A SCHENSTED INSERTION FOR TENSOR POWERS OF THE WEIL REPRESENTATION

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ABSTRACT. We give the first known example of a Robinson-Schensted type insertion for a class of infinite dimensional representations. The tensor powers of the Weil representation W of $\mathfrak{sp}(2n,\mathbb{C})$ decompose as a direct sum of certain highest weight $\mathfrak{sp}(2n,\mathbb{C})$-modules $L_k(\lambda)$ tensored with some corresponding finite dimensional irreducible $O(k,\mathbb{C})$-module V_λ. Concentrating only on the $\mathfrak{sp}(2n,\mathbb{C})$-module structure, we can see this as iterating the following decomposition:

$$L_k(\lambda)\otimes W \simeq \bigoplus_\mu L_{k+1}(\mu)\,. \tag{2}$$

If k is sufficently large relative to n and λ, then the module $L_k(\lambda)$ and all the terms on the right hand side belong to the holomorphic discrete series and our insertion algorithm allows us to give a weighted bijection proving the formal character identity corresponding to the decomposition (2). It seems likely that a correspondence along these lines can be given to combinatorially explain similar identities of formal characters for small k as well. We give a such a correspondence when $n=2$, but this remains open for general n.

RÉSUMÉ. On donne le premier exemple d'une insertion de type Robinson-Schensted pour une classe representations de dimension infinies. La puissance tensorielle de la representation de Weil decomposee en somme directe de certains $\mathfrak{sp}(2n,\mathbb{C})$-modules de highest weight est tensoree avec des $O(k,\mathbb{C})$-modules V_λ irreducibles de dimensions finies. En etudiant seulement les structures des $\mathfrak{sp}(2n,\mathbb{C})$-module on peut obtenir ces resultats par iteration de la decomposition suivante

$$L_k(\lambda)\otimes W \simeq \bigoplus_\mu L_{k+1}(\mu)\,. \tag{2}$$

Si k est suffisament grand par rapport a n et λ alors le module $L_k(\lambda)$ et tous les termes de droite appartiennent a la serie discrete holomorphe et notre algorithm d'insertion permet de donner une weighted bijection demontrant ainsi l'indentite du caractere formel correspondant a la decomposition (2). Il semblerait probable qu'une correspondence similaire pourait etre donnee de facon a donner une expliquation combinatoire sur les identites des caracteres formels pour petit k. Ceci est a l'heure actuelle un projet. On donne un tel correspondence pour $n=2$, mais le problème reste ouvert pour $n>2$.

1. INTRODUCTION

Our goal is to extend the notion of tableaux, insertion and the Robinson-Schensted correspondence (or the R-S correspondence for short) to a class of infinite-dimensional representations of the Lie algebra $\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{C})$. (Here $n\geq 2$ is a fixed positive integer; see § 2 for definitions of this and terms to follow.) The Weil representation of $\mathfrak{g}$ is an infinite-dimensional representation which can be realized on the space of polynomials $W=\mathbb{C}[x_1,x_2,\ldots,x_n]$ using multiplication operators and certain simple differential operators. We can reformulate results of M. Kashiwara and M. Vergne ([KV]) (see also the work of R. Howe [How]) to show that the centralizer of the action of $\mathfrak{g}$ on the k-th tensor power

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$W^{\otimes k}$ of W is given by an action of the orthogonal group $O(k,\mathbb{C})$, and that they constitute a dual pair, namely the space $W^{\otimes k}$ decomposes into a multiplicity-free sum:

$$W^{\otimes k} \simeq \bigoplus_{\lambda} L_k(\lambda) \otimes V_\lambda, \tag{1}$$

where each $L_k(\lambda)$ is an irreducible $\mathfrak{g}$-module, each V_λ is an irreducible $O(k,\mathbb{C})$-module (using the notation in [KV]), and the summation is for $\lambda \in \mathcal{L}_k$ where $\mathcal{L}_k = \{\, \lambda \in \mathbb{Y} \mid l(\lambda) \le n,\ \lambda'_1 + \lambda'_2 \le k \,\}$. Here $\mathbb{Y}$ denotes Young's lattice, the lattice of all partitions of nonnegative integers (ordered by inclusion). These $L_k(\lambda)$ all turn out to be irreducible highest weight modules (for a fixed choice of Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$). The highest weight of $L_k(\lambda)$ is

$$\begin{aligned}\Lambda(k,\lambda) &= -(\frac{k}{2} + \lambda_n)\epsilon_1 - (\frac{k}{2} + \lambda_{n-1})\epsilon_2 - \ldots - (\frac{k}{2} + \lambda_1)\epsilon_n \\ &= -\frac{k}{2}\omega_n - \lambda_n\epsilon_1 - \lambda_{n-1}\epsilon_2 - \cdots - \lambda_1\epsilon_n\,,\end{aligned}$$

where $\omega_n = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$.

The highest weights $\Lambda(k,\lambda)$ obtained in this manner from (1) for all $k \in \mathbb{Z}_{\ge 0}$ lie in (but do not fill) a cone $C_0 = \{\, \sum_{i=1}^n a_i\epsilon_i \in \widetilde{Q} \mid 0 > a_1 \ge a_2 \ge \cdots \ge a_n \,\}$ in the lattice $\widetilde{Q} = \{\, \sum_{i=1}^n a_i\epsilon_i \mid a_i \in \frac{1}{2}\cdot\mathbb{Z}\ (1 \le i \le n),$ and $a_i - a_{i+1} \in \mathbb{Z}\ (1 \le i \le n-1) \,\}$. Since the V_λ are finite-dimensional $O(k,\mathbb{C})$-modules, it is possible to describe their behavior in combinatorial terms. R. King and B. Wybourne exploited this connection to obtain some combinatorial results for the infinite-dimensional representations $L_k(\lambda)$ ([KW]).

Our model is the following version of the classical R-S correspondence which depicts the decomposition of the k-th tensor power of the natural representation of $\mathfrak{gl}(n,\mathbb{C})$:

$$[1,n]^k \xrightarrow{\sim} \coprod_{\substack{\lambda \vdash k \\ l(\lambda) \le n}} \mathrm{CST}^{(n)}(\lambda) \times \mathrm{SYT}(\lambda).$$

Here $\mathrm{CST}^{(n)}(\lambda)$ (resp. $\mathrm{SYT}(\lambda)$) denotes the set of column strict or semistandard tableaux (resp. standard tableaux) of shape λ with entries from $[1,n] = \{1,2,\ldots,n\}$. This bijection is constructed as a repetition of a procedure called *row insertion*, $\mathrm{CST}^{(n)}(\lambda) \times [1,n] \xrightarrow{\sim} \coprod_{\substack{\mu \dot{\supset} \lambda \\ l(\mu) \le n}} \mathrm{CST}^{(n)}(\mu)$, which depicts the decomposition of the tensor product $V_\lambda \otimes V_\square \simeq \bigoplus_\mu V_\mu$ (where V_λ and V_μ are here irreducible finite-dimensional $\mathfrak{gl}(n,\mathbb{C})$-modules, $V_\square$ is the space of natural representation of $\mathfrak{gl}(n,\mathbb{C})$, and μ runs over partitions such that μ/λ is one box and $l(\mu) \le n$). By analogy, our correspondence should consist of a repetition of a bijection reflecting the decomposition of the tensor product

$$L_k(\lambda) \otimes W \simeq \bigoplus_{\mu} L_{k+1}(\mu), \tag{2}$$

where μ runs over partitions in $\mathcal{L}_{k+1}$ such that μ/λ is a horizontal strip.

The first ingredient we need is a set $\mathrm{SIST}(\lambda)$ of certain tableaux which, with a notion of weight depending on k, gives the weight generating function equal to the formal character of $L_{\Lambda(k,\lambda)}$. Here, for any $\Lambda \in \mathfrak{h}^*$, L_Λ denotes the (unique) irreducible highest weight $\mathfrak{g}$-module with highest weight Λ.

This has been done for a subclass of these $\mathfrak{g}$-modules in [TY], namely for those having highest weights in a smaller cone $C_1 = \{\, \sum_{i=1}^n a_i\epsilon_i \in C_0 \mid -n > a_1 \,\}$, in other words those in the holomorphic discrete series, in which case L_Λ is actually a generalized Verma module:

If we put $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$ embedded in $\mathfrak{g}$ in a suitable fashion, then all weights in C_0 are $\mathfrak{k}$-dominant integral (with respect to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{k}$ of $\mathfrak{k}$). Therefore, if $\Lambda \in C_0$, we have a finite-dimensional irreducible $\mathfrak{k}$-module F_Λ with highest weight Λ. Let $\mathfrak{p}$ denote the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and with Levi part $\mathfrak{k}$. Then we can form the generalized Verma module $N_\Lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\Lambda$, where F_Λ is regarded as a $\mathfrak{p}$-module by defining the action of the nilpotent radical of $\mathfrak{p}$ to be trivial. If $\Lambda \in C_0 - C_1$, L_Λ is the irreducible quotient of N_Λ (still possibly isomorphic to N_Λ for some such Λ).

It can be easily seen that, in (2), if $L_k(\lambda)$ has highest weight in C_1, then so does every summand on the right hand side. Our result in § 3 presents a weight-preserving bijection

$$\mathrm{SIST}(\lambda) \times \mathbb{Z}^n_{\geq 0} \xrightarrow{\sim} \coprod_\mu \mathrm{SIST}(\mu)\,, \tag{3}$$

where μ runs through partitions in $\mathcal{L}_{k+1}$ such that μ/λ is a horizontal strip. Here, the generating function of $\mathbb{Z}^n_{\geq 0}$, with a proper definition of weights, gives the formal character of W, and that of $\mathrm{SIST}(\lambda)$ similarly gives the formal character of $L_k(\lambda) = L_{\Lambda(k,\lambda)}$ as stated above; so this bijection models the decomposition (2) for the case $\Lambda(k, \lambda) \in C_1$.

Unlike the case of finite-dimensional representations, an equality of formal characters is in general far from sufficient to draw out any conclusion on the decomposition of a tensor product. In our case, however, it is known that the tensor product decomposes as a direct sum of irreducible highest weight modules. (See [Kob].)

If we start from an $L_k(\lambda)$ in the holomorphic discrete series, we can repeat our procedure describing (2) and depict the decomposition of $L_k(\lambda) \otimes W^{\otimes k'}$. However, this is not enough to depict the whole decomposition of the tensor powers of W, since W itself (the very starting point) is a direct sum of two irreducible modules whose highest weights are not in C_1. (Recall that we have assumed that $n \geq 2$. If $n = 1$, then all $N_{\Lambda(k,\lambda)}$ in consideration are irreducible, so that (3) covers all desired cases.)

For $n = 2$, which is the smallest value of n having reducible $N_{\Lambda(k,\lambda)}$, we also fill this gap (§ 4). Namely we find subsets $\mathrm{SIST}(k, \lambda)$ of $\mathrm{SIST}(\lambda)$, for any (k, λ) with $\lambda \in \mathcal{L}_k$, whose weight generating function gives the formal character of $L_k(\lambda)$, including the case where $\Lambda(k, \lambda) \in C_0 - C_1$. We also present a modification of the bijection (3) which depicts the decomposition (2) for the case $\Lambda(k, \lambda) \in C_0 - C_1$. So for $n = 2$, we have associated the whole decomposition of the tensor powers of W with combinatorial bijections. For general n, we conjecture the existence of such subsets $\mathrm{SIST}(k, \lambda)$ and modified bijections that work for the cases where $\Lambda(k, \lambda) \in C_0 - C_1$.

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2. The Weil representation and its modules

We assume throughout the paper that $n \geq 2$ to avoid the degenerate case $n = 1$.

Definition 2.1 Fix the skew-symmetric form on $\mathbb{C}^{2n}$ given as $\langle v, w\rangle = v^t J w$, where J is the $2n \times 2n$ matrix given as follows:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The *symplectic Lie algebra* $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ is then given by

$$\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) | X^t J + JX = 0\}.$$

We may consider $\mathfrak{gl}(n,\mathbb{C})$ as living inside $\mathfrak{sp}(2n,\mathbb{C})$ via the following embedding

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.$$

We fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ in the manner read from Fig. 1 and its footnote. Let $\epsilon_i \in \mathfrak{h}^*$ be such that $\epsilon_i(E_{jj} - E_{n+j,n+j}) = \delta_{ij}$. The set of corresponding simple roots is is $\{\epsilon_i - \epsilon_{i+1}\,(1 \le i < n),\ 2\epsilon_n\}$.

Note that $\mathfrak{gl}(n,\mathbb{C})$ shares a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{sp}(2n,\mathbb{C})$. Also, $\mathfrak{gl}(n,\mathbb{C}) \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{gl}(n,\mathbb{C})$.

Definition 2.2 The *Weil representation* ρ of $\mathfrak{sp}(2n,\mathbb{C})$ on $W = \mathbb{C}[x_1, x_2, \dots, x_n]$ is given explicitely in Figure 1 by defining how various basis elements act on the polynomial ring. It extends, up to a twist by an automorphism of $\mathfrak{sp}(2n,\mathbb{C})$, the simpler action of $\mathfrak{gl}(n,\mathbb{C})$ given by sending $E_{ij} \mapsto x_i\partial_j$, where $\partial_j := \frac{\partial}{\partial x_j}$.

W has two independent highest weight vectors, namely 1 (weight $-\frac{1}{2}\omega_n$) and x_n (weight $-\frac{1}{2}\omega_n - \epsilon_n$). $U(\mathfrak{g}) \cdot 1$ (resp. $U(\mathfrak{g}) \cdot x_n$) is the space of all polynomials of even total degree (resp. odd total degree). We have $W = U(\mathfrak{g}) \cdot 1 \oplus U(\mathfrak{g}) \cdot x_n$, and that these two submodules are irreducible. In the notation defined below, they are $L_{\Lambda(1,(0))}$ and $L_{\Lambda(1,\square)}$. These two highest weights belong to $C_0 - C_1$. (See Definition 2.6).

Definition 2.3 Let Λ be a weight for $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$, i.e, $\Lambda \in \mathfrak{h}^*$. Let $\mathfrak{p}$ be the parabolic subalgebra which contains $\mathfrak{b}$ and whose Levi part is $\mathfrak{k} = \mathfrak{gl}(n,\mathbb{C})$ (thought of as embedded in $\mathfrak{sp}(2n,\mathbb{C})$ by the map described immediately after Definition 2.1). Λ is called $\mathfrak{k}$-*dominant integral* if $\Lambda = \sum_{i=1}^n c_i\epsilon_i$, $c_i - c_{i+1} \in \mathbb{Z}_{\ge 0}$ $(1 \le i \le n-1)$.

We define the following four modules.

- L_Λ = the unique irreducible highest weight $\mathfrak{g}$-module with highest weight Λ (for all $\Lambda \in \mathfrak{h}^*$)
- $M_\Lambda :=$ the Verma module for $\mathfrak{g}$ of highest weight Λ (for all $\Lambda \in \mathfrak{h}^*$).
- F_Λ =the finite dimensional irreducible $\mathfrak{k}$-module with highest weight Λ (for $\mathfrak{k}$-dominant integral Λ).
- N_Λ =the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\Lambda$ for $\mathfrak{g}$, where F_Λ is regarded as a $U(\mathfrak{p})$-module by defining the action of the nilpotent radical of $\mathfrak{p}$ to be trivial.

Definition 2.4 For $\lambda \in \mathbb{Y}$ with $l(\lambda) \le n$ and $k \in \mathbb{Z}_{\ge 0}$, we put:

$$\Lambda(k,\lambda) = -\frac{k}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) - \lambda_n\epsilon_1 - \lambda_{n-1}\epsilon_2 - \cdots - \lambda_1\epsilon_n.$$
$$= -\sum_{i=1}^n (\lambda_{n-i+1} + \frac{k}{2})\epsilon_i$$

Definition 2.5 We define the following lattice $\widetilde{Q}$ in $\mathfrak{h}^*$, which will contain all the weights we consider:

$$\widetilde{Q} = \left\{ \sum_{i=1}^n a_i\epsilon_i \mid a_i \in \frac{1}{2} \cdot \mathbb{Z}\ (1 \le i \le n),\ \text{and } a_i - a_{i+1} \in \mathbb{Z}\ (1 \le i \le n-1) \right\}$$

Unlike the usual weight lattice for $\mathfrak{sp}(2n,\mathbb{C})$ (see, e.g., [Hum]), which is just the one freely generated by the ϵ_i's, here we also need to include those half-integral linear combinations for which every coefficient a_i is not an integer.

Definition 2.6 We define two cones in the lattice $\widetilde{Q}$ as follows

$$C_0 = \{\, a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_n\epsilon_n \in \widetilde{Q} \mid 0 > a_1 \ge a_2 \ge \cdots \ge a_n \,\}.$$
$$C_1 = \{\, a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_n\epsilon_n \in \widetilde{Q} \mid -n > a_1 \ge a_2 \ge \cdots \ge a_n \,\}.$$

FIGURE 1. The Weil representation for $\mathfrak{sp}(2n,\mathbb{C})$.

Root	Element A	$n=2$ Example	Operator $\rho(A)$
Cartan SA	$E_{ii}-E_{n+i,n+i}$	p $\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	$-x_i\partial_i-\frac{1}{2}$
$\epsilon_i-\epsilon_j$ for $i<j$	$E_{ij}-E_{n+j,n+i}$	$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix}$	$x_j\partial_i$
$-\epsilon_i+\epsilon_j$ for $i<j$	$E_{ji}-E_{n+i,n+j}$	$\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&-1\\0&0&0&0\end{pmatrix}$	$x_i\partial_j$
$\epsilon_i+\epsilon_j$ for $i\neq j$	$E_{i,n+j}+E_{j,n+i}$	$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$\sqrt{-1}\partial_i\partial_j$
$-\epsilon_i-\epsilon_j$ for $i\neq j$	$E_{n+i,j}+E_{n+j,i}$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$	$\sqrt{-1}x_ix_j$
$2\epsilon_i$	$E_{i,n+i}$	$\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$\sqrt{-1}\partial_i^2$
$-2\epsilon_i$	$E_{n+i,i}$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	$\sqrt{-1}x_i^2$

In this table, the positive and negative root vectors alternate $+,-,+,-,+,-$ after the Cartan subalgebra entry. There is a Lie algebra automorphism of $\mathfrak{sp}(2n,\mathbb{C})$ which exchanges E_β and $E_{-\beta}$ for all positive roots β, and which act as -1 on $\mathfrak{h}$. This means that there is another version of the Weil representation given by composing this automorphism with ρ. It is this version which extends the representation $E_{ij}\mapsto x_i\partial_j$ of $\mathfrak{gl}(n,\mathbb{C})$.

Facts 2.7. *We have the following facts concerning the modules L_Λ.*

(1) *(*[KV] *and* [How]*) For $k\in\mathbb{Z}_{>0}$, we put $\mathcal{L}_k=\{\,\lambda\in\mathbb{Y}\mid l(\lambda)\le n$ and $\lambda'_1+\lambda'_2\le k\,\}$, where λ'_j denotes the length of the jth column of the Young diagram of λ. Then we have*

$$W^{\otimes k}\simeq\bigoplus_{\lambda\in\mathcal{L}_k}L_k(\lambda)\otimes V_\lambda$$

where $L_k(\lambda)\simeq L_{\Lambda(k,\lambda)}$ and V_λ is a finite-dimensional irreducible $O(k,\mathbb{C})$-module.

(2) *$C_1\subset\{\,\Lambda(k,\lambda)\mid\lambda\in\mathcal{L}_k,\ k\in\mathbb{Z}_{>0}\,\}\subset C_0$. (This is immediate. Set $k=2n$ or $k=2n+1$.)*

(3) *If $\Lambda \in \widetilde{Q}$, L_Λ is in the holomorphic discrete series if and only if $\Lambda \in C_1$. Moreover, in this case L_Λ is isomorphic to N_Λ.*

These conditions make it clear that, for k sufficiently large ($\geq 2n$), all the irreducible $\mathfrak{sp}(2n, \mathbb{C})$-modules appearing in the decomposition of $W^{\otimes k}$ are holomorphic discrete series modules. On the other hand, the two irreducible constituents of W (see after Definition 2.2) are not in the holomorphic discrete series.

Lemma 2.8. *The formal character of the Weil representation W is given as the sum of all monomials in t_1^{-1}, t_2^{-1}, ..., t_n^{-1} with an additional factor*

$$\operatorname{ch} W = \sum_{c \in \mathbb{Z}^n_{\geq 0}} t^{-c-(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})} = \frac{(t_1 t_2 \cdots t_n)^{-\frac{1}{2}}}{\prod_{i=1}^n (1 - t_i^{-1})}.$$

Namely, if we define the $\frac{1}{2}$-*adjusted weight* of a monomial x^c or $c \in \mathbb{Z}^n_{\geq 0}$ to be $c - \frac{1}{2}\omega_n$, then the weight generating function of $\mathbb{Z}^n_{\geq 0}$ for the $\frac{1}{2}$-adjusted weight equals the formal character of W.

Lemma 2.9.

$$\operatorname{ch} N_{\Lambda(k,\lambda)} = \frac{\operatorname{ch} F_{\Lambda(k,\lambda)}}{\prod_{1 \leq i \leq j \leq n}(1 - t_i^{-1} t_j^{-1})} = \frac{(t_1 t_2 \cdots t_n)^{-\frac{k}{2}} s_\lambda(t_1^{-1}, \ldots, t_n^{-1})}{\prod_{1 \leq i \leq j \leq n}(1 - t_i^{-1} t_j^{-1})}$$

where s_λ is the Schur function. Therefore, we can compute the character of $L_k(\lambda) = L_{\Lambda(k,\lambda)} = N_{\Lambda(k,\lambda)}$ if $\Lambda(k, \lambda) \in C_1$.

The following definitions of semi-infinite symplectic tableaux and weight are from [TY].

Definition 2.10 Fix a positive integer n, and let $\lambda \in \mathbb{Y}$ have length $\leq n$. Let Γ_n denote the totally ordered set $\{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$. The *semi-infinite complementary shape* to λ is the following convex subset of $\mathbb{Z} \times \mathbb{Z}$:

$$S_\lambda = \{(i, j) | 1 \leq i \leq n,\ -\infty < j_i < -\lambda_{n+1-i}\}$$

A *semi-infinite $\mathfrak{sp}(2n, \mathbb{C})$ tableau* of *complementary shape* λ is a map $T : S_\lambda \to \Gamma_n$, satisfying the following conditions:

(1) For each $i \in [n]$, there exists $N_i \in \mathbb{Z}$ such that $T(i, j) = i$ whenever $j < N_i$.
(2) Each row is weakly increasing: $T(i, j) \leq T(i, j + 1)$ whenever both are defined.
(3) Each column is strictly increasing: $T(i, j) < T(i + 1, j)$ whenever both are defined.

Note that the first two conditions imply that $T(i, j) \geq i$, which is part of the definition of (finite) symplectic tableaux used by, e.g., [KE], [Ber], and [KT] [Ber]).

We write $\operatorname{sch}(T) = \lambda$ to indicate that the semi-infinite tableau T has semi-infinite complementary shape λ, and denote the set of all SIST's of that shape by $\operatorname{SIST}(\lambda)$.

Definition 2.11 Define the *crude weight* of $T \in \operatorname{SIST}(\lambda)$ to be $\sum_{i=1}^n c_i \epsilon_i$ ($\in \mathfrak{h}^*$), which we abbreviate as $(c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n$ where

$$c_i := -(\text{number of } \bar{i}\text{'s in } T) - (\text{ number of columns of } T \text{ not containing } i) - \lambda_n.$$

The idea is that, as for the usual symplectic tableaux, the letter i (resp. $\bar{i}$) has weight ϵ_i (resp. $-\epsilon_i$), and that the reference tableau (whose crude weight is defined to be zero) is the one of complementary shape $\lambda = \emptyset$ with all entries in the ith row equal to i. For example, the crude weight of the tableau in Figure 2 is $(-13, -15, -13, -14)$. Also, for each $k \in \mathbb{Z}_{>0}$ we define the $\frac{k}{2}$-*adjusted weight* of T to be (the crude weight of T) $- \frac{k}{2}\epsilon_1 - \frac{k}{2}\epsilon_2 - \cdots - \frac{k}{2}\epsilon_n$. For example, the $\frac{7}{2}$-adjusted weight of the tableau in Figure 2 is $(-\frac{33}{2}, -\frac{37}{2}, -\frac{33}{2}, -\frac{35}{2})$.

FIGURE 2. A Semi-infinite symplectic tableau of complementary shape (9,9,4,1)

$\cdots$	1	1	1	1	1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3
$\cdots$	2	2	2	2	$\bar{2}$	$\bar{2}$	$\bar{2}$	3	3	3	3	3	$\bar{3}$			
$\cdots$	3	3	3	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	4								
$\cdots$	4	4	4	4	4	$\bar{4}$	$\bar{4}$	$\bar{4}$								

Lemma 2.12 ([TY]). *If $\lambda \in \mathbb{Y}$ with $l(\lambda) \le n$ and $k \in \mathbb{Z}_{>0}$, then the weight generating function of* $\mathrm{SIST}(\lambda)$ *for the $\frac{k}{2}$-adjusted weight equals the formal character of $N_{\Lambda(k,\lambda)}$. In particular, if $\Lambda(k,\lambda) \in C_1$, this also equals the formal character of $L_{\Lambda(k,\lambda)}$.*

3. INSERTION

Using the duality (2.7 (1)) and some knowledge of $O(k,\mathbb{C})$-modules, it is not difficult to see the decomposition (2) holds. The goal of this section is to give an insertion scheme which shows that the formal characters of both sides of (2) are equal.

Definition 3.1 We define an insertion operator

$$I_i : \mathrm{SIST}(\lambda) \longrightarrow \coprod_{\mu} \mathrm{SIST}(\mu)$$

where μ will turn out to be a shape such that μ/λ is a single box as follows. Remove the rightmost i in the ith row, making it a hole. Start sliding by jeu de taquin moving letters in the northwest direction, until a hole is left on the boundary of the SIST. (Or equivalently one could regard the slide starting from infinite left in the ith row.) The resulting tableau is a SIST because the sliding process preserves semistandardness. μ/λ is one box since the tableau loses one box of its shape at its outer corner.

In what follows, the *weight* of an ordered pair is the sum of the weights of its components.

Lemma 3.2. *Fix $\lambda \in \mathbb{Y}$ with $l(\lambda) \le n$. Then the map*

$$\mathrm{SIST}(\lambda) \times [1,n] \to \coprod_{\mu} \mathrm{SIST}(\mu)$$

defined by $(T,i) \mapsto I_i(T)$ is a weight-preserving bijection, where μ runs over partitions $\mu \in \mathbb{Y}$ with $l(\mu) \le n$ and such that μ/λ is one box, and the SIST's are assigned their crude weights, and the weight of $i \in [1,n]$ is defined to be $-\epsilon_i$.

Proof. This map is well-defined because given any hole in a column-strict tableaux there is a unique jeu de taquin slide starting at that hole to the boundary of the tableaux. The candidate for the inverse map is the reverse slide, which moves letters in the southeast direction, starting from the hole at μ/λ. (So in order to reverse this map, one must know λ as well as μ.) Any such reverse slide must eventually converge to some single row i, and the output of this inverse candidate will be the pair consisting of the resulting tableau and the letter i. That this gives the inverse map follows in the same way that it does for the case of finite jeux de taquin slides. The forward map forces one i to be lost in the tableaux, so by the definition of crude weight the map is weight-preserving. □

Example 3.3 Consider $I_1(T)$ where $T =$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \; .$$

We show the sliding step by step:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & \bar{2} & & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & \bar{2} & \bar{3} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \\ \cline{1-5} \end{array} \; .$$

$I_1(T)$ is the final tableau in the sequence above, and μ/λ is the leftmost empty box in row 3.

Definition 3.4 Let I_i^m denote the operator which repeats I_i m times, and for $c \in \mathbb{Z}^n_{\geq 0}$, set $I^c(T) = I_n^{c_n} \circ I_{n-1}^{c_{n-1}} \circ \cdots I_1^{c_1}(T)$. In other words, $I^c(T)$ is the map which successively slides c_1 1's to the left in T, c_2 2's to the left in T, and so on.

The main theorem of this section follows. It gives an analogue of Schensted's correspondence which shows the effect of tensoring a *stable* module $L_k(\lambda)$ (e.g., when $k \geq 2n$) with one copy of W.

Theorem 3.5. *Let $\lambda \in \mathbb{Y}$ with $l(\lambda) \leq n$. Then the map*

$$\mathrm{SIST}(\lambda) \times \mathbb{Z}^n_{\geq 0} \xrightarrow{\sim} \coprod_{\mu} \mathrm{SIST}(\mu)$$

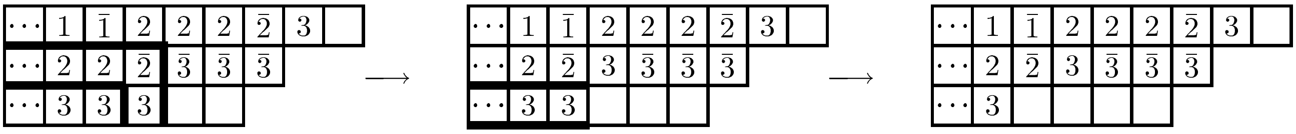
defined by $(T, c) \mapsto I^c(T)$ is a weight-preserving bijection, where μ runs over partitions $\mu \in \mathbb{Y}$ with $l(\mu) \leq n$ and such that μ/λ is a horizontal strip, and the SIST's and the n-tuples are assigned their crude weights.

Example 3.6 Consider $I^{(3,1,1)}(T)$ where T is the following tableau of weight $(-5, -5, -7)$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-7} \cdots & 3 & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-6} \end{array} \; .$$

We show the effect of successively inserting $1, 1, 1, 2, 3$, and indicate the sliding path by highlighting the sides of the squares involved. The sliding between the second and third tableaux is the same as that shown in Example 3.3 above.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-7} \cdots & 3 & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-6} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 & \\ \hline \cdots & 2 & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-7} \cdots & 3 & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-6} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 & \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} & \bar{3} \\ \cline{1-7} \cdots & 3 & 3 & 3 & \bar{3} & \\ \cline{1-6} \end{array} \longrightarrow$$



Note that the final tableau has weight $(-8, -6, -8)$, as it should.

Before beginning the proof of Theorem 3.5, we note the following

Lemma 3.7. *Let P be a skew tableau, and let C_1, C_2, ..., C_l be a sequence of squares such that C_i is an inner (resp. outer) cocorner of the skew shape* $\mathrm{shape}(P) \cup C_1 \cup \cdots \cup C_{i-1}$*. Recursively define P_i by $P_0 = P$ and P_i to be the result of the sliding which moves letters northwest (resp. southeast) performed on P_{i-1} starting from the hole at C_i. Let D_i be the square lost at the end of the sliding from P_{i-1} to P_i. If the C_i form a horizontal strip from right to left (resp. left to right), then the D_i also form a horizontal strip from right to left (resp. left to right).*

Proof. By a result of M. Haiman ([Hai], Lemma 2.7), we can exchange the role of the skew tableau which the sliding process actually moves and the skew tableau which designates the order of the starting positions of the sliding. Therefore it suffices to show that, if a sequence of slides is performed on a horizontal strip tableau numbered from left to right, then the result is also a horizontal strip tableau numbered from left to right. This is a well known fact. □

of Theorem 3.5. First we show that the result of $I^c(T)$ has shape complementary to μ with μ/λ being a horizontal strip. We can cut off the infinite stable part of T (leaving some of its stable part) to obtain a finite skew tableau T^0 whose inner shape is such that we can put a horizontal strip, adjacent to the inner shape of T^0, with c_i boxes in row i. Performing slides (moving letters northwest) along this horizontal strip, right to left, has the same effect as I^c, and by Lemma 3.7 we know that μ/λ is a horitontal strip (actually created from right to left).

Next we define a candidate for the inverse map. Let U be a SIST which belongs to the left-hand side. We can cut off the stable part, similarly to the above, in such a way that the lengths of adjacent rows of the inner shape differ by at least $|\mu/\lambda|$. We perform slides (moving letters southeast) along the horizontal strip μ/λ from right to left. By Lemma 3.7, the inner boundary of U moves southeast by a horizontal strip (vacation being created from right to left). Denote by c_i the number of boxes in this horizontal strip lying in row i. Then we see that if we iterate the inverse map of Lemma 3.2 to U in such a way that the outer shape loses μ/λ from right to left, then the letters obtained are c_n n's, c_{n-1} $(n-1)$'s, ..., c_1 1's, in this order. The output of our inverse candidate is defined to be the pair formed by the SIST obtained at the end of this iteration of the inverse to Lemma 3.2, and the n-tuple $(c_1, c_2, \ldots, c_n)$.

Now from the fact that map described as the inverse candidate in the proof of Lemma 3.2 is really the inverse, our map (in Theorem 3.5) and our inverse candidate (in this proof) are also inverses to each other. It preserves weights because it is an iteration of the weight-preserving map in Lemma 3.2 (and because of the definition of crude weights of n-tuples).

□

Corollary 3.8. *Let $k \in \mathbb{Z}_{\geq 2}$, and suppose that $\lambda \in \mathcal{L}_k$ and $\Lambda(k, \lambda) \in C_1$. Then all μ appearing on the right-hand side of Theorem 3.4 belong to $\mathcal{L}_{k+1}$, and for those μ we have $\Lambda(k+1, \mu) \in C_1$. Moreover, the map in Theorem 3.4 is also weight-preserving if we assign*

$\frac{k}{2}$*-adjusted weights to* $\mathrm{SIST}(\lambda)$*,* $\frac{1}{2}$*-adjusted weights to* $\mathbb{Z}^n_{\geq 0}$*, and* $\frac{k+1}{2}$*-adjusted weights to* $\mathrm{SIST}(\mu)$*. Thus interpreted, this bijection gives the formal character identity for* (2)*.*

Proof. If $\Lambda(k,\lambda) \in C_1$, then $\Lambda(k+1,\mu) \in C_1$ for any $\mu \supset \lambda$. The condition $\mu \in \mathcal{L}_{k+1}$ is more subtle. $l(\mu) \leq n$ is assured from the beginning (Theorem 3.5), so we only have to check if $\mu'_1+\mu'_2 \leq k+1$ holds. If $k+1 \geq 2n$, this is satisfied automatically. So suppose $k+1 \leq 2n-1$, or equivalently $\frac{k}{2} \leq n-1$. Now the condition $\Lambda(k,\lambda) \in C_1$ implies $\lambda_n + \frac{k}{2} > n$, so that we have $\lambda_n \geq 2$. This implies $\lambda'_1 + \lambda'_2 = 2n$, so that if $\lambda \in \mathcal{L}_k$, we must have $2n \leq k$, which is a contradiction. Therefore the case $k+1 \leq 2n-1$ never occurs, and we have $\mu \in \mathcal{L}_{k+1}$.

Since the map of Theorem 3.4 is weight-preserving for crude weights, it is also weight-preserving for the adjusted weights because the adjustments match. Using Lemma 2.7 and 2.9, and comparing with (2), we see that the weight generating function identity (for the adjusted weights) resulting from this bijection coincides with the formal character identity for (2). □

Definition 3.9 Following R. Proctor's notion of "N-orthogonal tableaux" [Pro], we propose a skew semistandard tableau of shape μ/λ with entries from $[k+1,l] = \{k+1, k+2, \ldots, l\}$ to be called a *skew* $O(l)/O(k)$*-tableau* if it satisfies what he calls the q-orthogonal condition for $q = k,\ k+1,\ \ldots,\ l$ in the following sense. For such q, let $\lambda^{(q)}$ denote the shape containing λ and all letter $\leq q$. Thus $\lambda = \lambda^{(k)} \subset \lambda^{(k+1)} \subset \cdots \subset \lambda^{(l)} = \mu$, and the differences of adjacent terms are horizontal strips. The q-orthogonal condition requires that the sum of the first two columns of $\lambda^{(q)}$ is at most q. Let $O(l)/O(k)(\mu/\lambda)$ denote the set of all $O(l)/O(k)$-tableaux of shape μ/λ.

Corollary 3.10. *Let* $k \in \mathbb{Z}_{\geq 2}$*, and let* $\lambda \in \mathcal{L}_k$*. Also assume that* $\Lambda(k,\lambda) \in C_1$*. Iterating the map in Theorem 3.4, we obtain a weight-preserving bijection*

$$\mathrm{SIST}(\lambda) \times \left(\mathbb{Z}^n_{\geq 0}\right)^{k'} \xrightarrow{\sim} \coprod_{\mu} \mathrm{SIST}(\mu) \times O(k+k')/O(k)(\mu/\lambda)$$

where μ *ranges over all partitions with* $l(\mu) \leq n$*. The SIST-component of the output is the result of inserting (by Lemma 3.2)* k' n*-tuples, and the skew-tableau component of the output is the one that records the complementary shapes of SIST's after each insertion of an* n*-tuple, namely the shape* $\lambda^{(k+i)}$ *is the shape complementary to the SIST after inserting the* i*th* n*-tuple.*

Example 3.11 Suppose $n = k = 3$ and $\lambda = (3,1)$. Successively inserting the monomials $t_1^2 t_2$, $t_1 t_2^2 t_3$, and t_2 into the following tableau of weight $(-3,-5,-2)$ proceeds as shown:

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & 3 \\ \hline \cdots & 2 & \bar{2} & \bar{2} & * \\ \hline \cdots & 3 & * & * & * \\ \hline \end{array} \quad \xrightarrow{(2,1,0)} \quad \begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & \bar{2} & \bar{2} & 3 \\ \hline \cdots & 2 & 3 & \cdot & \cdot & * \\ \hline \cdots & 3 & \cdot & * & * & * \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & \bar{2} & \bar{2} & 3 \\ \hline \cdots & 2 & 3 & * & * & * \\ \hline \cdots & 3 & * & * & * & * \\ \hline \end{array} \quad \xrightarrow{(1,2,1)} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & \bar{1} & \bar{2} & \bar{2} & 3 & \cdot \\ \hline \cdots & 2 & 2 & 3 & 3 & 3 & * & * & * \\ \hline \cdots & 3 & \cdot & \cdot & \cdot & * & * & * & * \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & \bar{1} & \bar{2} & \bar{2} & 3 & * \\ \hline \cdots & 2 & 2 & 3 & 3 & 3 & * & * & * \\ \hline \cdots & 3 & * & * & * & * & * & * & * \\ \hline \end{array} \quad \xrightarrow{(0,1,0)} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 1 & \bar{1} & \bar{2} & \bar{2} & 3 & * \\ \hline \cdots & 2 & 3 & 3 & 3 & \cdot & * & * & * \\ \hline \cdots & 3 & * & * & * & * & * & * & * \\ \hline \end{array}$$

The $*$'s keep track of the complementary shape at the start of each insertion, while the $\cdot$'s indicate the horizontal strip that is added by each inserted monomial. The final complementary shape here is $\mu = (7, 3, 1)$, and the final skew-orthogonal recording tableau of shape λ/μ is

*	*	*	4	5	5	5
*	4	4	6			
5						

4. The complete correspondence when $n = 2$

The previous section's insertion map gives the decomposition of tensoring with W only for those modules $L_{\Lambda(k,\lambda)}$ whose characters are the same as for the corresponding generalized Verma module $N_{\Lambda(k,\lambda)}$. This is always true when k is sufficiently large relative to n, but not for small values of k.

However, using the duality of these modules with the $O(k,\mathbb{C})$ modules, it is possible to describe $\operatorname{ch} L_{\Lambda(k,\lambda)}$ as an alternating sum of terms of the form $\operatorname{ch} N_{\Lambda(k,\mu)}$. For example, if we define $\hat{L}(k,\lambda) := \operatorname{ch} \hat{L}_{\Lambda(k,\lambda)}$ and $\hat{N}(k,\lambda) := \operatorname{ch} N_{\Lambda(k,\lambda)}$, then we have the following expressions.

For $n = 1$ all the modules are *stable*, i.e., $\hat{L}(k,\lambda) = \hat{N}(k,\lambda)$ for all k.

For $n = 2$ we have

$$\hat{L}(1,\emptyset) = \hat{N}(1,\emptyset) - \hat{N}(1,(2,2)) \tag{1}$$

$$\hat{L}(1,\square) = \hat{N}(1,\square) - \hat{N}(1,(2,1)) \tag{2}$$

$$\hat{L}(2,(m)) = \hat{N}(2,(m)) - \hat{N}(2,(m,2)) \tag{3}$$

In this section we use the above relations to define a subset of SIST tableaux, $\mathrm{SIST}(k,\lambda)$ which gives the weight generating function for $\hat{L}(k,\lambda)$. Then we give the following weight-preserving bijections f_k for $k = 0, 1, 2$:

$$\mathbb{Z}^2_{\geq 0} \xrightarrow{f_0} \mathrm{SIST}(1,\emptyset) \coprod \mathrm{SIST}(1,\square) \tag{4}$$

$$\mathrm{SIST}(1,\emptyset) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_1} \coprod_{r \geq 0} \mathrm{SIST}(2,(r)) \tag{5}$$

$$\mathrm{SIST}(1,\square) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_1} \mathrm{SIST}(2,(1,1)) \coprod_{r \geq 1} \mathrm{SIST}(2,(r)) \tag{6}$$

$$\mathrm{SIST}(2,(s)) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_2} \coprod_{r} \Big[\mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1))\Big] \quad (s \geq 2) \tag{7}$$

In all cases, $\operatorname{sch} f_i(T) - \operatorname{sch}(T)$ is a horizontal strip. By the last step we have reached the $k = 3$ stage, which is stable, so our usual definition of SIST gives the correct weight generating function, and the insertion of Section 3 works thereafter. For $k \geq 2$, the shapes $(1,1)$, $\emptyset$, and $\square$ are stable, so there is no need for a special f_2 bijection to handle these cases.

4.1. **Definition of** $\mathrm{SIST}(k,\lambda)$.

Definition 4.1 We call $T \in \mathrm{SIST}(\emptyset)$ $\genfrac{}{}{0pt}{}{22}{22}$-*deletable* if it contains the configuration $\genfrac{}{}{0pt}{}{22}{22}$, in which case removing the rightmost such configuration and shoving left will leave a valid SIST.

We call $T \in \mathrm{SIST}(\square)$ $\frac{2}{\bar{2}}$*-deletable* if it contains a 2 in row 1, a $\bar{2}$ in row 2, and if it remains a SIST when the rightmost 2 in $\mathrm{Row}_1\, T$ and the rightmost $\bar{2}$ in $\mathrm{Row}_2\, T$ are removed and any elements to the right of the deleted 2 are shoved one box left along the top row.

We call $T \in \mathrm{SIST}((s))$ $2\bar{2}$*-deletable* if $\mathrm{Row}_1\, T$ contains adjacent entries $2\bar{2}$, and if it remains a SIST when this pair is removed and any elements to the right of the deleted $\bar{2}$ are shoved two boxes left along the top row.

In our representation of tableaux, we will use $\cdot$'s to indicate cells that have been vacated due to sliding, as opposed to cells that are part of the complementary shape, which we indicate by $*$. We use $\times$ to indicate a cell that might be either a $\cdot$ or a $*$. To highlight that a certain configuration must occur as the rightmost columns of a tableau, we make the right boarder bold. In the bijections that follow, it will be necessary to keep track of the original complementary shape of the tableau as sliding proceeds.

Example 4.2 In the examples that follow, the tableaux on the left are deletable, and the ones on the right are not.

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & \bar{2} & \bar{2} \\ \hline \cdots & 2 & 2 & \cdot & \cdot & \cdot \\ \hline \end{array} \hookrightarrow \begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & \bar{2} & \cdot & \cdot \\ \hline \cdots & 2 & 2 & \cdot & \cdot & \cdot \\ \hline \end{array} \quad \text{BUT NOT} \quad \begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 2 & \bar{2} & \bar{2} \\ \hline \cdots & 2 & \bar{2} & \cdot & \cdot \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 \\ \hline \cdots & 2 & \bar{2} & * \\ \hline \end{array} \hookrightarrow \begin{array}{|c|c|c|c|}\hline \cdots & 1 & \bar{1} & \cdot \\ \hline \cdots & 2 & \cdot & * \\ \hline \end{array} \quad \text{BUT NOT} \quad \begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & 2 \\ \hline \cdots & 2 & 2 & 2 & * \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 1 & 2 & \bar{2} \\ \hline \cdots & 2 & \bar{2} & \bar{2} & * \\ \hline \end{array} \hookrightarrow \begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{2} & \cdot \\ \hline \cdots & 2 & \bar{2} & \cdot & * \\ \hline \end{array} \quad \text{BUT NOT} \quad \begin{array}{|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & \bar{1} \\ \hline \cdots & 2 & \bar{2} & \bar{2} & * \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 2 & 2 & 2 \\ \hline \cdots & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\ \hline \end{array} \hookrightarrow \begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & 2 & \cdot & \cdot \\ \hline \cdots & 2 & \bar{2} & \bar{2} & \cdot & \cdot \\ \hline \end{array} \quad \text{BUT NOT} \quad \begin{array}{|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & \bar{1} & 2 \\ \hline \cdots & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\ \hline \end{array}$$

A simple argument shows that $T \in \mathrm{SIST}(\square)$ is $\frac{2}{\bar{2}}$-deletable if and only if $2 \in \mathrm{Row}_1\, T$ and $\bar{2} \in \mathrm{Row}_2\, T$. Similarly, $T \in \mathrm{SIST}(\emptyset)$ is $\frac{22}{\bar{2}\bar{2}}$-deletable if and only if $\mathrm{Row}_1\, T$ contains at least two 2's. Finally, $T \in \mathrm{SIST}((s))$, where $s \geq 2$ is $2\bar{2}$-deletable if and only if $\mathrm{Row}_1\, T$ contains a $2\bar{2}$ pair which lies above empty cells in $\mathrm{Row}_2\, T$; in other words, the pair of elements lie in cells $T(1, -i)$ and $T(1, -i+1)$, where $2 \leq i \leq s$.

Definition 4.3 For $n = 2$, we define $\mathrm{SIST}(k, \lambda)$ as follows:

$$\begin{aligned} \mathrm{SIST}(1, \emptyset) &:= \{T \in \mathrm{SIST}(\emptyset) \mid T \text{ is not } \tfrac{22}{\bar{2}\bar{2}}\text{-deletable}\} \\ \mathrm{SIST}(1, \square) &:= \{T \in \mathrm{SIST}(\square) \mid T \text{ is not } \tfrac{2}{\bar{2}}\text{-deletable}\} \\ \mathrm{SIST}(2, (s)) &:= \{T \in \mathrm{SIST}((s)) \mid T \text{ is not } 2\bar{2}\text{-deletable}\}, \text{ for } s \geq 2. \end{aligned}$$

For all other pairs (k, λ), $\mathrm{SIST}(k, \lambda) := \mathrm{SIST}(\lambda)$.

Before giving the algorithms that define the maps f_i and their inverses, we need to define some operators that replace a given SIST with a different one of the same weight. Each step of f_i will consist of a normal sliding step, possibly augmented by a transformation. In order to keep the transformation straight, we will use the following notation.

- $S \mapsto T$ OR $S \overset{t_i}{\mapsto} T$ will indicate a slide that is part of a single step of the algorithm.

- $T \hookrightarrow U$ indicates a transformation other than a slide that is performed immediately after a slide as part of the same step of the algorithm.
- $S \longrightarrow U$ indicates a single step of the algorithm, including any nonslide transformation that may occur. If the entire step consists soley of an t_i slide, we write: $S \xrightarrow{t_i} U$.
- For the reverse algorithms, we use the same general conventions, but write the extracted monomial in square brackets above the arrow, e.g., $S \xrightarrow{[t_i]} U$, or $S \overset{[t_i]}{\mapsto} T$.

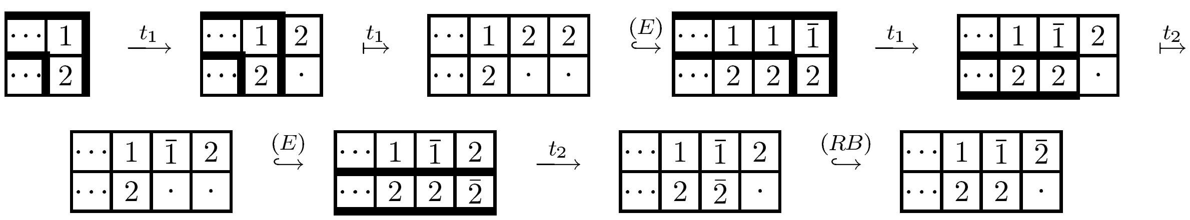
The first operation will replace $\frac{2}{2}$-deletable tableaux with elements of $\mathrm{SIST}(1, \square)$

Definition 4.4 Let $T \in \mathrm{SIST}(\square)$ with $T(1,-1) = 2$ and $T(2,-2) = \bar{2}$. Define $\mathrm{RaiseBar}\, T$ to be the tableau obtained from T by changing the $(1,-1)$-entry to $\bar{2}$, and changing the leftmost $\bar{2}$ in $\mathrm{Row}_2\, T$ to a 2. (In general, this may not be a SIST, because column-strictness may fail for the entry that was lowered, but as we use it below, we always get another SIST.) Inversely, if $T \in \mathrm{SIST}(\square)$ satisfies $T(1,-1) = \bar{2}$, then $\mathrm{LowerBar}\, T$ is defined to be the tableau obtained from T by changing the $(1,-1)$-entry to 2, and changing the rightmost 2 in $\mathrm{Row}_2\, T$ to a $\bar{2}$. (This operation always gives a well-defined SIST of the same shape.)

Definition 4.5 Let T be a tableau with $\mathrm{sch}\, T = (s)$ for some $s \geq 2$, and $i = 1$ or 2. To *explode* an $i\bar{i}$ pair in row i, means to place the horizontally adjacent entries $i\bar{i}$ in row i, immediately to the right of the rightmost i in row i. If $i = 1$, the added entries displace two entries to the cells immediately below. If $i = 2$, they displace the entries to the right of the explosion two place to the right. (While this procedure might not yield a valid SIST in general, it always will in the contexts where we apply it.)

Definition 4.6 Now define the map f_0 as follows. Slide according to the monomial until a shape other than $\emptyset$ or $\square$ is reached. (Call this a *shape violation* for f_0.) This will necessarily imply $\mathrm{sch}\, T = (2)$ because sliding always yields a horizontal strip. If the violation occurs while sliding t_i, then explode (E) an $i\bar{i}$ pair in row i. Continue until the entire monomial has been used. Finally, apply the RaiseBar (RB) operator if the result is $\frac{2}{2}$-deletable of $\mathrm{sch}(\square)$.

Example 4.7 Start with the monimial $t_1^3 t_2^2$. We get the following sequence of insertions:



Note that the output of this algorithm will always have complementary shape $\emptyset$ or $\square$ because anytime we reach a tableaux T with $\mathrm{sch}\, T = (2)$ it gets replaced by (E) with a tableaux U with $\mathrm{sch}\, U = \emptyset$. Since (RB) only applies to tableaux of $\mathrm{sch} = \square$, it will never take place immediately after (E), only after a slide.

The following lemma is easily shown by direct calculation.

Lemma 4.8. *The tableau T created from the monomial $t_1^j t_2^l$ by f_0 can be described as follows.*

a) *If j and l are even, then* $\mathrm{Row}_1\, T$ *consists of $j/2$ $\bar{1}$'s, and* $\mathrm{Row}_2\, T$ *consists of $l/2$ $\bar{2}$'s, and* $\mathrm{sch}\, T = \emptyset$.

b) *If j is even and l is odd, then* $\mathrm{Row}_1\, T$ *consists of $j/2$ $\bar{1}$'s and* $\mathrm{Row}_2\, T$ *consists of $(l-1)/2$ $\bar{2}$'s and* $\mathrm{sch}\, T = \square$.

c) *If j is odd and l is even and positive, then* $\mathrm{Row}_1 T$ *consists of* $(j-1)/2$ $\bar{1}$*'s followed by a* $\bar{2}$*, and* $\mathrm{Row}_2 T$ *consists of* $\frac{l}{2}-1$ $\bar{2}$*'s, and* $\mathrm{sch}\, T = \square$*. (This is the only case that* RaiseBar *is applied.) The special case j is odd and $l = 0$ gives the same result, except that $T(1,-1) = 2$ instead of a $\bar{2}$.*

d) *If j and l are odd, then* $\mathrm{Row}_1 T$ *consists of* $(j-1)/2$ $\bar{1}$*'s, followed by a* 2*, and* $\mathrm{Row}_2 T$ *consists of* $(l+1)/2$ $\bar{2}$*'s, and* $\mathrm{sch}\, T = \emptyset$*.*

Theorem 4.9. *The map f_0 is a weight-preserving bijection:*

$$\mathbb{Z}^2_{\geq 0} \xrightarrow{f_0} \mathrm{SIST}(1,\emptyset) \coprod \mathrm{SIST}(1,\square)$$

Proof. Define a map $g_0 : \mathrm{SIST}(1,\emptyset) \coprod \mathrm{SIST}(1,\square) \longrightarrow \mathbb{Z}^2_{\geq 0}$ by setting $g_0(T) = \mathrm{wt}(T)$, the weight of the tableaux T. We claim that f_0 and g_0 are inverses. If we start with a monomial m in $\mathbb{Z}^2_{\geq 0}$ and compute $T = f_0(m)$, then $\mathrm{wt}(T) = m$; for this would be the result of normal sliding, and it is easy to check that performing Explode or RaiseBar does not change the weight of a tableau. Hence, $g_0(f_0(m)) = m$. Conversely, suppose we start with an arbitrary $T \in \mathrm{SIST}(1,\emptyset) \coprod \mathrm{SIST}(1,\square)$. Let $w = \mathrm{wt}(T)$. If $\mathrm{sch}\, T = \emptyset$, then it is easy to see that T is the unique tableau of weight w which is not $\substack{22\\22}$-deletable. The only possible columns in T are $\substack{1\\2}$, $\substack{\bar{1}\\2}$, $\substack{1\\\bar{2}}$, $\substack{\bar{1}\\\bar{2}}$, and $\substack{2\\2}$, which have respective weights $(0,0)$, $(-2,0)$, $(0,-2)$, $(-2,-2)$, and $(-1,-1)$. Since T can contain either columns of $\substack{\bar{1}\\2}$ or columns of $\substack{1\\\bar{2}}$ but not both, each possible weight for a tableau of sch $= \emptyset$ can be made uniquely as a integral linear combination of these weights which uses the column $\substack{2\\2}$ at most once. Now Lemma 4.8 shows that $f_0(w)$ is not $\substack{22\\22}$-deletable, so we must have $T = f_0(w) = f_0(g_0(T))$.

If $\mathrm{sch}\, T = \square$, then the following similar but slightly more complicated argument shows that there is a unique way to make a tableau of weight $w(T)$, which is not $\substack{2\\2}$-deletable. Here we can no longer use $\substack{2\\2}$ columns, but we must use exactly one of the following columns to insure that the total degree is odd: $\substack{1\\\cdot}$, $\substack{\bar{1}\\\cdot}$, $\substack{2\\\cdot}$, or $\substack{\bar{2}\\\cdot}$, which have respective weights $(0,-1)$, $(-2,-1)$, $(-1,0)$, and $(-1,-2)$.

In the first case, the only possible columns are $\substack{1\\2}$, $\substack{1\\\bar{2}}$, and $\substack{1\\\cdot}$, giving a unique way to attain any weight of the form $(0,-l)$, for l odd and positive.

In the second, the possible nonzero weight columns are $\substack{\bar{1}\\2}$, $\substack{1\\\bar{2}}$, $\substack{\bar{1}\\\bar{2}}$ and $\substack{\bar{1}\\\cdot}$. As in the $\mathrm{sch}\, T = \emptyset$ case, only one of the first two types of columns can appear; thus, omitting the final column $\substack{\bar{1}\\\cdot}$, we have a unique way to attain any even weight. So this case gives a unique way to construct a tableau whose weight is of the form $(-j,-l)$ for j even and positive, l odd and positive.

In the third case, since our tableau cannot have a $\bar{2}$ in row 2, we may only have columns $\substack{1\\2}$, $\substack{\bar{1}\\2}$, and $\substack{2\\\cdot}$, giving a unique way to attain any weight of the form $(-j,0)$, for j odd and positive. In the fourth case, we find that since T can contain either columns of $\substack{\bar{1}\\2}$ or columns of $\substack{1\\\bar{2}}$ but not both, and no columns $\substack{2\\2}$, that the single $\substack{\bar{2}\\\cdot}$ column contributes $(-1,-2)$ to the weight, and the other columns contribute a unique way of writing any weight with both coordinates even, yielding a unique way to attain any weight of the form $(-j,-l)$, for j odd, l even and positive. $\square$

Definition 4.10 Define the map f_1 by iterating the following procedure. We start with a tableau $T \in \mathrm{SIST}(1,\emptyset)$ or $\mathrm{SIST}(1,\square)$ and a monomial m. Perform one slide according to the monomial. If we obtain a tableau T of the following form, we make one of the replacements shown, the *first* one that applies. Recall our convention that $\cdot$'s indicate cells that have been vacated due to sliding, while $*$ indicate cells that are part of the complementary shape of the original tableaux.

- (**Explode**) $\operatorname{sch} T = \mu = (2,1)$; we call this a *shape violation* for f_1, since $\mu'_1 + \mu'_2 > k = 2$. To repair this shape violation, we Explode a 2 in $\mathrm{Row}_1\, T$ and a $\bar{2}$ in $\mathrm{Row}_2\, T$, to wit, we make the following replacement (which makes the resulting tableau $\frac{2}{\bar{2}}$-deletable):

$$\begin{array}{|c|c|}\hline \bar{2} & \cdot \\ \hline \cdot & * \\ \hline \end{array} \overset{(E1)}{\hookrightarrow} \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \bar{2} & * \\ \hline \end{array} \quad \text{OR} \quad \begin{array}{|c|c|}\hline a & \cdot \\ \hline \cdot & * \\ \hline \end{array} \overset{(E2)}{\hookrightarrow} \begin{array}{|c|c|}\hline a & 2 \\ \hline \bar{2} & * \\ \hline \end{array}, \quad \text{for } a \leq 2.$$

  Then continue sliding according to the monomial.
- (**BarTur**) T has a *symplectic violation*, i.e., T is $2\bar{2}$-deletable. In this case, T must contain one of the two configurations below, to which we apply the following BarTur operation:

$$\begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \cdot & * \\ \hline \end{array} \overset{(B1)}{\hookrightarrow} \begin{array}{|c|c|}\hline 2 & \cdot \\ \hline \bar{2} & * \\ \hline \end{array} \quad \text{OR} \quad \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \cdot & \cdot \\ \hline \end{array} \overset{(B2)}{\hookrightarrow} \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} & \bar{2} \\ \hline \end{array}.$$

  (These make the resulting tableau $\frac{2}{\bar{2}}$-deletable or $\frac{22}{\bar{2}\bar{2}}$-deletable.) Then continue sliding according to the monomial.

Continue this procedure until the entire monomial has been used, and there are no further shape or symplectic violations. At each step at most one BarTur or Explode may occur before the next slide happens.

Lemma 4.11. *The procedure above gives a well-defined map*

$$\mathrm{SIST}(1,\emptyset) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_1} \coprod_{r \geq 0} \mathrm{SIST}(2,(r))$$

Proof. Since the original tableau T has $\operatorname{sch} T = \emptyset$, the only transformation we could apply is $(B2)$. Suppose that the algorithm described gives a sequence of slides:

$$T = T_0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow \cdots \longrightarrow T_m = f_1(T)\,.$$

We will show by induction that each T_i has the following properties:

A) $\operatorname{sch} T_i$ is a horizontal strip (possibly empty), created from right to left in the bottom row of T_i.
B) T_i is a SIST.
C) T_i is not $2\bar{2}$-deletable.

T_0 clearly enjoys these properties. Suppose T_i has these properties. If no BarTur is involved in the move $T_i \longrightarrow T_{i+1}$, then by the results of Section 3, T_{i+1} also satisfies (A–C). So now assume that the move involves BarTuring so that we have $T_i \mapsto U \hookrightarrow T_{i+1}$, where U is the tableau after sliding, before BarTuring. We have

$$T_i \mapsto \begin{array}{|c|c|c|c|c|}\hline \cdots & a & 2 & \bar{2} & \cdots \\ \hline \cdots & b & \cdot & \cdot & \cdots \\ \hline \end{array} \hookrightarrow \begin{array}{|c|c|c|c|c|}\hline \cdots & a & 2 & 2 & \cdots \\ \hline \cdots & b & \bar{2} & \bar{2} & \cdots \\ \hline \end{array}$$

Say that the rightmost 2 in $\mathrm{Row}_1\, U$ occurs in column j. We claim that the cell $U(2, j-1)$ (marked as containing b) could not be empty. For if the slide $T_i \mapsto U$ caused this cell to vacate, T_i itself would have been $2\bar{2}$-deletable, containing the configuration $\begin{smallmatrix} 2 & \bar{2} \\ \cdot & \cdot \end{smallmatrix}$ in columns j and $j+1$. On the other hand, if this configuration was caused by the slide, then either the 2 or $\bar{2}$ must have ascended from the second row, where it would have been resting in the middle of the the horizontal shape of $\operatorname{sch} T_i$, contradicting (A) for T_i. We conclude that $b = 2$ or $\bar{2}$, and that $\operatorname{sch} U$ is a horizontal strip starting at column j. So $\operatorname{sch} T_{i+1}$ is a horizontal strip starting at column $j+2$, all of whose cells lie in the bottom row. Now (A) follows by induction since we have just taken the leftmost cell away from $\operatorname{sch} T_i$. T_{i+1} is a SIST by inspection, and is not $2\bar{2}$-deletable since $T_{i+1}(1,k) = U(1,k) = \bar{2}$ for $k > j+1$. $\square$

Lemma 4.12. *The procedure above gives a well-defined map*

$$\mathrm{SIST}(1,\square) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_1} \coprod_{r \geq 1} \mathrm{SIST}(2,(r)) \coprod \mathrm{SIST}(2,(1,1))$$

Proof. Suppose we have a sequence of insertions as above, and assume that T_i satisfies the following conditions:

A) $\operatorname{sch} T_i = (1,1)$ or (r).
B) T_i is a SIST.
C) T_i is not $2\bar{2}$-deletable.

We wish to show that the same properties hold for T_{i+1}.

If $T_i \longrightarrow T_{i+1}$ involves no Explodes or BarTurs, then (A–B) are automatic, and we cannot get a $2\bar{2}$-deletable tableaux, which would force us to BarTur.

Now suppose we need to BarTur: $T_i \longrightarrow U \hookrightarrow T_{i+1}$, where the rightmost 2 in $\mathrm{Row}_1 T$ occurs in column $-j$. As in the lemma above we find that $U(2,-j-1)$ is nonempty. We get two cases, depending on whether $j = 2$ or $j > 2$:

$$\begin{array}{|c|c|c|c|}\hline \cdots & a & 2 & \bar{2} \\ \hline \cdots & b & \cdot & * \\ \hline \end{array} \overset{(B1)}{\hookrightarrow} \begin{array}{|c|c|c|c|}\hline \cdots & a & 2 & \cdot \\ \hline \cdots & b & \bar{2} & * \\ \hline \end{array} \quad (j=2) \qquad \text{OR} \qquad \begin{array}{|c|c|c|c|c|}\hline \cdots & a & 2 & \bar{2} & \cdots \\ \hline \cdots & b & \cdot & \cdot & \cdots \\ \hline \end{array} \overset{(B2)}{\hookrightarrow} \begin{array}{|c|c|c|c|c|}\hline \cdots & a & 2 & 2 & \cdots \\ \hline \cdots & b & \bar{2} & \bar{2} & \cdots \\ \hline \end{array}$$

Since we reached U by sliding without Explode, we know that $\operatorname{sch} T_i = (r)$, some $r \geq 2$ (not $(1,1)$), so after BarTuring, we get $\operatorname{sch} T_{i+1} = (r-1)$ or if $j = r = 2$, $\operatorname{sch} T_{i+1} = (1,1)$ In either case no $\begin{smallmatrix} 2 \\ \cdot \end{smallmatrix}$ column appears, whence (C). By inspection, T_{i+1} is a SIST.

Now suppose we need to Explode: $T_i \longrightarrow U \hookrightarrow T_{i+1}$ has two cases:

$$\begin{array}{|c|c|c|c|}\hline \cdots & b & \bar{2} & \cdot \\ \hline \cdots & c & \cdot & * \\ \hline \end{array} \overset{(E1)}{\hookrightarrow} \begin{array}{|c|c|c|c|}\hline \cdots & b & 2 & \bar{2} \\ \hline \cdots & c & \bar{2} & * \\ \hline \end{array} \qquad \text{OR} \qquad \begin{array}{|c|c|c|c|}\hline \cdots & b & a & \cdot \\ \hline \cdots & c & \cdot & * \\ \hline \end{array} \overset{(E2)}{\hookrightarrow} \begin{array}{|c|c|c|c|}\hline \cdots & b & a & 2 \\ \hline \cdots & c & \bar{2} & * \\ \hline \end{array}$$

Since T_i had no shape violation, $U(2,-3) = c$ is not empty. It is clear that $\operatorname{sch} T_{i+1} = \square$. Since $b < c \leq \bar{2}$ implies $b \leq 2$, we get that T_{i+1} is a SIST in the first case. In the second it follows from the condition $a \leq 2$. Finally, we see that T_{i+1} is not $2\bar{2}$-deletable because there is only one empty cell in the second row.

$\square$

Definition 4.13 Within a SIST, we use the symbol $\times$ to represent a cell which may contain either a $\cdot$ or $*$. We define a map g_1 by iterating the following procedure. We start with $T \in \mathrm{SIST}(2,(s))$ or $\mathrm{SIST}((1,1))$ and a complementary shape $\lambda = \emptyset$ or $\square$. Before performing a single reverse slide into the complementary shape, we make one of the following replacements (the first one that applies, if any), *provided the resulting tableau is a SIST.*

- (**BarTurBack**)

$$\begin{array}{|c|c|}\hline 2 & \cdot \\ \hline \bar{2} & * \\ \hline \end{array} \overset{(BB1)}{\hookrightarrow} \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \cdot & * \\ \hline \end{array} \qquad \text{OR} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} & \bar{2} \\ \hline \end{array} \overset{(BB2)}{\hookrightarrow} \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \cdot & \cdot \\ \hline \end{array}$$

- (**Implode**)

$$\begin{array}{|c|c|}\hline a & 2 \\ \hline \bar{2} & * \\ \hline \end{array} \overset{(I2)}{\hookrightarrow} \begin{array}{|c|c|}\hline a & \cdot \\ \hline \cdot & * \\ \hline \end{array}, \text{ for } a \leq 2. \qquad \text{OR} \qquad \begin{array}{|c|c|c|}\hline a & 2 & \bar{2} \\ \hline b & \bar{2} & * \\ \hline \end{array} \overset{(I1)}{\hookrightarrow} \begin{array}{|c|c|c|}\hline a & \bar{2} & \cdot \\ \hline b & \cdot & * \\ \hline \end{array}, \quad \text{for } a < 2,$$

Then perform a single reverse slide into the leftmost cell of the complementary shape of the resulting tableau, which extracts a term t_i. The cases overlap for tableaux ending $\begin{smallmatrix}2&2&2\\ \bar{2}&\bar{2}&*\end{smallmatrix}$ and $\begin{smallmatrix}2&2&\cdot\\ \bar{2}&\bar{2}&*\end{smallmatrix}$.

The procedure ends when a tableau $T \in \mathrm{SIST}(1,\lambda)$ is reached to which none of the replacements above applies. The output is the pair (T, m), where m is the product of the terms t_i extracted at each step.

Note that the proviso that the immediate result of replacement be a SIST only applies to $(BB2)$; e.g.,

$$\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline \bar{2}&\bar{2}&\bar{2}\\ \hline\end{array} \overset{(BB2)}{\hookrightarrow} \begin{array}{|c|c|c|}\hline 2&2&\bar{2}\\ \hline \bar{2}&\cdot&\cdot\\ \hline\end{array} \qquad \text{NOT} \qquad \begin{array}{|c|c|c|}\hline 2&\bar{2}&2\\ \hline \cdot&\cdot&\bar{2}\\ \hline\end{array}$$

Lemma 4.14. *The procedure above gives a well-defined map*

$$\coprod_{r\geq 1} \mathrm{SIST}(2,(r)) \coprod \mathrm{SIST}(2,(1,1)) \xrightarrow{g_1} \coprod_{\lambda=\emptyset,\Box} \mathrm{SIST}(1,\lambda) \times \mathbb{Z}^2_{\geq 0}$$

Proof. Given a pair (S,λ) where S is a SIST in the preimage of g_1 and $\lambda = \emptyset$ or $\Box$ is the complementary shape to slide into, let

$$S = S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow \cdots \longrightarrow S_m$$

represent the sequence of SISTs obtained by the procedure. We will show by induction that each S_i has the following properties:

A) $\operatorname{sch} S_i \backslash \lambda$ is a horizontal strip (possibly empty).
B) S_i is in $\mathrm{SIST}(2, \operatorname{sch} S_i)$.
C) $n(S_i) := |\operatorname{sch} S_i\backslash\lambda| + 2|\{\bar{2} \in S_i\}|$ is a strictly decreasing function of i.

The function n gives an upper bound on the number of steps remaining in the algorithm, since each implosion requires a $\bar{2}$ and leads to two more sliding steps.

Note we always have a well-defined case to apply to our tableau: say $S_i \hookrightarrow U \mapsto S_{i+1}$. To show (C), note that each BarTurBack and Implode transformation reduces by one the number of $\bar{2}$'s and increases $|\operatorname{sch} S_i\backslash\lambda|$ by two, which the following reverse slide reduces by one. Hence, $n(S_{i+1}) = n(S_i) - 1$. Since each BarTurBack and Implode transformation preserves (A) and (B), as does sliding, it follows by induction that each S_i satisfies all three properties.

The g_1 procedure must finish after finitely many steps because of property (C). We claim that the final output is in $\mathrm{SIST}(1,\lambda)$. If $\lambda = \Box$, then we need that the final SIST S_m is not $\genfrac{}{}{0pt}{}{2}{\bar{2}}$-deletable. If it were, then it would contain a $\genfrac{}{}{0pt}{}{2}{\bar{2}}$ column, forcing one of the Implode or BarTurBack cases to apply. But then S_m is not the last SIST of the procedure. On the other hand, if $\lambda = \emptyset$ and S_m is $\genfrac{}{}{0pt}{}{22}{\bar{2}\bar{2}}$-deletable, then S_m falls into the third BarTurBack case, and sliding would have continued. $\Box$

The following technical lemma will help us show that the maps f_1 and g_1 are inverses.

Lemma 4.15. *In applying the g_1 map step by step, we extract all t_2 terms before extracting any t_1 terms.*

Proof. The penultimate paragraph of the proof of Theorem 3.5 showed that this holds for each step of g_1 consisting of a pure reverse slide. Thus, it suffices to show the following: at no point in the g_1 algorithm do we have the sequence of moves

$$R \overset{[t_1]}{\mapsto} S \overset{(\tau)}{\hookrightarrow} T \overset{[t_2]}{\mapsto} U$$

where (τ) is one of the allowed replacements. We show this by contradiction in each case.

In case $(BB2)$ we would have:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 & 2 & c_1 & \cdots \\ \hline \cdots & b_3 & b_2 & b_1 & \bar{2} & c_2 & \cdots \\ \hline \end{array} \overset{[t_1]}{\mapsto} \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a_1 & 2 & 2 & \cdots \\ \hline \cdots & b_3 & b_2 & b_1 & \bar{2} & \bar{2} & \cdots \\ \hline \end{array} \overset{(BB2)}{\hookrightarrow} \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a_1 & 2 & \bar{2} & \cdots \\ \hline \cdots & b_3 & b_2 & b_1 & \cdot & \cdot & \cdots \\ \hline \end{array} \overset{[t_2]}{\mapsto} U$$

where the final step implies the inequalities $b_1 > 2$ and $b_{i+1} > a_i$ for $i \geq 2$ in order for the t_2 reverse slide to take place entirely along row two. These inequalities force the t_1 reverse slide to have the form shown, entirely across the top row until the last column shown explicitely; the column $\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}$ must be of the form $\begin{smallmatrix} \bar{2} \\ \cdot \end{smallmatrix}$, $\begin{smallmatrix} \bar{2} \\ * \end{smallmatrix}$, or $\begin{smallmatrix} \cdot \\ * \end{smallmatrix}$. Since $b_1 = \bar{2}$, we find the tableau R has the configuration $\begin{smallmatrix} 2 & 2 \\ \bar{2} & \bar{2} \end{smallmatrix}$, and $(BB2)$ should have been applied to the rightmost such, rather than a pure t_1 reverse slide. This contradicts the definition of g_1.

Following the same line of reasoning, in case $(BB1)$ we would have:

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 & \bar{2} & \cdot \\ \hline \cdots & b_3 & b_2 & b_1 & \cdot & * \\ \hline \end{array} \overset{[t_1]}{\mapsto} \begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a_1 & 2 & \cdot \\ \hline \cdots & b_3 & b_2 & b_1 & \bar{2} & * \\ \hline \end{array} \overset{(BB2)}{\mapsto} \begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a_1 & 2 & \bar{2} \\ \hline \cdots & b_3 & b_2 & b_1 & \cdot & * \\ \hline \end{array} \overset{[t_2]}{\mapsto} U .$$

The leftmost tableau R could not have occurred in the g_1 process since it is not a valid SIST, nor a configuration that arises from one of the g_1 transformations.

Following the same line of reasoning, in case $(I2)$ we would have:

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & a_2 & a & 2 & \cdot \\ \hline \cdots & b_3 & b_2 & \bar{2} & * \\ \hline \end{array} \overset{[t_1]}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a & 2 \\ \hline \cdots & b_3 & b_2 & \bar{2} & * \\ \hline \end{array} \overset{(I2)}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a & \cdot \\ \hline \cdots & b_3 & b_2 & \cdot & * \\ \hline \end{array} \overset{[t_2]}{\mapsto} U .$$

The leftmost tableau R could not have occurred in the g_1 process since it is not the output of any g_1-transformation, and the g_1 process would have applied $(BB1)$ rather than a simple reverse slide.

Finally, one can never have $(I1)$ followed by $[t_2]$ since

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & a_2 & a & 2 & \bar{2} \\ \hline \cdots & b_2 & b & \bar{2} & * \\ \hline \end{array} \overset{(I1)}{\mapsto} \begin{array}{|c|c|c|c|c|}\hline \cdots & a_2 & a & \bar{2} & \cdot \\ \hline \cdots & b_2 & b & \cdot & * \\ \hline \end{array} \overset{[t_2]}{\mapsto} U .$$

would force b to land below $\bar{2}$, which is impossible. ☐

Lemma 4.16. *For $\lambda = \emptyset$ or $\square$ the procedures f_1 and g_1 are inverses, to wit:*

$$g_1 \circ f_1(T, M) = \mathrm{id} \colon \ \mathrm{SIST}(1, \lambda) \times \mathbb{Z}^2_{\geq 0} \ \textit{and} \ f_1 \circ g_1 = \mathrm{id} \colon \coprod_{r \geq 1} \mathrm{SIST}(2, (r)) \coprod \mathrm{SIST}(2, (1, 1))$$

Proof. To show that $g_1 \circ f_1(T, M) = \mathrm{id}$, we must prove that at each step of the g_1 algorithm we recreate the same sequence of shapes that f_1 must have created to reach a given tableau T. In other words, even if T might have arisen *a priori* by some other combination of slides and f_1-transformations, we will show by tracing backwards the impossibility that f_1 could have created T in any other way. This will insure that g_1 correctly undoes the f_1 procedure step by step. In general T may arise from (1) a single t_1 or t_2 slide, or possibly the empty slide if $T \in \mathrm{SIST}(1, \lambda)$, (2) from an $(E1)$ or $(E2)$ explosion preceded immediately by a slide, or (3) from a $(B1)$ or $(B2)$ BarTur preceded immediately by slide.

Case $(BB1)$**:** Suppose T contains the configuration $\begin{smallmatrix} 2 & \cdot \\ \bar{2} & * \end{smallmatrix}$. By inspection, T is not the result of $(E1)$ or $(E2)$. If it came from a pure slide, then we must have:

$$\begin{array}{|c|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 & \cdot \\ \hline \cdots & b_2 & b_1 & \bar{2} & * \\ \hline \end{array} \xleftarrow{t_1} \begin{array}{|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 \\ \hline \cdots & b_1 & \bar{2} & * \\ \hline \end{array} \overset{(E2)}{\hookleftarrow} \begin{array}{|c|c|c|c|}\hline \cdots & a_2 & a_1 & \cdot \\ \hline \cdots & b_1 & \cdot & * \\ \hline \end{array} \xleftarrow{t_2}$$

where a quick inspection reveals that the middle tableau could only have come from $(E2)$. Since $a_i < b_i$ for all i, the preceding slide must be an t_2. But now we have an f_1 path in which an t_2 slide preceded an t_1 slide, contradicting the definition of f_1. Alternatively, if it came from $(B2)$ preceded by a slide: $T \overset{(B2)}{\hookleftarrow} T' \overset{t_i}{\longleftarrow} U$, then $\operatorname{sch} U = (2,1)$, which is impossible for a tableau in the image of f_1. Hence, T must have been created in the last step by a $(B1)$ transfomration.

Case $(BB2)$: This case is the most involved because the $(B2)$ and $(BB2)$ transformations can occur in columns other than the rightmost two. This requires us to consider several subcases.

Case $\lambda = \emptyset$: In this case the only columns occuring to the right of the rightmost $\substack{2\,2\\ \bar2\,\bar2}$ (which is the only columns where $(BB2)$ is allowed) are of the form $\substack{\bar2\\ \cdot}$. Tracing back, we find that at each stage the only possibilites are that we applied $(B2)$ or a pure t_1 slide. Since we are assuming by way of contradiction that T was not created by $(B2)$, the last step was an t_1 slide. If any move as we trace back was $(B2)$, then it must have been preceeded by an t_2 slide since $a_{i+s} \le a_i < b_i$ for any $s \in \mathbb{Z}_{\ge 0}$. But we must eventually apply $(B2)$ since there are a finite number of $\substack{\bar2\\ \cdot}$ columns to the right of $\substack{2\,2\\ \bar2\,\bar2}$, with one $\cdot$ being filled with each t_1 slide we trace back. Eventually we reach $\operatorname{sch} = \emptyset$, and the final two columns are $\substack{2\,2\\ \bar2\,\bar2}$. Therefore, we reach the contradiction of having an t_2 slide precede and t_1 slide.

Case $\lambda = \square$: Let the negative integer k denote the rightmost column in which $\substack{2\\ \bar2}$ appears. The only possible columns to the right of this are $\substack{\bar2\\ \cdot}$, $\substack{\bar2\\ *}$, or $\substack{\cdot\\ *}$. In particular, when $k = -2$, we know that T ends with: $\substack{2\,2\,\bar2\\ \bar2\,\bar2\,*}$ or $\substack{2\,2\,\cdot\\ \bar2\,\bar2\,*}$. We already showed that the latter case must come from $(B1)$, so we consider the former, tracing back:

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 & 2 & \bar2 \\ \hline \cdots & b_2 & b_1 & \bar2 & \bar2 & * \\ \hline \end{array} \overset{(E1)}{\hookleftarrow} \begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_2 & a_1 & 2 & \bar2 & \cdot \\ \hline \cdots & b_2 & b_1 & \bar2 & \cdot & * \\ \hline \end{array} \overset{t_1}{\longleftarrow} \begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_3 & a_2 & a_1 & 2 & \cdot \\ \hline \cdots & b_2 & b_1 & \bar2 & \bar2 & * \\ \hline \end{array} \overset{(B1)}{\hookleftarrow}$$

where the last $(B1)$ is forced by an earlier case, and must be preceded by an t_2 slide by the usual inequalities, yielding the same t_2 slide preceding an t_1 slide contradiction as before.

To handle the general case where $k < -2$ we note that as in the $\lambda = \emptyset$ case, until we slide into $(2,-2)$ the only possible backwards steps are $(B2)$ preceded by an t_1 slide, or a pure t_1 slide. Since we want to show that $(BB2)$ is correct, we suppose by way of contradiction that T was immediately created by a pure t_1 slide. The arguments of the last two paragraphs show that we eventually perform the $(BB2)$ operation preceded by an t_2 slide, which is a contradiction. This finishes the $(BB2)$ case.

In the $(I1)$ and $(I2)$ cases, it is clear by inspection that such a tableau could only arise from $(E1)$ or $(E2)$ respectively. This completes the proof that $g_1 \circ f_1(T, M) = \mathrm{id}$.

To show that $f_1 \circ g_1 = \mathrm{id}$ is much easier. By definition, g_1 is a sequence of moves each of which is a reverse slide, or a transformation followed by a reverse slide. Since each f_1-move is a slide or a slide followed by a transformation, and since sliding is reversible by the results of Section 3, we know that the initial slide of f_1 correctly reconstructs the tableau T created by the final reverse slide of g_1. We claim that if T falls into one of the cases calling for a transformation in the definition of f_1, that this T could only have arisen from its opposite transformation in g_1.

Case $(E1)$: If T ends with $\substack{\bar2\,\cdot\\ \cdot\,*}$, we claim that it must have arisen from $(I1)$. Indeed, since T contains a symplectic violation, it could not be the end result of any step of g_1 (Lemma 4.14). By inspection this configuration can not be created by $(BB1)$ or $(I2)$. If it were created by $(BB2)$, the symplectic violation would remain. The claim follows.

The $(E2)$, $(B1)$, and $(B2)$ cases follow similarly. The only new idea is that for the latter two cases, T is $2\bar{2}$-deletable, which is why it could not be the end result of any step of g_1. □

Definition 4.17 Define the map f_2 by iterating the following procedure. We start with a tableau in $\mathrm{SIST}(2,(s))$ with $s \geq 2$ and a monomial m. Perform one slide according to the monomial. If we obtain a tableau T which contains the configuration $\begin{smallmatrix} \cdot & \cdot \\ * & * \end{smallmatrix}$ (a shape violation) then explode a $2\bar{2}$ pair in $\mathrm{Row}_1 T$. The new $2\bar{2}$ pair must be placed just to the right of the rightmost 2 in $\mathrm{Row}_1 T$, possibly displacing a string of $\bar{2}$'s to the right.

Lemma 4.18. *For $s \geq 2$ the procedure above gives a well-defined map:*

$$\mathrm{SIST}(2,(s)) \times \mathbb{Z}^2_{\geq 0} \xrightarrow{f_2} \coprod_r \Big[\mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1))\Big].$$

Proof. Suppose we have a sequence of steps $T = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n = f_2(T)$ as described above, and assume that T_j satisfies the following condition:

$$T_j \in \mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1) \quad (*)$$

We wish to show that the same condition holds for T_{j+1}.

If the step $T_j \longrightarrow T_{j+1}$ involves no $(E3)$ then the condition holds by familiar properties of ordinary sliding. Now suppose we have $T_j \overset{t_i}{\mapsto} U \overset{(E2)}{\hookrightarrow} T_{j+1}$. The configuration $\begin{smallmatrix} \cdot & \cdot \\ * & * \end{smallmatrix}$ must occur in the two rightmost columns since U was obtained by sliding from a SIST. Further, these are the only columns of the form $\begin{smallmatrix} \cdot \\ * \end{smallmatrix}$ because if there were more than two, then U was created by a slide that left in hole a column $k < -2$, whence the last two columns of T_j were empty, contradicting the condition. So when we apply $(E3)$ to U,we get $\mathrm{sch}\, T_{j+1} = (s)$. It is easy to see that T_{j+1} is a SIST since Exploding $2\bar{2}$ in the first row preserves inequalities along the first row and column strictness. So T_{j+1} satisfies $(*)$. This argument also shows that all the explosions happen in the initial stages of f_1: a pair of t_1 slides entirely across the top row is followed by $(E3)$, returning the tableau to the initial shape. Once a slide descends into the second row, no more explosions can occur and the complementary shape monotonically increases.

Hence, by induction, f_2 is well defined. □

Definition 4.19 Define the map g_2 by iterating the following procedure. Given an integer $s \geq 2$ and a tableaux in $\mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1)$ for some $r \in \mathbb{Z}_{\geq 0}$, we check whether the $\mathrm{sch}\, T = (s)$. If not, then perform a single reverse slide into $\mathrm{sch} = (s)$. If so, and if T is $2\bar{2}$-deletable, then implode a $2\bar{2}$ pair in row 1 of T (shoving any elements to the right of the implosion leftwards two spaces) and perform a single reverse slide along $\mathrm{Row}_1 T$. The procedure terminates when $\mathrm{sch}\, T = (s)$ and T is not $2\bar{2}$-deletable, and the monomial extracted is the product of the terms extracted in each sliding step.

Lemma 4.20. *For $s \geq 2$ the procedure above gives a well-defined map:*

$$\coprod_r \Big[\mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1))\Big] \xrightarrow{g_2} \mathrm{SIST}(2,(s)) \times \mathbb{Z}^2_{\geq 0}.$$

Proof. Since only ordinary sliding is applied until $\mathrm{sch}\, T = (s)$, the procedure is clearly well-defined and always gives a SIST. At this point, if T is $2\bar{2}$-deletable, then it must contain a $2\bar{2}$ pair that lies above empty cells in (s); hence we can implode $2\bar{2}$, leaving a valid SIST, then reverse slide, giving a tableau of $\mathrm{sch} = (s,1)$. (In the next step, an ordinary reverse slide will be applied to this resulting tableau.)

To show that the procedure terminates, note that the number of implosions is bounded above by the minimum of the number of 2's in $\mathrm{Row}_1 T$ and the number of $\bar{2}$'s in $\mathrm{Row}_1 T$. Once we reach sch $= (s)$, all the remaining reverse slides take place entirely across the top row. The final output will be in $\mathrm{SIST}(2,(s))$ since the procedure continues until it reaches a tableau that is not $2\bar{2}$-deletable. □

Lemma 4.21. *For $s \geq 2$ the procedures f_2 and g_2 are inverses, to wit:*

$$g_2 \circ f_2(T,M) = \mathrm{id}\colon\ \mathrm{SIST}(2,(s)) \times \mathbb{Z}^2_{\geq 0} \text{ and } f_2 \circ g_2 = \mathrm{id}\colon \coprod_r \left[\mathrm{SIST}((s+r)) \coprod \mathrm{SIST}((s+r,1))\right]$$

Proof. The key idea is to note that whereas all the f_2 explosions take place as the initial steps of f_2 and reset the complementary shape to be (s), after which all steps are ordinary slides, that g_2 exactly reverses this, reverse sliding until it gets to complementary shape (s), then undoing all the initial explosions. □

This completes the full version of Schensted's correspondence for the Weil representation in the case $n = 2$, and leaves the case of general n as a tantilizing open problem, for which the techniques developed here may be useful.

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