)$ lifts to a unique isotropic $(m + k)$ - subspace Y of $Z + A$, containing Z, and such that $\varphi(Y) = W$. Note that Y is necessarily in $Z^{\perp}$, so that $\varphi(Y) = Y/Z$. The properties that Y is required to satisfy can be immediately verified by reversing the above argument. □

The following enumerative lemma is known, and in fact is [14, Proposition 4.2] and [4, Lemma 9.4.1]. The statement and proof in [4] is followed by a list of citations to articles in which this result was originally produced for the individual geometry types.

Lemma 3.

$$\begin{bmatrix} N \\ m \end{bmatrix}_0 = \frac{(q^{\frac{1}{2}N-\mu-m+1}, -q^{\frac{1}{2}N-\nu-m+1}; q)_m}{(q;q)_m} = \frac{(q^{\frac{1}{2}N-\mu}, -q^{\frac{1}{2}N-\nu}; q^{-1})_m}{(q;q)_m}$$

Proof. Using the notation in [12], the number of isotropic vectors not contained in, and orthogonal to, a given isotropic j-subspace is, by [12, Theorem 3.2],

$$\left[\begin{array}{c|c} N;\tau;q & 0 \\ \hline 0\cdots 0 & 0 \\ \vdots\ \vdots & \vdots \\ \underbrace{0\cdots 0}_{j\times j} & 0 \end{array}\right]$$

$$= \quad q^j\,[N-2j\,;\,\tau\,;\,q \mid 0] \quad = \quad q^j(q^{\frac{1}{2}N-\mu-j}-1)(q^{\frac{1}{2}N-\nu-j}+1).$$

It follows that the number sought is

$$\prod_{j=0}^{m-1} \frac{q^j(q^{\frac{1}{2}N-\mu-j}-1)(q^{\frac{1}{2}N-\nu-j}+1)}{q^m-q^j} = \prod_{j=0}^{m-1} \frac{(q^{\frac{1}{2}N-\mu-j}-1)(q^{\frac{1}{2}N-\nu-j}+1)}{q^{m-j}-1} =$$

$$\prod_{i=1}^{m} \frac{(q^{\frac{1}{2}N-\mu-m+i}-1)(q^{\frac{1}{2}N-\nu-m+i}+1)}{q^i-1} = \frac{(q^{\frac{1}{2}N-\mu-m+1}, -q^{\frac{1}{2}N-\nu-m+1}; q)_m}{(q;q)_m}.$$

□

The next two lemmas enumerate particular isotropic k-subspaces. These will be applied repeatedly in the next section, in combination with Lemma 2, in order to count the number of ways to extend a given isotropic subspace to a larger one, subject to certain restrictions.

Lemma 4. *The number of isotropic k-subspaces of V that trivially intersect a given coisotropic subspace of codimension m is*

$$\begin{bmatrix} m \\ k \end{bmatrix} \cdot q^{k(N-m-\frac{1}{2}k+\frac{1}{2}-\mu-\nu)}.$$

Proof. Let W denote the coisotropic subspace. First consider the special case where $k=m$. An isotropic m-subspace X that trivially intersects W is a complement of W. That is, $V = X \oplus W$. Note that $\dim(\mathrm{rad}(W)) = \dim(W^\perp) = m$. Now, by the modularity of $\mathcal{L}(V)$, $X^\perp = X^\perp \cap (X \oplus W) = X \oplus (X^\perp \cap W)$. So $\dim(X^\perp \cap W) = N-2m$. It follows that $X^\perp \cap W$ is a maximal nondegenerate subspace of W. Now, $X \cap W^\perp \subseteq X \cap W = 0$. So $X \cap W^\perp = 0$, and this is the unique maximal non-degenerate subspace of X. X and W are therefore pseudo-orthogonal complements in the sense defined in [11]. Conversely, every pseudo-orthogonal complement of W is isotropic. The result sought, in the special case $k=m$, then follows by [13, Theorem 5]. (The γ there equals $1-2\mu-2\nu$ here).

For general k, note first that a given suitable k-subspace Z is contained in multiple isotropic m-subspaces that trivially intersect W. To find the number of such, consider the mapping $\varphi : \mathcal{L}(V) \to \mathcal{L}(Z^\perp/Z)$ given by $\varphi(X) = (Z + (Z^\perp \cap X))/Z$. The coisotropic subspace $\varphi(W) = (Z \oplus (Z^\perp \cap W))/Z \cong Z^\perp \cap W = (Z \oplus W^\perp)^\perp$, which has dimension $N-m-k$. $Z^\perp/Z$ has dimension $N-2k$. By the first part of this proof, the number of isotropic

$(m-k)$ - subspaces of $Z^{\perp}/Z$ that trivially intersect $\varphi(W)$ is $q^{(m-k)(N-\frac{3}{2}m-\frac{1}{2}k+\frac{1}{2}-\mu-\nu)}$. By Lemma 2, this is also the number of isotropic m-subspaces Y of V that contain Z and have $Y \cap W \subseteq Z$. But $Y \cap W \subseteq Z$ if and only if $Y \cap W = 0$. Now note that each isotropic m-subspace that trivially intersects W contains $\left[{m \atop k}\right]$ isotropic k-subspaces (that trivially intersect W). The result sought, for general k, now follows by double counting suitable pairs (Z, Y), one time selecting Z first, the other time selecting Y first.

□

Lemma 5. *Consider a collection of coisotopic hyperplanes $H_1, H_2, ..., H_k$ of V in general position, and having the property that the intersection of any subset of these is coisotropic. The number of isotropic 1-subspaces of V that trivially intersect each of these hyperplanes is*

$$q^{N-k-\mu-\nu}(q-1)^{k-1}$$

Proof. For $S \subseteq \{1, 2, ..., k\}$, let $X_S = \bigcap_{j \in S} H_j$. Since the hyperplanes are in general position, $\dim(X_S) = N - |S|$. From Lemmas 3 and 4, it can be seen that the number of isotropic 1-subspaces contained in X_S is

$$\left[{N \atop 1} \right]_0 - \left[{|S| \atop 1} \right] \cdot q^{N-|S|-\mu-\nu} = \frac{1-q^{\frac{1}{2}N-\mu}+q^{\frac{1}{2}N-\nu}-q^{N-|S|-\mu-\nu}}{1-q}$$

Applying the Principle of Inclusion/Exclusion, the desired count is

$$\sum_{j=0}^{k}(-1)^j \binom{k}{j} \left[\frac{1-q^{\frac{1}{2}N-\mu}+q^{\frac{1}{2}N-\nu}-q^{N-j-\mu-\nu}}{1-q}\right]$$

$$=\sum_{j=0}^{k}(-1)^j \binom{k}{j} \left[\frac{q^{N-j-\mu-\nu}}{q-1}\right]$$

$$=\frac{q^{N-\mu-\nu}}{q-1}\sum_{j=0}^{k}\binom{k}{j}(-q^{-1})^j = q^{N-k-\mu-\nu}(q-1)^{k-1}.$$

□

The simple form of the count in Lemma 5 correctly suggests that a more intuitive proof is possible. With $S = \{1, 2, ..., k\}$, let Y be a pseudo-orthogonal complement of X_S. Y is isotropic and a hyperbolic partner of $\mathrm{rad}(X_S)$, as defined in [12, Section 2]. The decomposition $V = Y \oplus \mathrm{rad}(X_S) \oplus (Y \oplus \mathrm{rad}(X_S))^{\perp}$ can be used to obtain an intuitive proof. This involves considering "matched bases", as in the proof of [11, Proposition 1], for Y and $\mathrm{rad}(X_S)$, and an arbitrary basis for $(Y \oplus \mathrm{rad}(X_S))^{\perp}$. The details are left to the interested reader.

The next lemma is also known, and is a special case of [14, Proposition 4.10] and [4, Lemma 9.4.2]. It provides the sizes of the binary relations involved in the association scheme.

Lemma 6.

$$|\mathcal{R}_{k_\gamma}| = \begin{bmatrix} N \\ m \end{bmatrix}_0 \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} N-2m \\ k-\gamma \end{bmatrix}_0 \begin{bmatrix} k \\ \gamma \end{bmatrix} \cdot q^{k^2+\gamma(N-2m-2k+\frac{3}{2}\gamma+\frac{1}{2}-\mu-\nu)}$$

$$= \frac{(q^{\frac{1}{2}N-\mu}, -q^{\frac{1}{2}N-\nu}; q^{-1})_{m+k-\gamma}}{(q;q)_{m-k}\,(q;q)_\gamma\,(q;q)^2_{k-\gamma}} \cdot q^{k^2+\gamma(N-2m-2k+\frac{3}{2}\gamma+\frac{1}{2}-\mu-\nu)}$$

Proof. Fix an isotropic m-subspace U. There are $[{N \atop m}]_0$ choices this. Now consider selecting an isotropic m-subspace U' such that $(U, U') \in \mathcal{R}_{k_\gamma}$, by selecting this a portion at a time, beginning with the selection of an isotropic $(m-k)$ - subspace Z of U to play the role of $U \cap U'$. There are $[{m \atop k}]$ choices for Z.

After selecting Z, a subspace Y to play the role of $U^\perp \cap U'$ needs to be selected. This will be accomplished indirectly by selecting an isotropic subspace Y/Z of $Z^\perp/Z$. This selection must trivially intersect the k-subspace U/Z. It must also be a subspace of $U^\perp/Z$. By Lemma 2, these conditions are necessary and sufficient. Y/Z must therefore project injectively into the quotient space $(U^\perp/Z)/(U/Z)$ ($\cong U^\perp/U$). $\dim(U^\perp/U) = N - 2m$, and Y/Z must have dimension $k-\gamma$. So the number of choices for the isotropic subspace of $(U^\perp/Z)/(U/Z)$ is $[{N-2m \atop k-\gamma}]_0$. Each such choice lifts to $q^{k(k-\gamma)}$ choices for Y/Z, and each of these uniquely determines Y. So, given a choice for Z, there are $[{N-2m \atop k-\gamma}]_0\, q^{k(k-\gamma)}$ choices for Y.

After selecting Y, the rest of U' needs to be selected. This will be done by selecting an isotropic γ-subspace U'/Y of $Y^\perp/Y$, which has dimension $N-2m+2\gamma$. Again by Lemma 2, the selection must trivially intersect the coisotropic subspace $(Y^\perp \cap U^\perp)/Y = (Y+U)^\perp/Y$, which has the k-dimensional radical $(Y+U)/Y$ ($\cong U/(U\cap Y) = U/Z$), and so has codimension k. By Lemma 4, the number of choices for this selection is $[{k \atop \gamma}]\, q^{\gamma(N-2m+2\gamma-k-\frac{1}{2}\gamma+\frac{1}{2}-\mu-\nu)}$.

By multiplying the counts for the various portions, we see that the number of choices for U' is

$$\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} N-2m \\ k-\gamma \end{bmatrix}_0 \begin{bmatrix} k \\ \gamma \end{bmatrix} \cdot q^{k^2+\gamma(N-2m-2k+\frac{3}{2}\gamma+\frac{1}{2}-\mu-\nu)} =$$

$$\frac{(q;q)_m\,(q^{\frac{1}{2}N-m-\mu}, -q^{\frac{1}{2}N-m-\nu}; q^{-1})_{k-\gamma}}{(q;q)_{m-k}\,(q;q)_\gamma\,(q;q)^2_{k-\gamma}} \cdot q^{k^2+\gamma(N-2m-2k+\frac{3}{2}\gamma+\frac{1}{2}-\mu-\nu)}$$

Multiply this by the number of choices for U, namely $[{N \atop m}]_0$, expand and simplify to get the stated result.

□

3. The computation of $p^{1_0}_{i_\alpha j_\beta}$

In order to compute $p^{1_0}_{i_\alpha j_\beta}$, fix two isotropic m subspaces U and U', and assume that these intersect in $m-1$ dimensions and that they are orthogonal to each other. So $(U, U') \in \mathcal{R}_{1_0}$, and necessarily $1 \le m \le d-1$. Notice that $U + U' \subseteq U^\perp \cap U'^\perp$ and so the lattice $\mathcal{M}$ reduces to the one in Figure 3(a). The goal of this section is to count the number of isotropic m-subspaces U'' satisfying

1. $\dim(U \cap U'') = m - j$,
2. $\dim(U' \cap U'') = m - i$,
3. $\dim(U^\perp \cap U'') = m - \beta$,
4. $\dim(U'^\perp \cap U'') = m - \alpha$,

where $0 \le \alpha \le i \le m$ and $0 \le \beta \le j \le m$.

However, in order to effect a systematic enumeration of these subspaces, four additional conditions will be imposed on U''. These additional conditions will later be relaxed by summing over the four new

parameters to be introduced. Specifically, fix numbers h, e, η, ε and impose on U'' the following additional requirements:

5. $\dim(U \cap U' \cap U'') = m - h$,
6. $[(U + U') \cap U'' : U \cap U'' + U' \cap U''] = e$,
7. $\dim(U^\perp \cap U'^\perp \cap U'') = m - \eta$,
8. $[(U^\perp + U'^\perp) \cap U'' : U^\perp \cap U'' + U'^\perp \cap U''] = \varepsilon$.

Thus, the goal is then to solve the lattice problem mentioned in the introduction, but only for the special case $k = 1$ and $\gamma = 0$. The various parameters are required to satisfy certain conditions in order that a U'' as described above exists. Some of these conditions are given in the next lemma, while others will be addressed in Theorem 1.

Lemma 7. *If a subspace U'' exists that satisfies the above eight conditions, then the parameters $h, i, j, e, \eta, \alpha, \beta$ and ε must satisfy all of the following relationships:*

9. *h is between 1 and m, equals i or $i+1$, and equals j or $j+1$;*
10. *η is between 0 and m, equals*
α or $\alpha+1$, and equals β or $\beta+1$;
11. *e and ε are each either 0 or 1;*
12. $0 \le i + j - h - e - \eta \le d - m - 1$;
13. $0 \le \alpha + \beta - \eta - \varepsilon < h$;
14. $i - \alpha \le d - m$;
15. $j - \beta \le d - m$;
16. $0 \le h - 1 - \alpha - \beta + \eta + \varepsilon \le d - m$.

Proof. The first three items in this list are straightforward to check. For example, $e = [(U + U') \cap U'' : U \cap U'' + U' \cap U''] \le [(U + U') \cap U'' : U \cap U''] \le [U + U' : U] = [U' : U \cap U'] = 1$. The first inequality in item 12 follows from the fact that the dimension of $(U + U') \cap U''$ is required to be $(m - i) + (m - j) - (m - h) + e = m - i - j + h + e$, and so $0 \le [U^\perp \cap U'^\perp \cap U'' : (U + U') \cap U''] = i + j - h - e - \eta$. The second inequality in item 12 will be proved below.

Now, $\dim[(U^\perp + U'^\perp) \cap U''] = (m - \alpha) + (m - \beta) - (m - \eta) + \varepsilon = m - \alpha - \beta + \eta + \varepsilon \le m$, so $\alpha + \beta - \eta - \varepsilon \ge 0$. Since $\dim(U \cap U' \cap U'') = m - h$, $\alpha + \beta - \eta - \varepsilon \le h$. Suppose that equality holds here, *i.e.* suppose that $(U^\perp + U'^\perp) \cap U'' = U \cap U' \cap U''$. Let $\iota = [U \cap U' : U \cap U' \cap U''] = [U^\perp + U'^\perp + U''^\perp : U^\perp + U'^\perp]$. Then $\iota \ge [U^\perp + U'^\perp + U'' :$

$U^\perp + U'^\perp] = [U'' : (U^\perp + U'^\perp) \cap U''] = [U'' : U \cap U' \cap U'']$. Therefore $\dim(U'') = \dim(U \cap U' \cap U'') + [U'' : U \cap U' \cap U''] \leq (m-1-\iota)+\iota < m$, which is impossible. Therefore $\alpha+\beta-\eta-\varepsilon < h$. This establishes item 13.

To establish item 14, note that $U' + (U'^\perp \cap U'')$ is isotropic and has dimension $\dim(U')$ + $\dim(U'^\perp \cap U'')$ - $\dim(U' \cap U'') = m + (m-\alpha) - (m-i) = m+i-\alpha$, which cannot exceed d. Item 15 requires the obvious adjustment to the proof of item 14. The second inequality in item 12 is shown by means of the same sort of reasoning, but relies on the fact that $(U+U') + (U^\perp \cap U'^\perp \cap U'')$ is isotropic, so $\dim[(U+U') + (U^\perp \cap U'^\perp \cap U'')] = (m+1)+(m-\eta)-(m-i-j+h+e) = m+1+i+j-h-e-\eta \leq d$. Item 16 likewise results from the isotropic nature of $(U \cap U') + (U^\perp + U'^\perp) \cap U''$, which has dimension $(m-1)+(m-\alpha-\beta+\eta+\varepsilon)-(m-h) = m+h-1-\alpha-\beta+\eta+\varepsilon$. This cannot exceed d. But, $\dim(U \cap U' \cap U''^\perp) = N - \dim((U^\perp + U'^\perp)+U'') = N-(N-m+1)-m+(m-\alpha-\beta+\eta+\varepsilon) = m-1-\alpha-\beta+\eta+\varepsilon$, so that $[U \cap U' \cap U''^\perp : U \cap U' \cap U''] = h-1-\alpha-\beta+\eta+\varepsilon \geq 0$. □

Henceforth, it will be assumed that the parameters satisfy the conditions listed in Lemma 7. The following theorem is concerned with counting the number of possible U''. In this enumeration, $q \wedge x$ denotes q^x.

Theorem 1. *The number of isotropic m-subspaces U'' satisfying conditions 1 through 16 (above) is*

$$\frac{(q;q)_{m-1}\,(q^{N/2-\mu-m-1}, -q^{N/2-\nu-m-1}; q^{-1})_{i+j-h-e-\eta}}{(q;q)_{m-h}\,(q;q)_{\alpha+\beta-\eta-\varepsilon}\,(q;q)_{h-1-\alpha-\beta+\eta+\varepsilon}\,(q;q)_{i+j-h-e-\eta}} \cdot (q-1)^{e+\varepsilon-e\varepsilon}$$

$$\begin{gathered} \cdot\, q \wedge \{ i(h-i) + j(h-j) + (h-1)e + (i+j-h-e-\eta)(i+j-h-e+1) \\ + (N-2m-\mu-\nu-i+\alpha+\beta)(\eta-\alpha) + (N-2m-\mu-\nu-j+\alpha+\beta)(\eta-\beta) \\ + (\eta-\alpha)(\eta-\beta) + (N-2m-\mu-\nu-1-h+2\eta+e)\varepsilon \\ + (\alpha+\beta-\eta-\varepsilon)(N-2m-\mu-\nu-h+3[\alpha+\beta-\eta-\varepsilon+1]/2) \}, \end{gathered}$$

provided that at least one of the following five conditions is also satisfied:

- *A.* $\varepsilon = e = 0$ *and* $h = i = j$;
- *B.* $\varepsilon = e = 0$ *and* $\eta = \alpha = \beta$;
- *C.* $\varepsilon = e = 0$, $h = i$ *and* $\eta = \alpha$;
- *D.* $\varepsilon = e = 0$, $h = j$ *and* $\eta = \beta$;
- *E.* $h = i = j$ *and* $\eta = \alpha = \beta$.

In all other cases, the count is zero.

Proof. To count the number of suitable U'', we will consider selecting such a subspace a portion at a time, similar to the way in which U' was selected in Lemma 6. The restrictions stated in Lemma 7 will be assumed. It will also be assumed that i and j are both nonzero, the case where one of these is zero being trivial to check. However, we will not impose any of the five additional conditions, listed at the bottom of Theorem 1, at least not until the end of this proof.

The nodes of $\mathcal{M}$ will be taken in sequence, in a non-decreasing order, and at each stage the number of ways to select the intersection of U'' with the current node, given the previously selected portion of U'', will be determined. To begin this process, consider the selection of $U \cap U' \cap U''$, which must be an $(m-h)$-subspace of the $(m-1)$-subspace $U \cap U'$. The number of choices for this is simply

$$\begin{bmatrix} m-1 \\ m-h \end{bmatrix} = \begin{bmatrix} m-1 \\ h-1 \end{bmatrix} = \frac{(q;q)_{m-1}}{(q;q)_{h-1}\,(q;q)_{m-h}} \tag{1}$$

Now assume that a subspace Z_1 has been selected to play the role of $U \cap U' \cap U''$. Define a mapping between lattices of subspaces as follows:

$$\varphi_1 : \mathcal{L}(V) \to \mathcal{L}(Z_1^{\perp}/Z_1),\ \varphi_1(X) = (Z_1 + Z_1^{\perp} \cap X)/Z_1\ (\cong (Z_1^{\perp} \cap X)/(Z_1 \cap X))$$

In order to select the $U \cap U''$ portion of U'', it suffices by Lemma 2 to select a suitable subspace of the h-subspace $\varphi_1(U)\,(= U/Z_1\,)$ to play the role of $\varphi_1(U \cap U'')\,(= (U \cap U'')/Z_1\,)$, which is required to have dimension $h-j$, and to trivially intersect $\varphi_1(U \cap U')\,(= (U \cap U')/Z_1\,)$. $\varphi_1(U)$ and $\varphi_1(U \cap U')$ have dimensions h and $h-1$, respectively. If $h = j$, then there is nothing to select, and U'' will be required to satisfy $U \cap U'' = Z_1$. But if $h = j+1$, then the number of choices is $\left[{h \atop 1}\right] - \left[{h-1 \atop 1}\right] = q^{h-1} = q^j$. So in general, assuming as we are that h is j or $j+1$, the number of choices for $\varphi_1(U \cap U'')$ is simply

$$q^{j(h-j)} \tag{2}$$

Next, assume that a subspace Z_2 has been selected to play the role of $U \cap U''$ and let $\varphi_2 : \mathcal{L}(V) \to \mathcal{L}(Z_2^{\perp}/Z_2)$ be defined analogous to the definition of φ_1. The next step in the process is to select a subspace of $Z_2 + U'$ to play the role of $U \cap U'' + U' \cap U''$, which must have dimension $(m-j) + (m-i) - (m-h) = m+h-i-j$. Now $\dim(Z_2) = m-j$, and by Lemma 2, there is a bijection between $\{X \in \mathcal{L}(V) \,|\, X \text{is isotropic}, \dim(X) = m+h-i-j, Z_2 \le X \le Z_2 + U', X \cap U \le Z_2\}$ and the set of isotropic $(h-i)$ - subspaces of $\varphi_2(U')$ that trivially intersect $\varphi_2(U)$. Note that $\varphi_2(U') = \varphi_2(Z_2 + U') \cong U'/(U \cap U' \cap U'')$, which has dimension h. Selecting the desired subspace of $Z_2 + U'$ amounts to selecting an isotropic $(h-i)$ - subspace of $\varphi_2(U')$ that trivially intersects $\varphi_2(U)$. But $\varphi_2(U) \cap \varphi_2(U') = [Z_2^{\perp} \cap (Z_2 + U) \cap (Z_2 + U')]/Z_2 = [Z_2^{\perp} \cap U \cap (Z_2 + U')]/Z_2 = [Z_2^{\perp} \cap (Z_2 + U \cap U')]/Z_2 = \varphi_2(U \cap U') \cong (U \cap U')/(U \cap U' \cap U'')$, which has dimension $h-1$. Note that subspaces of $\varphi_2(U')$ that trivially intersect $\varphi_2(U \cap U')$ also trivially intersect $\varphi_2(U)$. So, similar to the previous count, and assuming h is i or $i+1$, the number of selections for $\varphi_2(U' \cap U'')$ is seen to be

$$q^{i(h-i)} \tag{3}$$

Continuing to the next node of $\mathcal{M}$, assume that a subspace Z_3 has been selected to play the role of $U \cap U'' + U' \cap U''$. If $e = 0$, then this subspace will also be $(U + U') \cap U''$. But if $e = 1$, then $(U + U') \cap U''$ must be one dimension larger. However, for this to be the case, it is necessary for h, i and j to all be equal. To see why, observe that $[(U + U') \cap U'' : U \cap U''] \le [U + U' : U] = [U' : U \cap U'] = 1$. Similarly, $[(U + U') \cap U'' : U' \cap U''] \le 1$. So unless $U \cap U'' = U' \cap U''$, it must be the case that $(U + U') \cap U'' = U \cap U'' + U' \cap U''$.

When $h=i=j$ and $e=1$, and so $Z_1=Z_2=Z_3$, the problem of selecting $(U+U')\cap U''$, again by Lemma 2, amounts to selecting a one-dimensional subspace inside the isotropic space $(U+U')/Z_1$ that trivially intersects each of the two hyperplanes U/Z_1 and U'/Z_1. The number of such selections is $\left[{h+1 \atop 1}\right]-2\left[{h \atop 1}\right]+\left[{h-1 \atop 1}\right]=(q-1)q^{h-1}$.

When $e=1$, but h,i and j are not all equal, there are of course no suitable choices for $(U+U')\cap U''$. Rather than imposing a further restriction on the parameters, at this point, this possibility will be allowed, and will be handled by saying that in general the number of choices for $(U+U')\cap U''$ is

$$\left[(q-1)q^{h-1}\cdot 0^{2h-i-j}\right]^e \tag{4}$$

where 0^0 is to be interpreted as 1. The "zero notation" here is simply a bookkeeping convenience, and essentially means that either $h=i=j$ or $e=0$ is required in order to have a nonzero count.

Assume next that some subspace Z_4 has been selected to serve as $(U+U')\cap U''$, and let φ_4 have the evident meaning. Z_4 has dimension $\dim(Z_3)+e=m+h+e-i-j$. Now $U^\perp\cap U'^\perp\cap U''$ can be selected by selecting $\varphi_4(U^\perp\cap U'^\perp\cap U'')\,(=(U^\perp\cap U'^\perp\cap U'')/Z_4\,)$, which must be isotropic and have dimension $(m-\eta)-(m+h+e-i-j)=i+j-h-e-\eta$. Next, note that $\varphi_4(V)\,(=Z_4^\perp/Z_4\,)$ has dimension $N-2m-2h-2e+2i+2j$, and $\varphi_4(U+U')\,(=(U+U')/Z_4\,)$ has dimension $(m+1)-(m+h+e-i-j)=i+j-h-e+1$. So $\varphi_4(U^\perp\cap U'^\perp)\,(=(U^\perp\cap U'^\perp)/Z_4\,)$ has dimension $N-2m-2h-2e+2i+2j-(i+j-h-e+1)=N-2m-h-e+i+j-1$.

The selection of $\varphi_4(U^\perp\cap U'^\perp\cap U'')$ must be made inside the coisotropic subspace $\varphi_4(U^\perp\cap U'^\perp)$, whose radical is $\varphi_4(U+U')$. The selection must also trivially intersect this latter subspace, and so must project injectively into the quotient space $\varphi_4(U^\perp\cap U'^\perp)/\varphi_4(U+U')$ $(\cong (U^\perp\cap U'^\perp)/(U+U')\,)$. This quotient space has dimension $N-2m-2$. Each isotropic $(i+j-h-e-\eta)$ - subspace of this quotient space lifts to $q^{(i+j-h-e-\eta)(i+j-h-e+1)}$ choices for $\varphi_4(U^\perp\cap U'^\perp\cap U'')$. The total number of selections for this latter subspace, and so for $U^\perp\cap U'^\perp\cap U''$ given the particular Z_4, is therefore

$$\begin{aligned}&\left[{N-2m-2 \atop i+j-h-e-\eta}\right]_0\cdot q^{(i+j-h-e-\eta)(i+j-h-e+1)}\\ &=\frac{(q^{\frac{1}{2}N-m-1-\mu},-q^{\frac{1}{2}N-m-1-\nu};q^{-1})_{i+j-h-e-\eta}}{(q;q)_{i+j-h-e-\eta}}\cdot q^{(i+j-h-e-\eta)(i+j-h-e+1)}\end{aligned} \tag{5}$$

Now suppose that $Z_5=U^\perp\cap U'^\perp\cap U''$ has been selected, and define φ_5 in the usual way. $U^\perp\cap U''$ can then be selected by selecting $\varphi_5(U^\perp\cap U'')\,(=(U^\perp\cap U'')/Z_5\,)$, which must have dimension $(m-\beta)-(m-\eta)=\eta-\beta$. This selection must be made inside the coisotropic subspace $\varphi_5(U^\perp)$, whose radical is $\varphi_5(U)\,(\cong U/(U\cap U'')\,)$. It is straightforward to check that $\dim(\varphi_5(V))=N-2m+2\eta$, $\dim(\varphi_5(U))=j$, $\dim(\varphi_5(U^\perp))=N-2m+2\eta-j$. By Lemma 2, it is further necessary and sufficient that the selection of $\varphi_5(U^\perp\cap U'')$ trivially intersect the coisotropic subspace $\varphi_5(U^\perp\cap U'^\perp)\,(=\varphi_5(U^\perp)\cap\varphi_5(U'^\perp)\,)$, which has radical $\varphi_5(U+U')\,(\cong (U+U')/[(U+U')\cap U'']\,)$. Again, it is straightforward to check that

$\dim(\varphi_5(U+U')) = i+j-h-e+1$ and $\dim(\varphi_5(U^\perp \cap U'^\perp)) = N-2m+2\eta-i-j+h+e-1$.

Each suitable choice for $\varphi_5(U^\perp \cap U'')$ projects injectively into the quotient space $\varphi_5(U^\perp)/\varphi_5(U)$ and must trivially intersect $\varphi_5(U^\perp \cap U'^\perp)/\varphi_5(U)$. Note that $[\varphi_5(U^\perp) : \varphi_5(U^\perp \cap U'^\perp)] = [\varphi_5(U + U') : \varphi_5(U)] = i - h - e + 1$. This is either 0 or 1. When it is 1, that is, when $i = h$ and $e = 0$, and when $\eta = \beta + 1$, then it is necessary to select a one-dimensional isotropic subspace of $\varphi_5(U^\perp)/\varphi_5(U)$ that trivially intersects the coisotopic hyperplane $\varphi_5(U^\perp \cap U'^\perp)/\varphi_5(U)$. Assume that this is the case for the moment. Note that $\dim[\varphi_5(U^\perp)/\varphi_5(U)] = N - 2m - 2j + 2\eta$. By Lemma 4, the number of such 1-subspaces is $q^{N-2m-2j+2\eta-1-\mu-\nu}$. Each of these lifts to q^j ($= \dim(\varphi_5(U))$) possible choices for $\varphi_5(U^\perp \cap U'')$. Putting this together, we see in general that the number of choices for $U^\perp \cap U''$, given a particular Z_5, is

$$\left[q^{N-2m-j+2\eta-1-\mu-\nu} \cdot 0^{h-i+e}\right]^{\eta-\beta}$$

which, because η is either β or $\beta + 1$, equals

$$\left[q^{N-2m-j+\alpha+\beta-\mu-\nu} \cdot 0^{h-i+e}\right]^{\eta-\beta} \cdot q^{(\eta-\alpha)(\eta-\beta)} \tag{6}$$

Next, assume that $Z_6 = U^\perp \cap U''$ has been selected. $U^\perp \cap U'' + U'^\perp \cap U''$ can then be selected by selecting $\varphi_6(U'^\perp \cap U'')$, where φ_6 is defined as usual. Now $\varphi_6(U'^\perp \cap U'')\,(= (U^\perp \cap U'' + U'^\perp \cap U'')/(U^\perp \cap U'') \cong (U'^\perp \cap U'')/(U^\perp \cap U'^\perp \cap U''))$ must have dimension $(m-\alpha)-(m-\eta) = \eta-\alpha$, and will be selected in a manner similar to that used previously to select Z_6.

Observe that $\varphi_6(U') = (Z_6 + Z_6^\perp \cap U')/Z_6 \cong (Z_6^\perp \cap U')/(U' \cap U'')$, which has dimension $i-\eta+\beta$, because $\dim(Z_6^\perp \cap U') = N$ - $\dim(Z_6+U'^\perp) = N$ - $\dim(Z_6)$ - $\dim(U'^\perp) + \dim(Z_5)$ $= N - (m-\beta) - (N-m) + (m-\eta) = m-\eta+\beta$. Now, $\dim(\varphi_6(V)) = N - 2m + 2\beta$, so $\dim[\varphi_6(U'^\perp)/\varphi_6(U')] = N - 2m - 2i + 2\eta$.

Note too that $\varphi_6(U)+\varphi_6(U') = \varphi_6(U+U') = [Z_6+Z_6^\perp\cap(U+U')]/Z_6 \cong [Z_6^\perp\cap(U+U')]/Z_4$. This has dimension $i+j-h-e-\eta+\beta+1$, since $\dim[Z_6^\perp\cap(U+U')] = N$ - $\dim[Z_6+(U^\perp\cap U'^\perp)]$ $= N$ - $\dim(Z_6)$ - $\dim(U^\perp\cap U'^\perp) + \dim(U^\perp\cap Z_5) = N-(m-\beta)-(N-m-1)+(m-\eta) = m-\eta+\beta+1$, and since $\dim(Z_4) = m+h+e-i-j$. It follows that $[\varphi_6(U'^\perp) : \varphi_6(U^\perp)\cap\varphi_6(U'^\perp)] = [\varphi_6(U'^\perp) : \varphi_6(U^\perp\cap U'^\perp)] = [\varphi_6(U+U') : \varphi_6(U')] = j-h-e+1$, which is either 0 or 1, and is 1 only when $h = j$ and $e = 0$.

By the same reasoning that was applied to select Z_6, it can now be seen that the number of choices for $U^\perp \cap U'' + U'^\perp \cap U''$, given the choice for Z_6, is

$$\left[q^{N-2m-i+\alpha+\beta-\mu-\nu} \cdot 0^{h-j+e}\right]^{\eta-\alpha} \tag{7}$$

since $\dim[\varphi_6(U'^\perp)/\varphi_6(U')] = N - 2m - 2i + 2\eta$ and $\dim(\varphi_6(U')) = i - \eta + \beta$.

So now assume that $Z_7 = U^\perp \cap U'' + U'^\perp \cap U''$ has been selected. If $\varepsilon = 0$ then this is also $(U^\perp + U'^\perp) \cap U''$. But if $\varepsilon = 1$, then $(U^\perp + U'^\perp) \cap U''$ needs to be selected to be one dimension larger. However, similar to the situation when selecting $(U + U') \cap U''$, this is only possible when $\eta = \alpha = \beta$, since $[U^\perp + U'^\perp : U^\perp]$ and $[U^\perp + U'^\perp : U'^\perp]$ are both one. Assume for the time being that $\eta = \alpha = \beta$ and $\varepsilon = 1$. Thus $Z_5 = Z_6 = Z_7$. Consider selecting the isotropic 1-subspace $\varphi_5((U^\perp + U'^\perp) \cap U'')$, which must be chosen inside $\varphi_5(U^\perp + U'^\perp)$, and, by Lemma 2, must trivially intersect both $\varphi_5(U^\perp)$ and $\varphi_5(U'^\perp)$, and any such selection lifts to an acceptable selection for $(U^\perp + U'^\perp) \cap U''$.

Recall that $\dim(\varphi_5(V)) = N - 2m + 2\eta$. Notice that $\varphi_5(U) + \varphi_5(U') = \varphi_5(U + U') \cong (U + U')/[(U + U') \cap U'']$, which has dimension $i + j - h - e + 1$. Also, $\varphi_5(U \cap U') \cong (U \cap U')/(U \cap U' \cap U'')$, which has dimension $h - 1$. However, $\dim[\varphi_5(U) \cap \varphi_5(U')] = \dim[\varphi_5(U)] + \dim[\varphi_5(U')]$ - $\dim[\varphi_5(U+U')] = j+i-(i+j-h-e+1) = h+e-1$. So in general, $\varphi_5(U\cap U')$ and $\varphi_5(U)\cap\varphi_5(U')$ are not the same, and in fact $[\varphi_5(U)\cap\varphi_5(U') : \varphi_5(U\cap U')] = e$. It follows that $[\varphi_5(U^\perp + U'^\perp) : \varphi_5(U^\perp) + \varphi_5(U'^\perp)] = e$.

Also note that $[\varphi_5(U^\perp + U'^\perp) : \varphi_5(U^\perp)] = [\varphi_5(U) : \varphi_5(U \cap U')] = j - h + 1$ and $[\varphi_5(U^\perp + U'^\perp) : \varphi_5(U'^\perp)] = [\varphi_5(U') : \varphi_5(U \cap U')] = i - h + 1$. Clearly $h = i = j$ is required in order to be able to select the required 1-subspace. So assume this as well for the time being. It is straightforward now to check that $\varphi_5(U^\perp) = \varphi_5(U'^\perp)$ if and only if $\varphi_5(U) = \varphi_5(U')$ if and only if $e = 1$.

In the $e = 1$ subcase, we are dealing with the same sort of enumeration problem considered earlier. Specifically, we must select an isotropic 1-subspace in the $(N-2m+2\eta-2h+2)$ - dimensional quotient space $\varphi_5(U^\perp + U'^\perp)/\varphi_5(U \cap U')$, one that trivially intersects the corresponding coisotropic hyperplane $\varphi_5(U^\perp)/\varphi_5(U \cap U')$. This then needs to be lifted to one of q^{h-1} choices for $\varphi_5((U^\perp + U'^\perp) \cap U'')$. By Lemma 4 or 5, the number of choices for $(U^\perp + U'^\perp) \cap U''$ is $q^{N-2m+2\eta-2h+2-1-\mu-\nu} \cdot q^{h-1} = q^{N-2m-h+2\eta-\mu-\nu}$. The $e = 0$ subcase is similar but requires trivially intersecting two coisotropic hyperplanes, $\varphi_5(U^\perp)/\varphi_5(U \cap U')$ and $\varphi_5(U'^\perp)/\varphi_5(U \cap U')$. By Lemma 5, the number of choices for $(U^\perp + U'^\perp) \cap U''$ is $(q - 1)\, q^{N-2m-h+2\eta-1-\mu-\nu}$.

Combining the subcases, we see that in general the number of choices for $(U^\perp+U'^\perp)\cap U''$, given a particular choice for Z_7, is

$$\left[(q-1)^{1-e} \cdot q^{N-2m-h+2\eta+e-1-\mu-\nu} \cdot 0^{2h-i-j+2\eta-\alpha-\beta}\right]^{\varepsilon} \tag{8}$$

As expected, suppose next that $Z_8 = (U^\perp + U'^\perp) \cap U''$ has been selected, and let φ_8 be the evident mapping.

The final task in selecting U'' is to extend Z_8 to all of U''. This can be accomplished by selecting $\varphi_8(U'')\,(= U''/Z_8\,)$, which must have dimension $\alpha + \beta - \eta - \varepsilon$, since $\dim(Z_8) = \dim(Z_7) + \varepsilon = (m - \beta) + (m - \alpha) - (m - \eta) + \varepsilon = m + \eta - \alpha - \beta + \varepsilon$. It must also trivially intersect the coisotropic subspace $\varphi_8(U^\perp + U'^\perp)$. The radical of this coisotropic subspace is $\varphi_8(U\cap U')\,(= (Z_8+U\cap U')/Z_8 \cong (U\cap U')/(U\cap U'\cap U'')\,)$, which has dimension $(m - 1) - (m - h) = h - 1$. Note that $\dim(\varphi_8(V)) = N - 2m - 2\eta + 2\alpha + 2\beta - 2\varepsilon$. By Lemma 4, the number of choices for $\varphi_8(U'')$ is

$$\begin{bmatrix} h-1 \\ \alpha+\beta-\eta-\varepsilon \end{bmatrix} \cdot q^{(\alpha+\beta-\eta-\varepsilon)[(N-2m-2\eta+2\alpha+2\beta-2\varepsilon)-(h-1)-\frac{1}{2}(\alpha+\beta-\eta-\varepsilon)+\frac{1}{2}-\mu-\nu]}$$

$$(9) \qquad = \frac{(q;q)_{h-1}}{(q;q)_{\alpha+\beta-\eta-\varepsilon}\,(q;q)_{h-\alpha-\beta+\eta+\varepsilon-1}} \cdot q^{(\alpha+\beta-\eta-\varepsilon)[N-2m-h+\frac{3}{2}(\alpha+\beta-\eta-\varepsilon+1)-\mu-\nu]}$$

The total number of allowable choices for U'' is of course obtained by multiplying the various counts ((1) through (9)) associated with the various stages in the above process used to select this subspace. Furthermore, each of the five conditions (A through E) listed in the theorem is sufficient to ensure that the exponents in all of the "zero notations" are zero. Conversely, it is straightforward to check that at least one of these conditions must hold in order for all of these exponents to be zero.

□

The next corollary offers a slightly different expression of the count in Theorem 1, in an effort to produce a more uniform approach to enumerating the values of $p^{1_0}_{i_\alpha j_\beta}$ for various cases, some of which are considered in Corollary 2.

Corollary 1. *The count in Theorem 1 can also be expressed as*

$$\frac{(q;q)_{m-1}\,(q^{d-m-1},-q^{d'-m-1};q^{-1})_{g-\kappa-r-\rho-e}}{(q;q)_{m-f-r}\,(q;q)_{\lambda-\rho-\varepsilon}\,(q;q)_{f-\lambda-1+r+\rho+\varepsilon}\,(q;q)_{g-\kappa-r-\rho-e}} \cdot (q-1)^{e+\varepsilon-e\varepsilon} \cdot q \wedge$$

$$\{\,[fg+g+d\kappa+d'\kappa-2m\kappa-2g\kappa+g\lambda-f\lambda+\kappa^2+\tfrac{1}{2}\lambda^2-\kappa+\tfrac{3}{2}\lambda]+4r\rho+r(f-g+2\kappa-2\lambda)$$
$$+\,\rho(d+d'-2m+f-3g+3\kappa-2\lambda)+e\varepsilon+e(f-2g+\kappa-1)+\varepsilon(2\kappa-3\lambda-1)\,\}\,,$$

assuming that at least one of the five conditions in Theorem 1 holds, and so either $e=\varepsilon=0$ *or* $r=\rho=0$, *where*

$$r=\begin{cases}0 & \text{if } h=i=j\\ 1 & \text{otherwise,}\end{cases} \qquad \rho=\begin{cases}0 & \text{if } \eta=\alpha=\beta\\ 1 & \text{otherwise,}\end{cases}$$

and where $d=N/2-\mu$, $d'=N/2-\nu$, $f=\min\{i,j\}$, $g=\max\{i,j\}$, $\kappa=\min\{\alpha,\beta\}$ *and* $\lambda=\max\{\alpha,\beta\}$.

Proof. First observe that $h=f+r$, $i+j-h=g-r$, $\eta=\kappa+\rho$ and $\alpha+\beta-\eta=\lambda-\rho$. The formula given in Theorem 1 is easily converted to the one given here, except that a portion of the power of q there needs to be handled in a special way. Specifically, $-i(\eta-\alpha)-j(\eta-\beta)$ can be seen to equal $(g-r)(\lambda-\kappa-2\rho)$, by the following argument. In order for $\eta-\alpha$ to be nonzero, it must be that $h=j$, since one of the conditions in Theorem 1 is presumably satisfied. In this case, $i=i+j-h=g-r$. Likewise, in order for $\eta-\beta$ to be nonzero, it must be that $j=g-r$. So $-i(\eta-\alpha)-j(\eta-\beta)=-(g-r)(2\eta-\alpha-\beta)=(g-r)(\lambda-\kappa-2\rho)$.

□

For given values of i,j,α and β, the numbers $p^{1_0}_{i_\alpha j_\beta}$ can now be computed by summing the count in Corollary 1 over the allowable values of r,e,ρ and ε, each of which must be either 0 or 1. The case where $0<\alpha=\beta<i=j<m$ involves the maximum number of terms (six), because here each of e,ε,r and ρ can be either 0 or 1, restricted only by the requirements that r and ρ cannot both be 1, and that either $e=\varepsilon=0$ or $r=\rho=0$. The need for these restrictions can be seen from the restrictions listed in Theorem 1 applied in the context that $i=j$ and $\alpha=\beta$. The other cases for $p^{1_0}_{i_\alpha j_\beta}$ involve fewer than six terms, and in fact if $i\neq j$ or $\alpha\neq\beta$, then there is just a single term to consider. Most of the numbers $p^{1_0}_{i_\alpha j_\beta}$ shall now be given explicitly in the next corollary and a subsequent remark.

Corollary 2. *Assuming that* $0 < \alpha < i < m$,

$$p^{1_0}_{i_\alpha i_\alpha} = \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha-1}}{(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha}} \cdot q^{i^2-i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2-\frac{1}{2}\alpha-1} \cdot X,$$

where

$$\begin{aligned} X = q^{d+d'-2m}\,&(2q^{i+2\alpha+1} - q^{i+\alpha+1} - q^{3\alpha+1} - q^{3\alpha} + q^{2\alpha}) \\ &+ (q^{d-m} - q^{d'-m})\,(q^{2i+\alpha+1} - q^{i+2\alpha} - q^{i+\alpha+1} + q^{i+\alpha}) \\ &+ q^{m+2i+1} - q^{m+i+\alpha+1} - q^{3i+1} - q^{3i} + 2q^{2i+\alpha} + q^{i+\alpha+1} - q^{i+\alpha}) \,. \end{aligned}$$

Also,

$$p^{1_0}_{i_\alpha i_{\alpha-1}} = \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha}}{(q;q)_{m-i}\,(q;q)_{\alpha-1}\,(q;q)^2_{i-\alpha}} \cdot q^{i^2+i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2-\frac{1}{2}\alpha-1},$$

$$p^{1_0}_{i_\alpha (i-1)_\alpha} = \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha-1}}{(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha-1}} \cdot q^{i^2+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{3}{2}\alpha-1},$$

$$p^{1_0}_{i_\alpha (i-1)_{\alpha-1}} = \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha-1}}{(q;q)_{m-i}\,(q;q)_{\alpha-1}\,(q;q)_{i-\alpha}\,(q;q)_{i-\alpha-1}} \cdot q^{i^2+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{1}{2}\alpha-1}.$$

Therefore,

$$p_{i_\alpha i_\alpha 1_0} = \frac{(q^d, q^{d'}; q^{-1})_{m+i-\alpha}}{(1-q)^2\,(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha}} \cdot q^{i^2-i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2-\frac{1}{2}\alpha} \cdot X,$$

$$p_{i_\alpha i_{\alpha-1} 1_0} = \frac{(q^d, q^{d'}; q^{-1})_{m+i-\alpha+1}}{(1-q)^2\,(q;q)_{m-i}\,(q;q)_{\alpha-1}\,(q;q)^2_{i-\alpha}} \cdot q^{i^2+i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2-\frac{1}{2}\alpha},$$

$$p_{i_\alpha (i-1)_\alpha 1_0} = \frac{(q^d, q^{d'}; q^{-1})_{m+i-\alpha}}{(1-q)^2\,(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha-1}} \cdot q^{i^2+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{3}{2}\alpha},$$

$$p_{i_\alpha (i-1)_{\alpha-1} 1_0} = \frac{(q^d, q^{d'}; q^{-1})_{m+i-\alpha}}{(1-q)^2\,(q;q)_{m-i}\,(q;q)_{\alpha-1}\,(q;q)_{i-\alpha}\,(q;q)_{i-\alpha-1}} \cdot q^{i^2+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{1}{2}\alpha}.$$

Proof. Using Corollary 1 to compute $p^{1_0}_{i_\alpha i_\alpha}$, we obtain

$$\sum_{r,e,\rho,\varepsilon} \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha-r-\rho-e}}{(q;q)_{m-i-r}\,(q;q)_{\alpha-\rho-\varepsilon}\,(q;q)_{i-\alpha-1+r+\rho+\varepsilon}\,(q;q)_{i-\alpha-r-\rho-e}} \cdot (q-1)^{e+\varepsilon-e\varepsilon} \cdot q\wedge$$

$$\{\, (i^2+i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\tfrac{3}{2}\alpha^2+\tfrac{1}{2}\alpha)+4r\rho+\rho(u+v-\alpha)+e\varepsilon-e(i-\alpha+1)-\varepsilon(\alpha+1)\,\}\,.$$

This can be rewritten in the form

$$\sum_{r,e,\rho,\varepsilon} \frac{(q;q)_{m-1}\,(q^{d-m-1}, -q^{d'-m-1}; q^{-1})_{i-\alpha-1}}{(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha}} \cdot q^{i^2+i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{1}{2}\alpha}$$

$$\cdot \Big\{\, \big[(1-q^{d-m-i+\alpha})(1+q^{d'-m-i+\alpha})\big]^{1-r\vee\rho\vee e}\,(1-q^\alpha)^{\rho\vee\varepsilon}\,(1-q^{m-i})^r\,(1-q^{i-\alpha})^{1+e-\varepsilon}$$

$$\cdot\,(q-1)^{e\vee\varepsilon}\;\cdot\;q^{4r\rho+\rho(u+v-\alpha)+e\varepsilon-e(i-\alpha+1)-\varepsilon(\alpha+1)}\Big\}\,,$$

where the operation $\vee$ is defined by $\sigma\vee\tau=\sigma+\tau-\sigma\tau$. Symbolic manipulation software, such as Mathematica[1] and Maple[2], can then be used to quickly establish that the sum of the quantity in curly braces here, over the allowable values of r, e, ρ and ε, equals $q^{-2i-\alpha-1}$ times X.

The numbers $p^{1_0}_{i_\alpha i_{\alpha-1}}, p^{1_0}_{i_\alpha (i-1)_\alpha}$ and $p^{1_0}_{i_\alpha (i-1)_{\alpha-1}}$ can likewise be computed using Corollary 1. Because of the restrictions imposed by Theorem 1 on the parameters (r,ρ,e,ε), only one term is nonzero in each of these cases. The only nonzero term in the computation of $p^{1_0}_{i_\alpha i_{\alpha-1}}$ occurs when $\rho=1$, $r=e=\varepsilon=0$. The only nonzero term in the computation of $p^{1_0}_{i_\alpha (i-1)_\alpha}$ occurs when $r=1$, $\rho=e=\varepsilon=0$. The only nonzero term in the computation of $p^{1_0}_{i_\alpha (i-1)_{\alpha-1}}$ occurs when $r=\rho=1$, $e=\varepsilon=0$. The remaining claims follows from Lemma 6 and the fact that $p_{i_\alpha j_\beta 1_0}=|\mathcal{R}_{1_0}|\,p^{1_0}_{i_\alpha j_\beta}$.

□

Remark 2. *Using the values from Corollary 2, it is straightforward, possibly with the aid of symbolic manipulation software, to check that*

$$\sum_{j_\beta} p^{1_0}_{i_\alpha j_\beta} \;=\; p^{1_0}_{i_\alpha i_\alpha}+p^{1_0}_{i_\alpha i_{\alpha-1}}+p^{1_0}_{i_\alpha i_{\alpha+1}}+p^{1_0}_{i_\alpha (i-1)_\alpha}+p^{1_0}_{i_\alpha (i+1)_\alpha}+p^{1_0}_{i_\alpha (i-1)_{\alpha-1}}+p^{1_0}_{i_\alpha (i+1)_{\alpha+1}}$$

$$=\frac{(q;q)_{m-1}\,(q^{d-m-1},-q^{d'-m-1};q^{-1})_{i-\alpha-1}}{(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha}}\cdot q^{i^2-i+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2-\frac{1}{2}\alpha-1}$$

$$\cdot\Big\{\,q^{d+d'-2m}(2q^{i+2\alpha+1}-q^{i+\alpha+1}-q^{3\alpha+1}-q^{3\alpha}+q^{2\alpha})$$

$$+\,(q^{d-m}-q^{d'-m})(q^{2i+\alpha+1}-q^{i+2\alpha}-q^{i+\alpha+1}+q^{i+\alpha})$$

$$+\,(q^{m+2i+1}-q^{m+i+\alpha+1}-q^{3i+1}-q^{3i}+2q^{2i+\alpha}+q^{i+\alpha+1}-q^{i+\alpha})$$

$$+\,(1-q^{d-m-i+\alpha})(1+q^{d'-m-i+\alpha})(1-q^{\alpha})q^{2i}\;+\;(1-q^{i-\alpha})^2q^{d+d'-2m+3\alpha+1}$$

$$+\,(1-q^{i-\alpha})^2q^{i+2\alpha}\;+\;(1-q^{d-m-i+\alpha})(1+q^{d'-m-i+\alpha})(1-q^{m-i})q^{3i+1}$$

$$+\,(1-q^{\alpha})(1-q^{i-\alpha})q^{i+\alpha}\;+\;(1-q^{m-i})(1-q^{i-\alpha})q^{d+d'-2m+i+2\alpha+1}\Big\}$$

$$=\frac{(q;q)_m\,(q^{d-m},-q^{d'-m};q^{-1})_{i-\alpha}}{(q;q)_{m-i}\,(q;q)_\alpha\,(q;q)^2_{i-\alpha}}\cdot q^{i^2+d\alpha+d'\alpha-2m\alpha-2i\alpha+\frac{3}{2}\alpha^2+\frac{1}{2}\alpha}\;=\;\Big|\mathcal{R}_{i_\alpha}\Big|\Big/\begin{bmatrix}N\\m\end{bmatrix}_0 .$$

This is as required, since the summation must count all of the isotropic m-subspaces U'' for which $(U',U'')\in\mathcal{R}_{i_\alpha}$, given an isotropic m-subspace U'.

[1] *Mathematica* is a registered trademark of Wolfram Research, Inc.
[2] *Maple* is a registered trademark of Waterloo Maple Inc.

Notice that Corollary 2 and Remark 2 pertain to the intersection numbers $p^{1_0}_{i_\alpha j_\beta}$ for which $0 < \alpha < i < m$. It is staightforward to use Theorem 1 or Corollary 1 and to reason as in the proof of Corollary 2, in order to compute the remaining intersection numbers. Such computations can then be double checked along the lines of Remark 2. Some of these remaining intersection numbers are as follows, as the reader can verify.

Remark 3. *Assume that $0 < i < m$. Then*

(1) *the only nonzero cases of $p^{1_0}_{i_0 j_\beta}$ are*

$$p^{1_0}_{i_0 i_0} = \frac{(q;q)_{m-1}(q^{d-m-1}, -q^{d-m-1}; q^{-1})_{i-1}}{(q;q)_{m-i}(q;q)_i(q;q)_{i-1}} \cdot q^{i^2-1}$$

$$\cdot \left[(1-q^{d-m-1})(1+q^{d'-m-1})q^{i+1} + (1-q^{m-i})q^{i+1} + (1-q^i)(q-1)\right],$$

$$p^{1_0}_{i_0 i_1} = \frac{(q;q)_{m-1}(q^{d-m-1}, -q^{d-m-1}; q^{-1})_{i-1}}{(q;q)_{m-i}(q;q)^2_{i-1}} \cdot q^{d+d'-2m+i^2-i},$$

$$p^{1_0}_{i_0 (i+1)_0} = \frac{(q;q)_{m-1}(q^{d-m-1}, -q^{d-m-1}; q^{-1})_{i}}{(q;q)_{m-i-1}(q;q)^2_{i}} \cdot q^{i^2+2i},$$

$$p^{1_0}_{i_0 (i+1)_1} = \frac{(q;q)_{m-1}(q^{d-m-1}, -q^{d-m-1}; q^{-1})_{i-1}}{(q;q)_{m-i-1}(q;q)_i(q;q)_{i-1}} \cdot q^{d+d'-2m+i^2},$$

$$p^{1_0}_{i_0 (i-1)_0} = \frac{(q;q)_{m-1}(q^{d-m-1}, -q^{d-m-1}; q^{-1})_{i-1}}{(q;q)_{m-i}(q;q)^2_{i-1}} \cdot q^{i^2-1} \quad \textit{and}$$

(2) *the only nonzero cases of $p^{1_0}_{i_i j_\beta}$ are*

$$p^{1_0}_{i_i i_i} = \frac{(q;q)_{m-1}}{(q;q)_{m-i}(q;q)_{i-1}} \cdot (q-1) \cdot q^{di+d'i-2mi+\frac{1}{2}i^2+\frac{1}{2}i-1},$$

$$p^{1_0}_{i_i (i+1)_i} = \frac{(q;q)_{m-1}}{(q;q)_{m-i-1}(q;q)_{i}} \cdot q^{di+d'i-2mi+\frac{1}{2}i^2+\frac{3}{2}i},$$

$$p^{1_0}_{i_i i_{i-1}} = \frac{(q;q)_{m-1}}{(q;q)_{m-i}(q;q)_{i-1}} \cdot q^{di+d'i-2mi+\frac{1}{2}i^2+\frac{1}{2}i-1}.$$

As a final thought, it might be noted that the partial results obtained above concerning the intersection numbers of the association schemes being discussed might yield partial results concerning the eigenvalues of these association schemes. Presumably, for each $\mathcal{R}_{i_\alpha}$, the Bose-Mesner algebra of the association scheme discussed in this article has a corresponding "natural" idempotent E_{i_α}. Let F_{i_α} denote its rank. Of course each $\mathcal{R}_{i_\alpha}$ has a corresponding adjacency matrix A_{i_α} in the Bose-Mesner algebra. The eigenvalue matrix $(P_{i_\alpha j_\beta})$ is defined by $A_{i_\alpha} = \Sigma_{j_\beta} P_{i_\alpha j_\beta} E_{j_\beta}$, and its entries (the eigenvalues) are known to be related to the intersection numbers by the formula $p_{i_\alpha j_\beta k_\gamma} = \Sigma_{t_\omega} P_{i_\alpha t_\omega} P_{j_\beta t_\omega} P_{k_\gamma t_\omega} F_{t_\omega}$. Perhaps it is possible at this point to gain some information about the eigenvalues from a knowledge of the numbers $p_{i_\alpha j_\beta 1_0}$.

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